J. Appl. Math. & Informatics Vol. **29***(2011), No. 3 - 4, pp. 1049 - 1057*
Website: http://www.kcam.biz

FUZZY BE-ALGEBRAS†

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Abstract. In this paper, we fuzzify the concept of BE-algebras, investigate some of their properties. We give a characterization of fuzzy BE-algebras, and discuss a characterization of fuzzy BE-algebras in terms of level subalgebras of fuzzy BE-algebras.

AMS Mathematics Subject Classification : 06F35, 03G25.
Key words and phrases : (fuzzy) BE-algebra, level subalgebra, normal fuzzy BE-algebra.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [5,6]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3,4] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [12] introduced the notion of d-algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras [13,14] which is equivalent in some sense to the groups. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim [10] introduced a new notion, called an BH-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras. A. Walendziak obtained another equivalent axioms for B-algebra [15]. H. S. Kim, Y. H. Kim and J. Neggers [9] introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. C. B. Kim and H. S. Kim [7] introduced the notion of a BM-algebra which is a specialization of B-algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0-commutative B-algebra. In [8], H.S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition

Received November 30, 2010. Revised January 21, 2011. Accepted February 8, 2011.
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of a filter in BE-algebras. In [1,2], S. S. Ahn and K. S. So introduced the notion of ideals in BE-algebras, and then discussed several characterizations of such ideals. Also they generalized the notion of upper sets in BE-algebras, and discussed several properties of the characterizations of generalized upper sets $A_n(u, v)$ while relating them to the structure of ideals in transitive and self distributive BE-algebras.

In this paper, we fuzzify the concept of BE-algebras, investigate some of their properties. We give a characterization of fuzzy BE-algebras, and discuss a characterization of fuzzy BE-algebras in terms of level subalgebras of fuzzy BE-algebras.

2. Preliminaries.

We recall some definitions and results discussed in [1,2,8].

Definition 2.1. An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation "$\leq$" on X defined by $x \leq y$ if and only if $x * y = 1$. A non-empty subset A of X is said to be a *subalgebra* of a BE-algebra X if it is closed under the operation "$*$". Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in A$.

Proposition 2.1. *If $(X; *, 1)$ is a BE-algebra, then $x * (y * x) = 1$ for any $x, y \in X$.*

Example 2.2. *Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:*

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

Definition 2.2. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is said to be a *filter* of X if

(F1) $1 \in F$;
(F2) $x * y \in F$ and $x \in F$ imply $y \in F$.

In Example 2.2, $F_1 := \{1, a, b\}$ is a filter of X, but $F_2 := \{1, a\}$ is not, since $a * b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

Definition 2.3. A BE-algebra $(X, *, 1)$ is said to be *self distributive* if $x*(y*z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Example 2.3. *Let $X := \{1, a, b, c, d\}$ be a set with the following table:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to see that X is a BE-algebra satisfying self distributivity.

Note that the BE-algebra in Example 2.2 is not self distributive, since $d*(a*0) = d*d = 1$, while $(d*a)*(d*0) = 1*a = a$.

Proposition 2.4. *If $(X; *, 1)$ is a self-distributive BE-algebra, then it is transitive.*

Proposition 2.5. *Let $X := (X; *, 1)$ be a BE-algebra and F be a filter of X. If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.*

Proposition 2.6. *Let X be a self distributive BE-algebra. Then for any $x, y, z \in X$,*

(1) *if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$;*
(2) *$y * z \leq (z * x) * (y * x)$.*

3. Fuzzy BE-algebras.

In what follows, let X be a BE-algebra unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called a *fuzzy BE-algebra* of X if it satisfies: for all $x, y \in X$.

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}.$$

A fuzzy set μ in X is a function $\mu : X \to [0, 1]$. We note that $x * x = 1$ for all $x \in X$ and so if μ is a fuzzy BE-algebra of X, then $\mu(1) \geq \mu(x)$ for all $x \in X$.

