SEMILOCAL CONVERGENCE OF NEWTON'S METHOD FOR SINGULAR SYSTEMS WITH CONSTANT RANK DERIVATIVES

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ABSTRACT. We provide a semilocal convergence result for approximating a solution of a singular system with constant rank derivatives, using Newton's method in an Euclidean space setting. Our approach uses more precise estimates and a combination of two Lipschitz–type conditions leading to the following advantages over earlier works [13], [16], [17], [29]: tighter bounds on the distances involved, and a more precise information on the location of the solution. Numerical examples are also provided in this study.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$(1.1) \qquad F'(x)^+ F(x) = 0,$$

where, F is a continuously Fréchet–differentiable operator defined on an open, nonempty, convex subset $\mathcal{D}$ of $\mathcal{X}$ in $\mathcal{Y}$, where X and $\mathcal{Y}$ are an Euclidean spaces $(\dim \mathcal{X} = i;\ \dim \mathcal{Y} = j;\ i, j \in \mathbb{N})$, and for arbitrary $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, A^+ denotes a Moore–Penrose generalized inverse of A, defined as the unique linear operator in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfying

$$A\ A^+\ A = A, \quad A^+\ A\ A^+ = A^+, \quad (A^+\ A)^T = A^+\ A, \quad (A\ A^+)^T = A\ A^+.$$

The field of computational sciences has seen a considerable development in mathematics, engineering sciences, and economic equilibrium theory. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$,

for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. We note that in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method.

Newton's method (NM)

$$(1.2) \qquad x_{n+1} = x_n - F'(x_n)^+ \, F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}),$$

is undoubtedly the must popular way for generating a sequence $\{x_n\}$ approximating a solution $x^\star$. A survey on local as well as semilocal convergence results for Newton–type methods can be found in [1]–[29] and the references there.

In this study, we are motivated by the recent elegant study by Xu and Li [29] (which improved earlier results [9]), and optimization considerations. Using a combination of two Lipschitz–type conditions, we provide a new semilocal convergence result for Newton's method (1.3) with the following advantages (under the same hypotheses and computational cost) over [29]:

(a) Tighter error bounds estimates on $\| x_{n+1} - x_n \|$ and $\| x_n - x^\star \|$ $(n \geq 0)$;

(b) A more precise information on the location of the solution $x^\star$.

Numerical examples and applications are also provided in this study.

It is well known that the solution set of equation (1.1) may contain elements not in the solution set of equation ([4], [8])

$$(1.3) \qquad\qquad\qquad\qquad F(x) = 0.$$

The paper is organized as follows: Preliminary results are provided in Section 2. The semilocal convergence of Newton's method is given in Section 3. Finally, the numerical examples and applications are provided in Section 4.

2. Preliminary Results

In order for us to make the study as self contained as possible, we re–introduce some basic notions and results [4], [21], [29]. Let $\mathcal{X}$, $\mathcal{Y}$ denote the two Euclidean spaces, and $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ as defined in the introduction. We suppose $\operatorname{rank} F'(x) \leq r$ $(x \in \mathcal{D})$ and $r \leq \min\{i, j\}$ is a positive integer. Denote by $I_{\mathcal{X}}$ the identity on $\mathcal{X}$ and $U(z, R)$ the open ball with radius $R > 0$ and center $z \in \mathcal{X}$. Finally, for matrix $\mathcal{M}$, denote by $Ker\,\mathcal{M}$ and $Im\,\mathcal{M}$ the kernel and image of $\mathcal{M}$, respectively, and Π_E the orthogonal projection onto a subspace E of $\mathcal{X}$. We have the following statments:

$$(2.1) \qquad \mathcal{M}^+ \mathcal{M} = \Pi_{Ker\,\mathcal{M}^\perp} \quad \text{and} \quad \mathcal{M}\mathcal{M}^+ = \Pi_{Im\,\mathcal{M}}.$$

We will use two lemmas of perturbations bounds for Moore–Penrose inverse, whose proofs can be found in [21].

