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A NOTE ON THE PARAMETRIZATION OF MULTIWAVELETS OF DGHM TYPE†

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Abstract. Multiwavelet coefficients can be constructed from the multiscaling coefficients by using the factorization for paraunitary matrices. In this paper we present a procedure for parametrizing all possible multiwavelet coefficients corresponding to the multiscaling coefficients of DGHM type.

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1. Introduction

In [2,1], Daubechies gave perfect formulas for the constructions of wavelets. However, it seems that there is not such a good formula of similar structure for multiwavelets. Donovan et al.[3,6] described a method for constructing orthogonal multiwavelets associated with orthogonal scaling functions of some type. And also Donovan et al. [4] presented an alternate construction of multiwavelet coefficients of DGHM type. This paper gives a procedure for parametrizing all possible multiwavelet coefficients associated with two scaling functions of the same type as [6,7].

2. Parametrization of Multiwavelets of DGHM Type

Assume that $\Phi = [\phi_1, \phi_2]^T$ is an orthonormal scaling vector of DGHM type, that is

(1) Φ is continuous and supported on $[-1, 1]$
(2) ϕ_1 is not supported on either $[-1, 0]$ or $[0, 1]$, and $\phi_1(0) \neq 0$
(3) ϕ_2 is supported on $[0, 1]$

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Then the scaling vector Φ satisfies the following refinement equation:

$$\Phi(x) = \sqrt{2} \sum_{k=-2}^{1} C(k)\Phi(2x-k).$$

where the 2×2 matrices $C(k) = \begin{bmatrix} c_{11}(k) & c_{12}(k) \\ c_{21}(k) & c_{22}(k) \end{bmatrix}$ are the scaling coefficients for Φ. Since ϕ_2 is supported on $[0,1]$, we have

$$c_{21}(k), c_{22}(k) = 0, \text{ for } k = -2, -1.$$

By the fact that ϕ_1 is continuous and nonzero at 0, we have $c_{11}(0) = \frac{1}{\sqrt{2}}$. Let

$$\begin{aligned} C_{1L} &= [c_{12}(-2)\ c_{11}(-1)\ c_{12}(-1)], \\ C_{1R} &= [c_{12}(0)\ c_{11}(1)\ c_{12}(1)], \text{ and} \\ C_2 &= [c_{22}(0)\ c_{21}(1)\ c_{22}(1)]. \end{aligned}$$

Then the orthonormality of Φ implies

(1) C_{1L}, C_{1R} and C_2 are orthogonal.

(2) $\| \phi_1 \|^2 = \| C_{1L} \|^2 + \frac{1}{2} + \| C_{1R} \|^2 = 1$

Now we construct the multiwavelet coefficients. If we assume that $\Psi = [\psi_1, \psi_2]^T$ is a wavelet vector supported on $[-1,1]$, then it must be of the form

$$\Psi(x) = \sqrt{2} \sum_{k=-2}^{1} D(k)\Phi(2x-k),$$

where the 2×2 matrices $D(k) = \begin{bmatrix} d_{11}(k) & d_{12}(k) \\ d_{21}(k) & d_{22}(k) \end{bmatrix}$ are the multiwavelet coefficients for Ψ. Let

$$\begin{aligned} D_{iL} &= [d_{i2}(-2)\ d_{i1}(-1)\ d_{i2}(-1)], \\ D_{iR} &= [d_{i2}(0)\ d_{i1}(1)\ d_{i2}(1)], i = 1, 2. \end{aligned}$$

And let

$$\begin{aligned} \Phi_L &= \sqrt{2}[\phi_2(2\cdot+2)\ \phi_1(2\cdot+1)\ \phi_2(2\cdot+1)]^T, \\ \Phi_R &= \sqrt{2}[\phi_2(2\cdot)\ \phi_1(2\cdot-1)\ \phi_2(2\cdot-1)]^T. \end{aligned}$$

Then

$$\psi_i = D_{iL}\Phi_L + d_{i1}(0)\sqrt{2}\phi_1(2\cdot) + D_{iR}\Phi_R.$$

Since ψ_i is orthogonal to ϕ_2 and $\phi_1(\cdot-1), D_{iR}$ is orthogonal to C_{1L} and C_2. Hence D_{iR} must be a multiple of C_{1R}. Similarly, D_{iL} must be a multiple of C_{1L}. So Ψ must be of the form

$$\psi_i = \alpha_{iL}C_{1L}\Phi_L + d_{i1}(0)\phi_1(2\cdot) + \alpha_{iR}C_{1R}\Phi_R, \text{ for some } \alpha_{iL}, \alpha_{iR} \in \mathbf{R}.$$