Proposition 3.1. *Let μ be a fuzzy BE-algebra of X and let $a \in X$. If μ is decreasing, then it is constant.*

Proof. We note that $\mu(x) \leq \mu(1)$ for all $x \in X$. Since $x \leq 1$ for all $x \in X$, $\mu(x) \geq \mu(1)$ because μ is decreasing. Hence $\mu(x) = \mu(1)$ for all $x \in X$. Thus μ is constant. □

Example 3.2. *Let $X := \{1, a, b, c, d, 0\}$ be the BE-algebra as in Example 2.2 and let $A := \{1, a, b\}$. Let $t_1, t_2 \in [0, 1]$ be such that $t_1 > t_2$. Define a mapping $\mu : X \to [0, 1]$ by $\mu(1) = \mu(a) = \mu(b) = t_1$ and $\mu(c) = \mu(d) = \mu(0) = t_2$. Then μ is a fuzzy BE-algebra of X.*

Theorem 3.3. *Let μ be a fuzzy set in a BE-algebra X. Then μ is a fuzzy BE-algebra of X if and only if for every $\alpha \in [0,1]$, the level subset μ_α is a subalgebra of X, when $\mu_\alpha \neq \emptyset$.*

Proof. Let μ be a fuzzy BE-algebra of X and let $x, y \in \mu_\alpha$ for every $\alpha \in [0,1]$ with $\mu_\alpha \neq \emptyset$. Then $\mu(x*y) \geq \min\{\mu(x), \mu(y)\} \geq \alpha$, which implies that $\mu(x*y) \geq \alpha$. Hence $x * y \in \mu_\alpha$. Thus μ_α is a subalgebra of X. Conversely, assume that μ_α is a subalgebra of X for every $\alpha \in [0,1]$ with $\mu_\alpha \neq \emptyset$. Let $x, y \in X$ and let $\mu(x) = \alpha_1$ and $\mu(y) = \alpha_2$. Then $x \in \mu_{\alpha_1}$ and $y \in \mu_{\alpha_2}$. Without loss of generality, we may assume that $\alpha_1 \leq \alpha_2$. Then $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$ and so $y \in \mu_{\alpha_1}$. Since μ_{α_1} is a subalgebra of X, we have $x * y \in \mu_{\alpha_1}$. Hence $\mu(x * y) \geq \alpha_1 = \min\{\mu(x), \mu(y)\}$. Therefore μ is a fuzzy BE-algebra of X. □

Definition 3.2. Let μ be a fuzzy BE-algebra of X. Each subalgebra μ_α of X, $\alpha \in [0,1]$, is called a *level subalgebra* of μ, when $\mu_\alpha \neq \emptyset$.

Theorem 3.4. *Let A be a subalgebra of a BE-algebra X and let $\mu : X \to [0,1]$ be a fuzzy set defined by, for all $x \in X$,*

$$\mu(x) = \begin{cases} \alpha_0 & \text{if } x \in A \\ \alpha_1 & \text{if } x \notin A \end{cases}$$

where $\alpha_0, \alpha_1 \in [0,1], \alpha_0 > \alpha_1$. Then μ is a fuzzy BE-algebra of X.

Proof. Let $x, y \in X$. If at least one of x and y does not belong to A, then $\mu(x * y) \geq \alpha_1 = \min\{\mu(x), \mu(y)\}$, since α_1 is the minimum value of μ. If $x, y \in A$, then $x * y \in A$. Hence $\mu(x * y) = \min\{\mu(x), \mu(y)\}$. Therefore μ is a fuzzy BE-algebra of X. □

Corollary 3.5. *Any subalgebra of a BE-algebra of X can be realized as a level subalgebra of some fuzzy BE-algebra of X.*

Proof. Let A be a subalgebra of X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

where α is fixed number in $(0,1]$. Taking $\alpha_0 = \alpha$ and $\alpha_1 = 0$ in Theorem 3.4, we know that μ is a fuzzy BE-algebra of X, and obviously $\mu_\alpha = A$. This completes the proof. □

We can generalize Theorem 3.4 as follows:

Theorem 3.6. *Let $\{A_n\}_{n=0}^{\infty}$ be a strictly decreasing sequence of subalgebras of BE-algebra $X = \mathrm{A}_0$ and let $\{\alpha_n\}_{n=0}^{\infty}$ be a strictly increasing sequence in $(0,1)$. Then there is a fuzzy BE-algebra μ of X such that $\mu_{\alpha_n} = A_n$ for all $n = 0, 1, 2, \cdots$.*

Proof. Define a fuzzy set $\mu : X \to [0,1]$ by

$$\mu(x) = \begin{cases} \alpha_n & \text{if } x \in A_n - A_{n+1} \\ lim_{n\to\infty}\alpha_n & \text{if } x \in \cap_{n=1}^{\infty} A_n. \end{cases}$$

It is easily seen that μ is a fuzzy BE-algebra of X and that $\mu_{\alpha_n} = A_n$ for all $n = 0, 1, 2, \cdots$. □

Proposition 3.7. *If μ is a fuzzy BE-algebra of X, then the set $X_\mu := \{x \in X | \mu(x) = \mu(1)\}$ is a subalgebra of X.*

Proof. Noticing that $\mu(x) \le \mu(1)$ for all $x \in X$, we have

$$\mu_{\mu(1)} = \{x \in X | \mu(x) \ge \mu(1)\} = \{x \in X | \mu(x) = \mu(1)\} = X_\mu.$$

By Theorem 3.3, we know that X_μ is a subalgebra of X. □

Theorem 3.8. *Let μ be a fuzzy BE-algebra of X. Then two level subalgebras $\mu_{\alpha_1}, \mu_{\alpha_2}$ with $\alpha_1 < \alpha_2$ of μ are equal if and only if there is no $x \in X$ such that $\alpha_1 \le \mu(x) < \alpha_2$.*

Proof. Suppose that $\alpha_1 < \alpha_2$ and $\mu_{\alpha_1} = \mu_{\alpha_2}$. If there exists $x \in X$ such that $\alpha_1 \le \mu(x) < \alpha_2$, then μ_{α_2} is a proper subset of μ_{α_1}. This is impossible.

Conversely, suppose that there is no $x \in X$ such that $\alpha_1 \le \mu(x) < \alpha_2$. Note that $\alpha_1 < \alpha_2$ implies $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$. If $x \in \mu_{\alpha_1}$, then $\mu(x) \ge \alpha_1$, and so $\mu(x) \ge \alpha_2$ because $\mu(x) \not< \alpha_2$. Hence $x \in \mu_{\alpha_2}$, which says that $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$. Thus $\mu_{\alpha_1} = \mu_{\alpha_2}$. This completes the proof. □

Remark 3.1. *As a consequence of Theorem 3.8, the level subalgebras of a fuzzy BE-algebra μ of X which has a countable image form a chain. But $\mu(x) \le \mu(1)$ for all $x \in X$, and so $\mu_{\mu(1)}$ is the smallest level subalgebra of a fuzzy BE-algebra, but not always $\mu_{\mu(1)} = \{1\}$ as shown in the following example. Thus we have a chain*

$$X = \mu_{\alpha_0} \supseteq \mu_{\alpha_1} \supseteq \cdots \supseteq \mu_{\alpha_k} \supseteq \cdots,$$

where $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \cdots$ and $\mu(1) = lim_{n\to\infty}\alpha_n$.

Example 3.9. *Let A be a proper subalgebra of a BE-algebra X and let μ be a fuzzy BE-algebra of X in the proof to Corollary 3.5. Then $Im(\mu) = \{0, \alpha\}$, and two level subalgebras of μ are $\mu_0 = X$ and $\mu_\alpha = A$. Thus we have $\mu(1) = \alpha$ but $\mu_\alpha = A \ne \{1\}$.*

Corollary 3.10. *Let μ be a fuzzy BE-algebra of X. If $Im(\mu) = \{\alpha_1, \cdots, \alpha_n\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, then the family of subalgebras μ_{α_i} of $\mu (i = 1, 2, \cdots, n)$ constitutes all the level subalgebras of μ.*