Lemma 2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $i \times j$ matrices, with $\operatorname{rank}\mathcal{N} = \operatorname{rank}\mathcal{M} = r$, and $\parallel \mathcal{M}^+ \parallel \parallel \mathcal{N} - \mathcal{M} \parallel < 1$. Then, the following holds*

$$\parallel \mathcal{N}^+ - \mathcal{M}^+ \parallel \leq c\, \frac{\parallel \mathcal{M}^+ \parallel^2 \parallel \mathcal{N} - \mathcal{M} \parallel}{1 - \parallel \mathcal{M}^+ \parallel \parallel \mathcal{N} - \mathcal{M} \parallel}$$

where,

$$(2.2) \qquad c = \begin{cases} \dfrac{1 + \sqrt{5}}{2} & \text{if} & r < \min\{i, j\} \\ \sqrt{2} & \text{if} & r = \min\{i, j\}\ (i \neq j) \\ 1 & \text{if} & r = i = j. \end{cases}$$

We only study singular case when $r < \min\{i, j\}$ (i.e., $c = \dfrac{1 + \sqrt{5}}{2}$ in Lemma 2.1). The case $r = \min\{i, j\}$ can also studied similarly.

Lemma 2.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $i \times j$ matrices, with $\operatorname{rank}(\mathcal{M} + \mathcal{N}) \leq \operatorname{rank}\mathcal{M} = r$, and $\parallel \mathcal{M}^+ \parallel \parallel \mathcal{N} \parallel < 1$. Then, the following hold*

$$\operatorname{rank}(\mathcal{M} + \mathcal{N}) = r$$

and

$$\parallel (\mathcal{M} + \mathcal{N})^+ \parallel \leq \frac{\parallel \mathcal{M}^+ \parallel}{1 - \parallel \mathcal{M}^+ \parallel \parallel \mathcal{N} \parallel}.$$

We need the following definition.

Definition 2.3. Let $R > 0$ be such that $U(x_0, R) \subseteq \mathcal{D}$, and $L_0(u)$, $L(u)$ be positive nondecreasing functions on $[0, +\infty)$.

(a) Mapping F' satisfies the center–Lipschitz condition with L_0–average in $U(x_0, R)$, if, for any $x \in U(x_0, R)$, the following holds

(2.3) $$\| F(x_0)^+ \| \, \| F'(x) - F'(x_0) \| \leq \int_0^{\|x-x_0\|} L_0(u) \ du.$$

(b) Mapping F' satisfies the center–Lipschitz condition in the inscribed sphere with L–average in $U(x_0, R)$, if, for any $x, y \in U(x_0, R)$ with $\| x - x_0 \| + \| y - x \| < R$, the following holds

(2.4) $$\| F(x_0)^+ \| \, \| F'(y) - F'(x) \| \leq \int_{\|x-x_0\|}^{\|x-x_0\|+\|y-x\|} L(u) \ du.$$

Remark 2.4. (i) By Definition 2.3, if F' satisfies the center–Lipschitz condition in the inscribed sphere with L–average in $U(x_0, R)$, then, F' satisfies the center–Lipschitz condition with L–average in $U(x_0, R)$.

(ii) Note that

(2.5) $$L_0(u) \leq L(u) \qquad u \in [0, +\infty),$$

and $\dfrac{L(u)}{L_0(u)}$ can be arbitrarily large [1]–[7].

(iii) If equality holds in (2.5), then our Definition 2.3 coincides with [29, Definition 2.2, p. 691].

We need the following results.

Lemma 2.5. *Assume that F' satisfies the center–Lipschitz condition with L_0–average in $U(x_0, R)$, and for $x \in U(x_0, R)$:*

$$\mathrm{rank} \, F'(x) \leq \mathrm{rank} \, F'(x_0) = r \ \ \text{and} \ \ \int_0^{\|x-x_0\|} L_0(u) \ du < 1.$$

Then, for $x \in U(x_0, R)$, the following hold:

(2.6) $$\mathrm{rank} \, F'(x) = r,$$

(2.7) $$\| F'(x) \| \leq \| F'(x_0) \| + \frac{1}{\| F'(x_0)^+ \|} \int_0^{\|x-x_0\|} L_0(u) \ du,$$

and

(2.8) $$\| F'(x)^+ \| \leq \frac{\| F'(x_0)^+ \|}{1 - \int_0^{\|x-x_0\|} L_0(u) \ du}.$$

Proof. (a) We have:

$$(2.9) \qquad \Pi_{Ker\, F'(x_0)} + F'(x_0)^+ \, F'(x) = I_{\mathcal{X}} - F'(x_0)^+ \, (F'(x_0) - F'(x)),$$

and

$$(2.10) \qquad \| \, F'(x_0)^+ \, \| \, \| \, F'(x_0) - F'(x) \, \| \le \int_0^{\|x-x_0\|} L_0(u) \, du < 1.$$

Hence $\Pi_{Ker\, F'(x_0)} + F'(x_0)^+ \, F'(x)$ is nonsingular. Using (2.1), we obtain

$$\begin{aligned} \Pi_{Im\, F'(x_0)} \, F'(x) &= F'(x_0) \, F'(x_0)^+ \, F'(x) + F'(x_0) \, \Pi_{Ker\, F'(x_0)} \\ &= F'(x_0) \, (F'(x_0)^+ \, F'(x) + \Pi_{Ker\, F'(x_0)}). \end{aligned}$$

That is, we get

$$\mathrm{rank}\, (\Pi_{Im\, F'(x_0)} \, F'(x)) = \mathrm{rank}\, F'(x_0) = r,$$

which together with $\mathrm{rank}\, F'(x) \le \mathrm{rank}\, F'(x_0) = r$, imply (2.6).