Since C_{1L} is orthogonal to C_{1R}, ψ_i is orthogonal to the nonzero shifts of a ψ_j for $i \neq j$. If we note the remaining orthogonality conditions

$$\langle \psi_i, \phi_1 \rangle = 0, \text{ and } \langle \psi_i, \psi_j \rangle = \delta_{ij}, i = 1, 2,$$

then we can complete the construction of orthonormal wavelets ψ_i, by finding the solutions $\alpha_{iL}, d_{iL}(0), \alpha_{iR}$ of the following equations:

(1) $\| C_1L \|^2 + \| C_{1R} \|^2 = \frac{1}{2}$

(2) $\alpha_{iL} \| C_{1L} \|^2 + \frac{1}{\sqrt{2}} d_{i1}(0) + \alpha_{iR} \| C_{1R} \|^2 = 0$

(3) $\alpha_{iL}\alpha_{jL} \| C_{1L} \|^2 + d_{i1}(0) d_{j1}(0) + \alpha_{iR}\alpha_{jR} \| C_{1R} \|^2 = \delta_{ij}$.

It can be modified to the problem to find two solutions $(x_i, y_i, z_i), i = 1, 2$, of equations

(1) $\| C_{1L} \|^2 + \| C_{1R} \|^2 = \frac{1}{2}$

(2) $\| C_{1L} \|^2 x + \frac{1}{\sqrt{2}} y + \| C_{1R} \|^2 z = 0$

(3) $\| C_{1L} \|^2 x^2 + y^2 + \| C_{1R} \|^2 z^2 = 1$

such that $(\| C_{1L} \|^2 x_i, y_i, \| C_{1R} \|^2 z_i), i = 1, 2$ are orthogonal.

Let Q be a 3×2 matrix defined by

$$\begin{bmatrix} \| C_{1L} \| & 0 \\ -\sqrt{2} \| C_{1L} \|^2 & -\sqrt{2} \| C_{1R} \|^2 \\ 0 & \| C_{1R} \| \end{bmatrix}$$

Then Q^TQ is symmetric 2×2 matrix, i.e.

$$Q^TQ = \begin{bmatrix} \| C_{1L} \|^2 + 2 \| C_{1L} \|^4 & 2 \| C_{1L} \|^2 \| C_{1R} \|^2 \\ 2 \| C_{1R} \|^2 \| C_{1L} \|^2 & \| C_{1R} \|^2 + 2 \| C_{1R} \|^4 \end{bmatrix}$$

This matrix Q^TQ has in fact distinct positive eigenvalues by the following lemma.

Lemma 1. *Q^TQ has distinct positive eigenvalues.*

Proof. The characteristic polynomial $f(\lambda)$ of Q^TQ is

$$\begin{aligned} f(\lambda) &= \det(Q^TQ - \lambda I) \\ &= \lambda^2 - (1 - 4 \| C_{1L} \|^2 \| C_{1R} \|^2)\lambda + 2 \| C_{1L} \|^2 \| C_{1R} \|^2 \end{aligned}$$

Putting $a = \| C_{1L} \|^2$,

$$\begin{aligned} D(a) &= 1 - 16 \| C_{1L} \|^2 \| C_{1R} \|^2 + 16 \| C_{1L} \|^4 \| C_{1R} \|^4 \\ &= 1 - 16a(\frac{1}{2} - a) + 16a^2(\frac{1}{2} - a)^2 \\ &= 16a^4 - 16a^3 + 20a^2 - 8a + 1 \\ D'(a) &= 64a^3 - 48a^2 + 40a - 8 = 8(4a - 1)(2a^2 - a + 1) \end{aligned}$$

Since $D(\frac{1}{4}) = \frac{1}{16} > 0$ and $2a^2 - a + 1 > 0$, D is positive for all a. Hence $f(\lambda) = 0$ has distinct roots. And since $\| C_{1L} \|^2$ and $\| C_{1R} \|^2$ are less than $\frac{1}{2}, 1 - 4 \| C_{1L} \|^2 \| C_{1R} \|^2 > 0$. Hence Q^TQ has distinct positive eigenvalues. □