Proof. Let $\alpha \in [0,1]$ and $\alpha \notin Im(\mu)$. If $\alpha < \alpha_1$, then $\mu_{\alpha_1} \subseteq \mu_\alpha$. Since $\mu_{\alpha_1} = X$, we have $\mu_\alpha = X$ and $\mu_\alpha = \mu_{\alpha_1}$. If $\alpha_i < \alpha < \alpha_{i+1}$ $(1 \le i \le n-1)$, then there is no $x \in X$ such that $\alpha \le \mu(x) < \alpha_{i+1}$. Using Theorem 3.8, we obtain $\mu_\alpha = \mu_{\alpha_{i+1}}$. This shows that for any $\alpha \in [0,1]$ with $\alpha \le \mu(1)$, the level subalgebra μ_α is in $\{\mu_{\alpha_i} | 1 \le i \le n\}$. □

Theorem 3.11. *Let μ be a fuzzy BE-algebra of X with $Im(\mu) = \{\alpha_i | i \in \wedge\}$ and $\mathcal{A} = \{\mu_{\alpha_i} | i \in \wedge\}$ where $\wedge$ is an arbitrary index set. Then*

(i) *There exists a unique $i_0 \in \wedge$ such that $\alpha_{i_0} > \alpha_i$ for all $i \in \wedge$.*
(ii) *$X_\mu = \cap_{i\in\wedge}\mu_{\alpha_i} = \mu_{\alpha_{i_0}}$.*
(iii) *$X = \cup_{i\in\wedge}\mu_{\alpha_i}$.*
(iv) *The members of $\mathcal{A}$ form a chain.*
(v) *If μ attains its infimum on all subalgebras of X, then $\mathcal{A}$ contains all level subalgebras of μ.*

Proof. (i) Since $\mu(1) \in Im(\mu)$, there exists a unique $i_0 \in \wedge$ such that $\alpha_{i_0} = \mu(1) \geq \mu(x)$ for all $x \in X$ so that $\alpha_{i_0} \geq \alpha$ for all $i \in \wedge$.
(ii) We know that

$$\begin{aligned}\mu_{\alpha_{i_0}} &= \{x \in X | \mu(x) \geq \alpha_{i_0}\}\\ &= \{x \in X | \mu(x) = \alpha_{i_0}\}\\ &= \{x \in X | \mu(x) = \mu(1)\}\\ &= X_\mu.\end{aligned}$$

Since $\alpha_{i_0} \geq \alpha_i$ for all $i \in \wedge$, therefore clearly $\mu_{\alpha_{i_0}} \subseteq \mu_{\alpha_i}$ for all $i \in \wedge$. Hence $\mu_{\alpha_{i_0}} \subseteq \cap_{i\in\wedge}\mu_{\alpha_i}$, and so $\mu_{\alpha_{i_0}} = \cap_{i\in\wedge}\mu_{\alpha_i}$, because $i_0 \in \wedge$.
(iii) It is sufficient to show that $X \subseteq \cup_{i\in\wedge}\mu_{\alpha_i}$. Let $x \in X$. Then $\mu(x) \in Im(\mu)$ and so there exists $i(x) \in \wedge$ such that $\mu(x) = \alpha_{i(x)}$. This implies $x \in \mu_{\alpha(i)} \subset \cup_{i\in\wedge}\mu_{\alpha_i}$. This proves (iii).
(iv) Note that for any $i, j \in \wedge$, either $\alpha_i \geq \alpha_j$ or $\alpha_i \leq \alpha_j$; hence $\mu_{\alpha_i} \subseteq \mu_{\alpha_j}$ or $\mu_{\alpha_j} \subseteq \mu_{\alpha_i}$. Therefore the members of $\mathcal{A}$ form a chain.
(v) Assume that μ attains its infimum on all subalgebras of X. Let μ_α be a level subalgebra of μ. If $\alpha = \alpha_i$ for some $i \in \wedge$, then clearly $\mu_\alpha \in \mathcal{A}$. Assume that $\alpha \neq \alpha_i$ for all $i \in \wedge$. Then there is no $x \in X$ such that $\mu(x) = \alpha$. Let $A = \{x \in X | \mu(x) > \alpha\}$. Obviously $1 \in A$, and so $A \neq \emptyset$. Let $x, y \in A$. Then $\mu(x) > \alpha$ and $\mu(y) > \alpha$. Since μ is a fuzzy BE-algebra of X, it follows that