(b) Using (2.3), we get

$$\begin{aligned} \| \, F'(x) \, \| &\le \| \, F'(x_0) \, \| + \| \, F'(x) - F'(x_0) \, \| \\ &\le \| \, F'(x_0) \, \| + \frac{1}{\| \, F'(x_0)^+ \, \|} \int_0^{\|x-x_0\|} L_0(u) \, du. \end{aligned}$$

(c) Set $\mathcal{M} = F'(x_0)$, and $\mathcal{N} = F'(x) - F'(x_0)$. In view of (2.10), $\mathrm{rank}\, \mathcal{M} = r$, $\| \, \mathcal{M}^+ \, \| \, \| \, \mathcal{N} \, \| \le \int_0^{\|x-x_0\|} L_0(u) \, du < 1.$ By Lemma 2.2, we get

$$\| \, F'(x)^+ \, \| = \| \, (\mathcal{M} + \mathcal{N})^+ \, \| \le \frac{\| \, \mathcal{M}^+ \, \|}{1 - \| \, \mathcal{M}^+ \, \| \, \| \, \mathcal{N} \, \|} \le \frac{\| \, F'(x_0)^+ \, \|}{1 - \int_0^{\|x-x_0\|} L_0(u) \, du}.$$

That completes the proof of Lemma 2.5. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 2.6. If $L_0(u) = L(u)$, Lemma 2.5 coincides with [29, Lemma 2.3, p. 692].

Lemma 2.7 ([29])**.** *Assume that F' satisfies the center–Lipschitz condition in the inscribed sphere with L–average in $U(x_0, R)$, and $\int_0^R L(u) \, du < 1$. Let $x \in U(x_0, R)$, and y such that $\| \, x - x_0 \, \| + \| \, y - x \, \| < R$, and $\max\{\mathrm{rank}\, F'(x), \mathrm{rank}\, F'(y)\} \le \mathrm{rank}\, F'(x_0) = r$. Then*

$$\| \, F'(y)^+ - F'(x)^+ \, \|$$

$$(2.11) \qquad \le \frac{1 + \sqrt{5}}{2} \frac{\| \, F'(x)^+ \, \|^2 \, \| \, F'(x_0)^+ \, \|^{-1} \int_0^{\|y-x\|} L(\| \, x - x_0 \, \| + u) \, du}{1 - \| \, F'(x)^+ \, \| \, \| \, F'(x_0)^+ \, \|^{-1} \int_0^{\|y-x\|} L(\| \, x - x_0 \, \| + u) \, du}.$$

Lemma 2.8 ([20])**.** *Let $\varrho \geq 0$, and define functions ψ_1 and ψ_2 on $(0, +\infty)$ by:*

(2.12) $\qquad \psi_1(t) = \dfrac{1}{t} \displaystyle\int_0^t L(u)\, du \quad \text{and} \quad \psi_2(t) = \dfrac{1}{t} \displaystyle\int_0^t (t-u)\, L(\varrho + u)\, du.$

Then ψ_1 and ψ_2 are positive nondecreasing on $(0, +\infty)$.

Lemma 2.9 ([27])**.** *Assume that F' satisfies the center–Lipschitz condition in the inscribed sphere with L–average in $U(x_0, R)$. Let $x, y \in U(x_0, R)$ with $\parallel x - x_0 \parallel + \parallel y - x \parallel < R$. Then, the following hold:*

$$\parallel F(x) - F(y) - F'(x)\,(x-y) \parallel$$

(2.13)
$$\leq \frac{1}{\parallel F'(x_0)^+ \parallel} \int_0^{\|y-x\|} (\parallel y - x \parallel - u)\, L(\parallel x - x_0 \parallel + u)\, du.$$

and

(2.14)
$$\parallel F(y) - F(x) \parallel$$
$$\leq \frac{1}{\parallel F'(x_0)^+ \parallel} \int_0^{\|y-x\|} (\parallel y - x \parallel - u)\, L(\parallel x - x_0 \parallel + u)\, du + \parallel F'(x) \parallel \parallel y - x \parallel .$$

3. Semilocal Convergence of Newton's Method

Definition 3.1. Let

(3.1) $$\mathcal{S} = \{x \,:\, F'(x)^+\, F(x) = 0\}.$$

Newton's method converges, in general to a point in $\mathcal{S}$ rather than a solution of the equation $F(x) = 0$, when F is a singular system.