By the lemma, let λ and $\mu (\lambda \neq \mu)$ be positive eigenvalues of Q^TQ, and v_1, v_2 the corresponding unit eigenvectors. Then Q^TQ can be diagonalized as

$$P^TQ^TQP = D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

where $P = [v_1, v_2]$ is an orthogonal matrix. Let

$$X_i = [||C_{1L}||x_i, y_i, ||C_{1R}||z_i]^T, i = 1, 2.$$

Then $X_i, i = 1, 2$ are orthonormal if and only if $D^{\frac{1}{2}} P^T [x_i, z_i]^T \equiv [\bar{x}_i, \bar{z}_i]^T$ are orthonormal, which follows from the fact that

$$\begin{aligned}
\langle X_i, X_j \rangle &= \langle Q[x_i, z_i]^T, Q[x_j, z_j]^T \rangle \\
&= \langle [x_i, z_i]^T, Q^T Q[x_j, z_j]^T \rangle \\
&= \langle [x_i, z_i]^T, PDP^T[x_j, z_j]^T \rangle \\
&= \langle P^T[x_i, z_i]^T, DP^T[x_j, z_j]^T \rangle \\
&= \langle D^{\frac{1}{2}} P^T[x_i, z_i]^T, D^{\frac{1}{2}} P^T[x_j, z_j]^T \rangle \\
&= \langle [\bar{x}_i, \bar{z}_i]^T, [\bar{x}_j, \bar{z}_j]^T \rangle, i = 1, 2
\end{aligned}$$

Let $\begin{bmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{z_1} & \bar{z_2} \end{bmatrix}$ be the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then $[X_1, X_2] = QPD^{-\frac{1}{2}} R$. From these arguments we obtain the following result.

Theorem 1. *Given orthonormal scaling vector $\Phi = [\phi_1, \phi_2]^T$ of the DGHM type satisfying refinement equation*

$$\Phi(x) = \sqrt{2} \sum_{k=-2}^{1} C(k)\Phi(2x - k), C(k) = \begin{bmatrix} c_{11}(k) & c_{12}(k) \\ c_{21}(k) & c_{22}(k) \end{bmatrix},$$

the orthonormal wavelet vector $\Psi = [\psi_1, \psi_2]^T$ can be constructed as following steps:

(1) $C_{1L} = [c_{12}(-2)\ c_{11}(-1)\ c_{12}(-1)], C_{1R} = [c_{12}(0)\ c_{11}(1)\ c_{12}(1)]$

(2) Define a 3×2 matrix Q by

$$Q = \begin{bmatrix} \| C_{1L} \| & 0 \\ -\sqrt{2} \| C_{1L} \|^2 & -\sqrt{2} \| C_{1R} \|^2 \\ 0 & \| C_{1R} \| \end{bmatrix}$$

(3) Diagonalize $Q^T Q$:

$$P^T Q^T Q P = D,$$

where P is an orthogonal matrix and D is diagonal.

$$(4) \begin{bmatrix} \alpha_{1L} & \alpha_{2L} \\ d_{11}(0) & d_{21}(0) \\ \alpha_{1R} & \alpha_{2R} \end{bmatrix} = \begin{bmatrix} \frac{1}{\|c_{1L}\|} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\|c_{1R}\|} \end{bmatrix} QPD^{-\frac{1}{2}} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(5) $\Psi(x) = \sqrt{2} \sum_{k=-2}^{1} D(k)\Phi(2x - k)$ *is a wavelet vector with multiwavelet coefficients* $D(k)$ *:*

$$D(-2) = \begin{bmatrix} 0 & \alpha_{1L} \cdot c_{12}(-2) \\ 0 & \alpha_{2L} \cdot c_{12}(-2) \end{bmatrix}, D(-1) = \begin{bmatrix} \alpha_{1L} c_{11}(-1) & \alpha_{1L} c_{12}(-1) \\ \alpha_{2L} c_{11}(-1) & \alpha_{2L} c_{12}(-1) \end{bmatrix},$$

$$D(0) = \begin{bmatrix} d_{11}(0) & \alpha_{1R}\cdot c_{12}(0) \\ d_{21}(0) & \alpha_{2R}\cdot c_{12}(0) \end{bmatrix}, D(1) = \begin{bmatrix} \alpha_{1R}c_{11}(1) & \alpha_{1R}c_{12}(1) \\ \alpha_{2R}c_{11}(1) & \alpha_{2R}c_{12}(1) \end{bmatrix}.$$

Example 1. *The DGHM multiscaling coefficients $C(k)$ are given in [6] as*

$$C(-2) = \frac{1}{20}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, C(-1) = \frac{1}{20}\begin{bmatrix} -3\sqrt{2} & 9 \\ 0 & 0 \end{bmatrix},$$

$$C(0) = \frac{1}{20}\begin{bmatrix} 10\sqrt{2} & 9 \\ 0 & 6\sqrt{2} \end{bmatrix}, C(1) = \frac{1}{20}\begin{bmatrix} -3\sqrt{2} & -1 \\ 16 & 6\sqrt{2} \end{bmatrix}.$$

We follow the steps of above theorem to get wavelet coefficients $D(k)$.