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} > \alpha$$

so that $\mu(x*y) > \alpha$, i.e., $x*y \in A$. Hence A is a subalgebra of X. By hypothesis, there exists $y \in A$ such that $\mu(y) = \inf\{\mu(x) | x \in X\}$. Now $\mu(y) \in Im(\mu)$ implies $\mu(y) = \alpha_i$ for some $i \in \wedge$. Obviously $\alpha_i \geq \alpha$, and so by assumption $\alpha_i > \alpha$. Note that there is no $z \in X$ such that $\alpha \leq \mu(z) < \alpha_i$. It follows from Theorem 3.8 that $\mu_\alpha = \mu_{\alpha_i}$. Hence $\mu_\alpha \in \mathcal{A}$. This completes the proof. □

4. Normal fuzzy BE-algebras.

Definition 4.1. A fuzzy BE-algebra μ of X is said to be *normal* if there exists $x \in X$ such that $\mu(x) = 1$.

Let μ and σ be any two fuzzy subsets of a set X. Then μ is said to be *contained in* σ, denoted by $\mu \subseteq \sigma$, if $\mu(x) \leq \sigma(x)$ for all $x \in X$. If $\mu(x) = \sigma(x)$ for all $x \in X$, μ and σ are said to be *equal* and we write $\mu = \sigma$. We note that if μ

is a normal fuzzy BE-algebra of X, then $\mu(1) = 1$. Hence we have the following characterization.

Theorem 4.1. *A fuzzy BE-algebra μ of X is normal if and only if $\mu(1) = 1$.*

Theorem 4.2. *If μ is a fuzzy BE-algebra of X, then the fuzzy set μ^+ of X defined by $\mu^+(x) := \mu(x) + 1 - \mu(1)$ for all $x \in X$ is a normal fuzzy BE-algebra of X containing μ.*

Proof. Assume that μ is a fuzzy BE-algebra of X and let $x, y \in X$. Then

$$\begin{aligned}\mu^+(x * y) &= \mu(x * y) + 1 - \mu(1)\\ &\geq \min\{\mu(x), \mu(y)\} + 1 - \mu(1)\\ &= \min\{\mu(x) + 1 - \mu(1), \mu(y) + 1 - \mu(1)\}\\ &= \min\{\mu^+(x), \mu^+(y)\}\end{aligned}$$

and $\mu^+(1) = \mu(1) + 1 - \mu(1) = 1$. Hence μ^+ is a normal fuzzy BE-algebra of X, and clearly $\mu \subset \mu^+$. This completes the proof. □

Theorem 4.3. *Let μ and ν be fuzzy BE-algebras of X. If $\mu \subset \nu$ and $\mu(1) = \nu(1)$, then $X_\mu \subset X_\nu$.*

Proof. Assume that $\mu \subset \nu$ and $\mu(1) = \nu(1)$. If $x \in X_\mu$ then $\nu(x) \geq \mu(x) = \mu(1) = \nu(1)$. Noticing that $\nu(x) \leq \nu(1)$ for all $x \in X$, we have $\nu(x) = \nu(1)$, i.e., $x \in X_\nu$. This completes the proof. □

Corollary 4.4. *If μ and ν are normal fuzzy BE-algebras of X satisfying $\mu \subset \nu$, then $X_\mu \subset X_\nu$.*

Theorem 4.5. *A fuzzy BE-algebra μ of X is normal if and only if $\mu^+ = \mu$.*

Proof. The sufficiency is obvious. Assume that μ is a normal fuzzy BE-algebra of X and let $x \in X$. Then $\mu^+(x) = \mu(x) + 1 - \mu(1) = \mu(x)$, and hence $\mu^+ = \mu$, ending the proof. □