Definition 3.2. It is convenient for us to define some constants. Let $a \in [0, 1)$, $b = \dfrac{1}{1-a}$,

$$K = \parallel F'(x_0)^+ \parallel \parallel F'(x_0) \parallel, \qquad \beta = \parallel F'(x_0)^+ \parallel \parallel F(x_0) \parallel,$$

$$\delta_1 = \int_0^\beta L(b\,\beta + u)\, du, \qquad \overline{\delta_1} = \int_0^\beta L_0(b\,\beta + u)\, du,$$

$$\delta_2 = \int_0^{b\,\beta} L(u)\, du, \qquad \overline{\delta_2} = \int_0^{b\,\beta} L_0(u)\, du,$$

$$\Delta_1 = \int_0^\beta (\beta - u)\, L(b\,\beta + u)\, du, \qquad \overline{\Delta_1} = \int_0^\beta (\beta - u)\, L_0(b\,\beta + u)\, du,$$

$$\Delta_2 = \int_0^{b\,\beta} (b\,\beta - u)\, L(b\,\beta + u)\, du, \qquad \overline{\Delta_2} = \int_0^{b\,\beta} (b\,\beta - u)\, L_0(b\,\beta + u)\, du$$

$$p = \frac{\Delta_1}{\beta\,(1 - \overline{\delta_2})} + \frac{1 + \sqrt{5}}{2}\,\frac{\delta_1\,(\beta + \beta\,\overline{\delta_2} + (b+1)\,\beta\,K + \Delta_1 + \overline{\Delta_2})}{\beta\,(1 - \overline{\delta_2})\,(1 - \delta_1 - \overline{\delta_2})},$$

and

$$p_0 = \frac{\overline{\Delta_1}}{\beta} + \frac{1 + \sqrt{5}}{2}\,\frac{\overline{\delta_1}\,(\beta + \beta\,K + \overline{\Delta_1})}{\beta\,(1 - \overline{\delta_1})}.$$

We shall provide a semilocal convergence analysis for Newton's method.

Theorem 3.3. *Let* $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ *satisfying (2.3), (2.4) in* $\mathcal{D}_0 = U(x_0, b\,\beta)$, *and* $\operatorname{rank} F'(x) \leq \operatorname{rank} F'(x_0)$ *far all* $x \in \mathcal{D}_0$. *Assume:*

(3.2) $$\delta_1 + \overline{\delta_2} < 1 \quad \text{and} \quad p \leq a.$$

Then, sequence $\{x_n\}$ *generated by Newton's method is well defined, and converges to a point* $x^\star \in \mathcal{S}$. *Moreover, the following estimates hold:*

(3.3) $$\| x_2 - x_1 \| \leq p_0 \| x_1 - x_0 \| \leq p \| x_1 - x_0 \|,$$

(3.4) $$\| x_{n+1} - x_n \| \leq a^n \| x_1 - x_0 \|, \quad (n \geq 2),$$

and

(3.5) $$\| x_0 - x^\star \| \leq b \| x_1 - x_0 \| .$$

Proof. We shall show (3.3), and

(3.6) $$\| x_{k+1} - x_k \| \leq a \| x_k - x_{k-1} \|, \quad k \geq 2$$

by induction on k.

For $x \in \mathcal{D}_0$, we have

(3.7) $$\int_0^{\|x - x_0\|} L_0(u)\,du \leq \int_0^{b\,\beta} L_0(u)\,du = \overline{\delta_2} < 1.$$

Using (2.6), we get $\operatorname{rank} F'(x) \leq \operatorname{rank} F'(x_0)$ for $x \in \mathcal{D}_0$. We also have:

$$x_1 - x_0 = -F'(x_0)^+ F(x_0) \in \operatorname{Ker} F'(x_0)^\perp \quad \text{and} \quad F'(x_0)^+ F'(x_0) = \Pi_{\operatorname{Ker} F'(x_0)^\perp},$$

so,

$$x_2 - x_1 = F'(x_0)^+ F'(x_0)\,(x_1 - x_0) + F'(x_0)^+ F(x_0) - F'(x_1)^+ F(x_1)$$