(1) $C_{1L} = \left[-\frac{1}{20} \ \ -\frac{3\sqrt{2}}{20} \ \ \frac{9}{20}\right], C_{1R} = \left[\frac{9}{20} \ \ -\frac{3\sqrt{2}}{20} \ \ -\frac{1}{20}\right]$

(2) $Q = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{1}{2} \end{bmatrix}$

(3) $Q^TQ = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix}$*, and it has eigenvalues* $\frac{1}{2}, \frac{1}{4}$ *with the corresponding unit eigenvectors* $\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T, \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]^T$*,respectively. Thus Q^TQ can be diagonalized as*

$$Q^TQ = PDP^T, \text{where } P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

(4)

$$\begin{bmatrix} \alpha_{1L} & \alpha_{2L} \\ d_{11}(0) & d_{21}(0) \\ \alpha_{1R} & \alpha_{2R} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sqrt{2} \\ -\frac{1}{\sqrt{2}} & 0 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta + \sqrt{2}\sin\theta & -\sin\theta + \sqrt{2}\cos\theta \\ -\frac{1}{\sqrt{2}}\cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ \cos\theta - \sqrt{2}\sin\theta & -\sin\theta - \sqrt{2}\cos\theta \end{bmatrix}.$$

Hence the multiwavelet coefficients of DGHM type are

$$D(-2) = \begin{bmatrix} 0 & (\cos\theta + \sqrt{2}\sin\theta)\cdot c_{12}(-2) \\ 0 & (-\sin\theta + \sqrt{2}\cos\theta)\cdot c_{12}(-2) \end{bmatrix},$$

$$D(-1) = \begin{bmatrix} (\cos\theta + \sqrt{2}\sin\theta)c_{11}(-1) & (\cos\theta + \sqrt{2}\sin\theta)c_{12}(-1) \\ (-\sin\theta + \sqrt{2}\cos\theta)c_{11}(-1) & (-\sin\theta + \sqrt{2}\cos\theta)c_{12}(-1) \end{bmatrix},$$

$$D(0) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\cos\theta & (\cos\theta - \sqrt{2}\sin\theta)\cdot c_{12}(0) \\ \frac{1}{\sqrt{2}}\sin\theta & (-\sin\theta - \sqrt{2}\cos\theta)\cdot c_{12}(0) \end{bmatrix},$$

$$D(1) = \begin{bmatrix} (\cos\theta - \sqrt{2}\sin\theta)c_{11}(1) & (\cos\theta - \sqrt{2}\sin\theta)c_{12}(1) \\ (-\sin\theta - \sqrt{2}\cos\theta)c_{11}(1) & (-\sin\theta - \sqrt{2}\cos\theta)c_{12}(1) \end{bmatrix}.$$

In particular, if we take $\theta = 0$ then we obtain DGHM multiwavelets [7] as

$$D(-2) = \frac{1}{20}\begin{bmatrix} 0 & -1 \\ 0 & -\sqrt{2} \end{bmatrix}, D(-1) = \frac{1}{20}\begin{bmatrix} -3\sqrt{2} & 9 \\ -6 & 9\sqrt{2} \end{bmatrix},$$
$$D(0) = \frac{1}{20}\begin{bmatrix} -10\sqrt{2} & 9 \\ 0 & -9\sqrt{2} \end{bmatrix}, D(1) = \begin{bmatrix} -3\sqrt{2} & -1 \\ 6 & \sqrt{2} \end{bmatrix}.$$

If ϕ_1 is symmetric, then $\| c_{1L} \| = \| c_{1R} \| = \frac{1}{2}$. From the step (4) in example 2.1, we know that the multiwavelet functions are symmetric or antisymmetric if and only if $\theta = \frac{\pi}{2}n, n \in \mathbf{Z}$. Thus we obtain the following corollary.

Corollary 1. *If $\Phi = [\phi_1, \phi_2]^T$ is a scaling vector of DGHM type with symmetric ϕ_1 , then there exists a unique symmetric or antisymmetric wavelet vector $\Psi = [\psi_1, \psi_2]^T$ up to sign.*

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