Theorem 4.6. *If μ is a fuzzy BE-algebra of X, then $(\mu^+)^+ = \mu^+$.*

Proof. For any $x \in X$, we have $(\mu^+)^+(x) = \mu^+(x) + 1 - \mu^+(1) = \mu^+(x)$, completing the proof. □

Theorem 4.7. *Let μ be a fuzzy BE-algebra of X. If there exists a fuzzy BE-algebra ν of X satisfying $\nu^+ \subset \mu$, then μ is normal.*

Proof. Suppose there exists a fuzzy BE-algebra ν of X such that $\nu^+ \subset \mu$. Then $1 = \nu^+(1) \leq \mu(1)$, whence $\mu(1) = 1$. The proof is complete. □

Corollary 4.8. *Let μ be a fuzzy BE-algebra of X. If there exists a fuzzy BE-algebra ν of X satisfying $\nu^+ \subset \mu$, then $\mu^+ = \mu$.*

Theorem 4.9. *Let μ be a fuzzy BE-algebra of X and let $f : [0, \mu(1)] \to [0, 1]$ be an increasing function. Define a fuzzy set $\mu_f : X \to [0, 1]$ by $\mu_f(x) := f(\mu(x))$ for all $x \in X$. Then μ_f is a fuzzy BE-algebra of X. In particular, if $f(\mu(1)) = 1$, then μ_f is normal, and if $f(\alpha) \geq \alpha$ for all $\alpha \in [0, \mu(1)]$, then $\mu \subset \mu_f$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \mu_f(x * y) &= f(\mu(x * y)) \\ &\geq f(\min\{\mu(x), \mu(y)\}) \\ &= \min\{f(\mu(x)), f(\mu(y))\} \\ &= \min\{\mu_f(x), \mu_f(y)\}. \end{aligned}$$

Hence μ_f is a fuzzy BE-algebra of X. If $f(\mu(1)) = 1$ then clearly μ_f is normal. Assume that $f(\alpha) \geq \alpha$ for all $\alpha \in [0, \mu(1)]$. Then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves that $\mu \subset \mu_f$. □

Theorem 4.10. *Let μ be a non-constant normal fuzzy BE-algebra of X, which is maximal in the poset of normal fuzzy BE-algebras under set inclusion. Then μ takes only the values 0 and 1.*

Proof. Note that $\mu(1) = 1$. Let $x \in X$ be such that $\mu(x) \neq 1$. It sufficient to show that $\mu(x) = 0$. Assume that there exists $a \in X$ such that $0 < \mu(a) < 1$. Define a fuzzy set $\nu : X \to [0, 1]$ by $\nu(x) := \frac{1}{2}\{\mu(x) + \mu(a)\}$ for all $x \in X$. Then clearly ν is well-defined. Let $x, y \in X$. Then

$$\begin{aligned} \nu(x * y) &= \frac{1}{2}\{\mu(x * y) + \mu(a)\} \\ &= \frac{1}{2}\mu(x * y) + \frac{1}{2}\mu(a) \\ &\geq \frac{1}{2}\min\{\mu(x), \mu(y)\} + \frac{1}{2}\mu(a) \\ &= \min\{\frac{1}{2}(\mu(x) + \mu(a)), \frac{1}{2}(\mu(y) + \mu(a))\} \\ &= \min\{\nu(x), \nu(y)\}, \end{aligned}$$

which proves that ν is a fuzzy BE-algebra of X. Now we have

$$\begin{aligned} \nu^+(x) &= \nu(x) + 1 - \nu(1) \\ &= \frac{1}{2}\{\mu(x) + \mu(a)\} + 1 - \frac{1}{2}\{\mu(1) + \mu(a)\} \\ &= \frac{1}{2}\{\mu(x) + 1\} \end{aligned}$$

and so $\nu^+(1) = \frac{1}{2}\{\mu(1) + 1\} = 1$. Hence ν^+ is a normal fuzzy BE-algebra of X. From $\nu^+(a) > \mu(a)$ it follows that μ is not maximal. This is a contradiction and the proof is complete. □

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