(3.8) $$= F'(x_0)^+ \left(F'(x_0)\,(x_1 - x_0) + F(x_0) - F(x_1) \right)$$

$$+ \left(F'(x_0)^+ - F'(x_1)^+ \right) \left(F(x_1) - F(x_0) \right) + \left(F'(x_0)^+ - F'(x_1)^+ \right) F(x_0).$$

Since $\| x_1 - x_0 \| \leq \beta$, and $L_0(u)$ is nondecreasing, we deduce from Lemmas 2.6–2.8 that

(3.9)

$\| x_2 - x_1 \|$

$$\leq \int_0^{\|x_1-x_0\|} (\| x_1 - x_0 \| -u) L_0(\| x_1 - x_0 \| +u) du$$

$$+ \frac{1+\sqrt{5}}{2} \frac{\int_0^{\|x_1-x_0\|} L_0(u) du}{1 - \int_0^{\|x_1-x_0\|} L_0(u)\, du} \left(\int_0^{\|x_1-x_0\|} (\| x_1 - x_0 \| -u)\, L_0(\| x_1 - x_0 \| +u) du \right.$$

$$\left. +K \,\| x_1 - x_0 \| \right) + \frac{1+\sqrt{5}}{2} \frac{\beta \int_0^{\|x_1-x_0\|} L_0(u)\, du}{1 - \int_0^{\|x_1-x_0\|} L_0(u)\, du}$$

$$\leq \left(\frac{\overline{\Delta_1}}{\beta} + \frac{1+\sqrt{5}}{2} \frac{\overline{\delta_1}}{1-\overline{\delta_1}} \left(\frac{\overline{\Delta_1}}{\beta} + K + 1 \right) \right)\, \| x_1 - x_0 \|$$

$$= \left(\frac{\overline{\Delta_1}}{\beta} + \frac{1+\sqrt{5}}{2} \frac{\overline{\delta_1}}{\beta\,(1-\overline{\delta_1})}\,(\beta + \beta\,K + \overline{\Delta_1}) \right)\, \| x_1 - x_0 \|$$

$$\leq p_0 \,\| x_1 - x_0 \| \leq a \,\| x_1 - x_0 \|,$$

which implies (3.4), and (3.5) for $k = 1$.

Assume that (3.6) holds for $k = 2, \cdots, k$. Using (3.8) by replacing x_2, x_1 by x_{n+1} and x_n respectively, we get

$$\| x_{n+1} - x_n \| \leq \| F'(x_{n-1})^+ \| \,\| F'(x_{n-1})\,(x_n - x_{n-1}) + F(x_{n-1}) - F(x_n) \|$$

(3.10)
$$+ \| F'(x_{n-1})^+ - F'(x_n)^+ \| \,\| F(x_n) - F(x_{n-1}) \|$$

$$+ \| F'(x_{n-1})^+ - F'(x_n)^+ \| \,\| F(x_{n-1}) \|$$

$$= \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3.$$

We shall estimate $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$. By induction hypothesis (3.6), we have

$$(3.11) \quad \| x_i - x_0 \| \leq \sum_{k=1}^n \| x_k - x_{k-1} \| \leq b \,\| x_1 - x_0 \| \leq b\,\beta \qquad \text{for } i = n,\, n-1.$$

Thus, (2.8) implies

(3.12)
$$\| F'(x_{n-1})^+ \| \leq \frac{\| F'(x_0)^+ \|}{1 - \int_0^{\|x_{n-1}-x_0\|} L_0(u)\, du} \leq \frac{\| F'(x_0)^+ \|}{1 - \int_0^{b\,\beta} L_0(u)\, du} = \frac{\| F'(x_0)^+ \|}{1 - \overline{\delta_2}}.$$

Using (3.11), we have

$$(3.13) \qquad\qquad L_0(\| x_n - x_0 \| +u) \leq L_0(b\,\beta + u), \quad \forall\, u \geq 0.$$

By induction hypothesis, (3.13), and Lemma 2.8, we obtain

$$\int_0^{\|x_n - x_{n-1}\|} (\| x_n - x_{n-1} \| -u) \, L(\| x_n - x_0 \| +u) \, du$$

$$\leq \frac{1}{\| x_n - x_{n-1} \|} \int_0^{\|x_n - x_{n-1}\|} (\| x_n - x_{n-1} \| -u) \, L(b\,\beta + u) \, du \; \| x_n - x_{n-1} \|$$

$$\leq \frac{\Delta_1}{\beta} \; \| x_n - x_{n-1} \| .$$

By (2.13), we deduce the following estimate of $\mathcal{B}_1$:

$$(3.14) \quad \mathcal{B}_1 \leq \frac{\| F'(x_{n-1})^+ \|}{\| F'(x_0)^+ \|} \int_0^{\|x_n - x_{n-1}\|} (\| x_n - x_{n-1} \| -u) \, L(\| x_n - x_0 \| +u) \, du$$

$$\leq \frac{\Delta_1}{\beta\,(1 - \overline{\delta_2})} \; \| x_n - x_{n-1} \| .$$

We shall now estimate $\mathcal{B}_2$ and $\mathcal{B}_3$. By induction hypothesis, (3.13), and Lemma 2.8, we have:

$$\int_0^{\|x_n - x_{n-1}\|} L(\| x_{n-1} - x_0 \| +u) \, du$$

$$\leq \frac{1}{\| x_1 - x_0 \|} \int_0^{\|x_1 - x_0\|} L(b\,\beta + u) \, du \; \| x_n - x_{n-1} \|$$

$$\leq \frac{\delta_1}{\beta} \; \| x_n - x_{n-1} \| .$$

From Lemma 2.7, we have:

(3.15)
$$\| F'(x_n)^+ - F'(x_{n-1})^+ \|$$

$$\leq \frac{1 + \sqrt{5}}{2} \frac{\| F'(x_{n-1})^+ \|^2 \, \| F'(x_0)^+ \|^{-1} \int_0^{\|x_n - x_{n-1}\|} L(\| x_{n-1} - x_0 \| +u) \, du}{1 - \| F'(x_{n-1})^+ \| \, \| F'(x_0)^+ \|^{-1} \int_0^{\|x_n - x_{n-1}\|} L(\| x_{n-1} - x_0 \| +u) \, du}$$

$$\leq \frac{1 + \sqrt{5}}{2} \frac{\| F'(x_0)^+ \|}{(1 - \overline{\delta_2})^2} \frac{\delta_1}{\beta} \frac{\| x_n - x_{n-1} \|}{1 - \dfrac{\delta_1}{\beta\,(1 - \overline{\delta_2})} \| x_n - x_{n-1} \|}$$

$$\leq \frac{1 + \sqrt{5}}{2} \frac{\| F'(x_0)^+ \| \, \delta_1}{\beta\,(1 - \overline{\delta_2})\,(1 - \delta_1 - \overline{\delta_2})} \; \| x_n - x_{n-1} \| .$$

We also have:

$$\| F(x_n) - F(x_{n-1}) \|$$

$$\leq \frac{1}{\| F'(x_0)^+ \|} \int_0^{\|x_n - x_{n-1}\|} (\| x_n - x_{n-1} \| -u) \, L(\| x_{n-1} - x_0 \| +u) \, du$$

(3.16) $$+ \| F'(x_{n-1}) \| \| x_n - x_{n-1} \|$$

$$\leq \frac{\Delta_1}{\| F'(x_0)^+ \|} \left(K + \int_0^{\|x_{n-1} - x_0\|} L_0(u) \, du \right) \beta$$

$$\leq \frac{\beta \, \overline{\delta_2} + \beta \, K + \Delta_1}{\| F'(x_0)^+ \|}.$$

By (3.15), and (3.16), we obtain the estimate of $\mathcal{B}_2$:

(3.17) $$\mathcal{B}_2 \leq \frac{1 + \sqrt{5}}{2} \frac{\| F'(x_0)^+ \| \, \delta_1}{\beta \, (1 - \overline{\delta_2}) \, (1 - \delta_1 - \overline{\delta_2})} \, \| x_n - x_{n-1} \| \, \frac{\beta \, \overline{\delta_2} + \beta \, K + \Delta_1}{\| F'(x_0)^+ \|}$$

$$= \frac{1 + \sqrt{5}}{2} \frac{\delta_1 \, (\beta \, \overline{\delta_2} + \beta \, K + \Delta_1)}{\beta \, (1 - \overline{\delta_2}) \, (1 - \delta_1 - \overline{\delta_2})} \, \| x_n - x_{n-1} \| .$$

For estimating $\mathcal{B}_3$, we first estimate $\| F(x_{n-1}) \|$. By (2.13), we have

(3.18)

$$\| F(x_{n-1}) \|$$

$$\leq \| F(x_0) \| + \| F'(x_0) \| \| x_{n-1} - x_0 \| + \| F(x_{n-1}) - F(x_0) - F'(x_0) \, (x_{n-1} - x_0) \|$$

$$\leq \| F(x_0) \| + b \, \beta \, \| F'(x_0) \| + \frac{1}{\| F'(x_0)^+ \|} \int_0^{\|x_{n-1} - x_0\|} (\| x_{n-1} - x_0 \| -u) \, L_0(u) \, du$$

$$\leq \| F(x_0) \| + b \, \beta \, \| F'(x_0) \| + \frac{\overline{\Delta_2}}{\| F'(x_0)^+ \|}.$$

Hence, by (3.15), and (3.18), we obtain estimate for $\mathcal{B}_3$:

$$\mathcal{B}_3 \leq \frac{1 + \sqrt{5}}{2} \frac{\| F'(x_0)^+ \| \, \delta_1}{\beta \, (1 - \overline{\delta_2}) \, (1 - \delta_1 - \overline{\delta_2})} \, \| x_n - x_{n-1} \| \left(\| F(x_0) \| \right.$$

(3.19) $$+ b \, \beta \, \| F'(x_0)^+ \| + \frac{\overline{\Delta_2}}{\| F'(x_0)^+ \|} \right)$$

$$\leq \frac{1 + \sqrt{5}}{2} \frac{\delta_1 \, (\beta + b \, \beta \, K + \overline{\Delta_2})}{\beta \, (1 - \overline{\delta_2}) \, (1 - \delta_1 - \overline{\delta_2})} \, \| x_n - x_{n-1} \| .$$

Using (3.14), (3.17), and (3.19), we conclude

$$\| x_{n+1} - x_n \| \leq \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3$$

$$\leq p \, \| x_n - x_{n-1} \| \leq a \, \| x_n - x_{n-1} \|,$$

and the induction is completed.

By (3.4), $\{x_n\}$ is a complete sequence, and as such that it converges to $x^\star$ in $\mathcal{X}$. Letting $n \longrightarrow \infty$ in (1.3), we get $F'(x^\star)^+ F(x^\star) = 0$, and $x^\star \in \mathcal{S}$.

Finally, letting $n \longrightarrow \infty$ in

$$\| x_{n+1} - x_0 \| \leq \sum_{i=1}^{n+1} \| x_i - x_{i-1} \| \leq (\sum_{i=1}^{n+1} a^{i-1}) \| x_1 - x_0 \| \leq b \| x_1 - x_0 \|,$$

we deduce estimate (3.5).

That completes the proof of Theorem 3.3. $\qquad\square$

Remark 3.4. In view of the proof of Theorem 3.3, δ_2, Δ_2 can replace $\overline{\delta_2}$, and $\overline{\Delta_2}$, respectively.

4. Special Cases and Applications

Let us compare our results with the corresponding in [29].

Case $a = \dfrac{1}{2}$**.**

(a) If $L_0(u) = L(u)$ ($u \in [0, +\infty)$), then, our Theorem 3.3 is reduces to [29, Theorem 3.1, p. 695].

(b) If

(4.1) $$L_0(u) < L(u) \qquad (u \in [0, +\infty)),$$

then, our condition (3.2) is weaker, and the error bounds tighter than the corresponding ones in [29]. Indeed, let us define

$$d_1 = \int_0^{\beta} L(2\beta + u) \, du, \quad d_2 = \int_0^{2\beta} L(u) \, du$$

$$d_3 = \int_0^{\beta} (\beta - u) L(2\beta + u) \, du, \quad d_4 = \int_0^{2\beta} (2\beta - u) L(2\beta + u) \, du,$$

and

$$d = \frac{d_3}{\beta(1 - d_2)} + \frac{1 + \sqrt{5}}{2} \frac{d_1(\beta + \beta d_2 + 3\beta K + d_3 + d_4)}{\beta(1 - d_2)(1 - d_1 - d_2)}.$$

The conditions in [29, Theorem 3.1, p. 695] are

(4.2) $$p_1 + p_2 < 1 \quad \text{and} \quad d \leq \frac{1}{2}.$$

But, in view of (4.1), we have:

$$\delta_1 < d_1, \quad \overline{\delta_2} < d_2, \quad \overline{\Delta_1} < \Delta_1 < d_3, \quad \overline{\Delta_2} < \Delta_2 < d_4,$$

so,

$$p < d.$$

These advantages are obtained under the same computational cost as in [29], since in practice the computation of function L requires that of L_0.

Case $a \neq \dfrac{1}{2}$. This case has not been examined in [29].

(a) If $\dfrac{1}{2} < a < 1$, then $b > 2$. This case allows larger range of choices for δ_1, $\overline{\delta_2}$, Δ_1, and $\overline{\Delta_2}$.

(b) If $0 \leq a < \dfrac{1}{2}$, we obtain $b < 2$, and a smaller ratio of convergence. Moreover, in this case, the solution $x^\star$ is in $\mathcal{D}_0 \subseteq U(x_0, 2\beta)$.

Application 4.1. Let $L_0(u) = L_0$, and $L(u) = L$ for all $u \in [0, \infty)$, where L_0 and L are constants, with $0 < L_0 \leq L$. Then, using Definition 3.2, we obtain

$$\delta_1 = L\beta, \quad \overline{\delta_1} = L_0\beta, \quad \delta_2 = bL\beta, \quad \overline{\delta_2} = bL_0\beta$$

$$\Delta_1 = \frac{L}{2}\beta^2, \quad \overline{\Delta_1} = \frac{L_0}{2}\beta^2, \quad \Delta_2 = bL\beta^2, \quad \text{and} \quad \overline{\Delta_2} = bL_0\beta^2.$$

Then, Theorem 3.3 reduces to a Kantorovich–type result.

Let us provide a numerical example, where $L_0 = L$, $a < \dfrac{1}{2}$, and $b < 2$.

Example 4.2. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, be equipped with the ℓ_1–norm. Define $F : \mathcal{X} \longrightarrow \mathcal{Y}$ by:

$$F(x) = (\theta_1 - \theta_2, .5\,(\theta_1 - \theta_2)^2)^T, \qquad x = (\theta_1, \theta_2)^T.$$

F is $\mathcal{C}^1$ on $\mathcal{X}$, and for all $x = (\theta_1, \theta_2)^T \in \mathcal{X}$, we have:

$$F'(x) = \begin{pmatrix} 1 & -1 \\ \theta_1 - \theta_2 & \theta_2 - \theta_1 \end{pmatrix}.$$

Hence, rank $F'(x) = 1$ for all $x \in \mathcal{X}$. The Moore–Penrose inverse of $F'(x)$ is

$$F'(x)^+ = \frac{1}{2\,(1 + (\theta_1 - \theta_2)^2)} \begin{pmatrix} 1 & \theta_1 - \theta_2 \\ -1 & \theta_2 - \theta_1 \end{pmatrix}.$$

Let $x_0 = (.54, .5)^T$, and

$$a = .49, \quad b = 1.9607843, \quad \| F'(x_0) \| = 1.04, \quad \text{and} \quad \| F(x_0) \| = .0408.$$

Then,

$$\| F'(x_0)^+ \| = .99840256.$$

We have also for $x = (\theta_1, \theta_2)^T \in \mathcal{X}$, and $y = (\xi_1, \xi_2)^T \in \mathcal{X}$

$$\| F'(x_0)^+ \| \| F'(x) - F'(y) \| \leq .99840256 \, (|\theta_1 - \xi_1| + |\theta_2 - \xi_2|) \leq .99840256 \, \| x - y \| \, .$$

We get

$$L = L_0 = .99840256, \quad \delta_1 = .040669753,$$

$$K = 1.038338658, \quad \delta_2 = .079744613,$$

$$\beta = .040734824, \quad \Delta_1 = .000828338, \quad \Delta_2 = .003248383,$$

$$\delta_1 + \delta_2 = .120414366 < 1,$$

and

$$p = .367942966 < a = .49.$$

Hence, the conclusion of Theorem 3.3 hold. That is Newton's method converges to a point $x^\star \in \overline{\mathcal{D}_0}$, satisfying $F'(x^\star) F(x^\star) = 0$, with ratio a. The same example in [29] showed that $x^\star \in \overline{U}(x_0, 2\,\beta) \supseteq \overline{\mathcal{D}_0}$, with ration of convergence $a < .5$.

Application 4.3. Theorem 3.3 reduces to a Smale–type result if we define functions L_0, L on $[0, \dfrac{1}{\gamma_0})$, and $[0, \dfrac{1}{\gamma})$, respectively, by

$$L_0(u) = \frac{2\,\gamma_0}{(1 - \gamma_0\,u)^3}$$

and

$$L(u) = \frac{2\,\gamma}{(1 - \gamma\,u)^3}.$$

If $\gamma_0 < \gamma$, then, our Theorem 3.3 improves the corresponding results in [29, Section 4]. We leave the details to the motivated reader.

Finally, note that examples where $L_0 < L$ can be found in [4], in the more general setting of a Banach space.

CONCLUSION

Using a combination of two center–Lipschitz–type conditions with average, we provided a semilocal convergence analysis for Newton's method to approximate a locally unique solution of a singular system of equations with constant rank derivatives in Euclidean space setting. Numerical examples and applications further validating the results are also provided in this study.

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