J. Appl. Math. & Informatics Vol. 29(2011), No. 3 - 4, pp. 1027 - 1035
Website: http://www.kcam.biz

FIXED AND PERIODIC POINT THEOREMS IN QUASI-METRIC SPACES

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Abstract. In this paper, we introduce the concept of generalized weak q-contractivity for multivalued maps defined on quasi-metric spaces. A new fixed point theorem for these maps is established. The convergene of iterate schem of the form $x_{n+1} \in Fx_n$ is investigated. And a new periodic point theorem for weakly q-contractive self maps of quasi-metric spaces is proved.

AMS Mathematics Subject Classification : 47H10, 54H25.
Key words and phrases : Fixed point, weak q-contractive multivalued map, Q-function, quasi-metric space.

1. Introduction and preliminaries

The authors [1] introduced the notion of weakly contractive mappings for single valued maps on Hilbert spaces. This concept is one of generalizations of contraction maps. They proved the existence of fixed points for weakly contractive maps in Hilbert spaces. The author [12] extended some of results in [1] to arbitrary Banach spaces. In fact, weakly contractive maps are closely related to maps of Boyd and Wong type ones [6] and Reich type ones [10].

In [3], the author introduced the notion of weak contractivity for multivalued maps and proved some fixed point theorems for weakly contractive multivalued maps with inwardness or weakly inwardness conditions.

In [4], the authors proved the existence of coincidence points and common fixed points for two single valued maps satisfying generalized weakly contractive conditions.

Recently, in [2] the authors gave the notions of q-contractivity for multivalued and single valued maps in quasi-metric spaces, and gave some fixed point theorems in quasi-metric spaces.

Received October 19, 2010. Revised March 10, 2011. Accepted March 22, 2011. *Corresponding author. †The second author was supported by Hanseo University.

In this paper we introduce the notion of generalized weakly q-contractive multivalued maps defined on a quasi-metric space, and we give a new fixed point theorems for these maps. We also give a new fixed point theorems for weakly q-contractive multivalued maps, which is a extension of theorem 3.1 in [3] and theorem 1 in [12] to quasi-metric spaces. We obtain a generalization of theorem 6.1 in [2]. With a Q-function, we obtain quasi-metric space versions of Nadler's and Banach's fixed point theorem. Also, we investigate the convergene of iterate schem of the form $x_{n+1} \in Fx_n$ with error estimates, where F is a weakly q-contractive multivalued map. And we prove a new periodic point theorem for weakly q-contractive self maps of quasi-metric spaces.

For the convenience, recall the following well known definition of a quasi-metric space.

Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a *quasi-metric* on X if the following are satisfied:

$(d1)$ $d(x, y) \geq 0$ for all $x, y \in X$;
$(d2)$ $d(x, y) = 0$ if and only if $x = y$;
$(d3)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A nonempty set X together with a quasi-metric d is called a *quasi-metric space* and it is denoted by (X, d). Note that the notion of a quasi-metric space is a generalization of the notion of a metric space.

Throughout the paper, unless otherwise specified, X is assumed to be a quasi-metric space with a quasi-metric d.

We know that each quasi-metric d on X generartes a T_0 topology on X. For a quasi-metric d on X, the conjugate quasi-metric d^{-1} on X of d is defined by $d^{-1}(x, y) = d(y, x)$. We denote by d^u the metric $d \vee d^{-1}$, that is, $d^u(x, y) = max\{d(x, y), d(y, x)\}$, for all $x, y \in X$.

A sequence $\{x_n\}$ of points of X is called *left K-Cauchy* [11] if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \in \mathbb{N}$. A sequence $\{x_n\}$ of points of X *converges* to some point $x \in X$ if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x, x_n) < \epsilon$ for all $n \geq n_0$.

X is called *left K-complete* [11, 13] if every left K-Cauchy sequence in X is convergent with respect to d. X is called *Smyth-complete* [8, 14] if every left K-Cauchy sequence in X is convergent with respect to d^u.

Remark 1.1. *Every Smyth-complete quasi-metric space is left K-complete. In general, it is known that the converse is not true.*

A function $q : X \times X \to [0, \infty)$ is called *Q-function* [2] on X if the following are satisfied:
$(Q1)$ for all $x, y, z \in X$, $q(x, z) \leq q(x, y) + q(y, z)$;
$(Q2)$ if $x \in X$ and $\{y_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(y, y_n) = 0$ and $q(x, y_n) \leq M$ for some $M > 0$, then $q(x, y) \leq M$;

$(Q3)$ for any $\epsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \epsilon$.

We denote by $K(X)$ the family of nonempty compact subsets of (X, d^u) and by $C(X)$ the family of nonempty closed subsets of (X, d). Let $q : X \times X \to [0, \infty)$ be a Q-function on X.

We define $H_q : C(X) \times C(X) \to [0, \infty)$ by

$$H_q(A, B) = max\{sup_{b\in B} q(A, b), sup_{a\in A} q(a, B)\}, \quad A, B \in C(X),$$

where $q(A, b) = inf\{q(a, b) : a \in A\}$ and $q(a, B) = inf\{q(a, b) : b \in B\}$.

Let $D_q(A, B) = sup_{a\in A} q(a, B)$. Then $D_q(A, B) \leq H_q(A, B)$.

Lemma 1.1. [2] *Let $q : X \times X \to [0, \infty)$ be a Q-function on X, and let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. If $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, \infty)$ such that $\alpha_n \to 0$ and $\beta_n \to 0$, then the following are satisfied:*
(i) If $q(x_n, y) \leq \alpha_n$ and $q(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$;
(ii) If $q(x_n, y) \leq \alpha_n$ and $q(x_n, y_n) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} d(y, y_n) = 0$;
(iii) If $q(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a left K-Cauchy sequence;
(iv) If $q(y, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a left K-Cauchy sequence.

From now on, let $\varphi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that
(φ_1) $\varphi(0) = 0$;
(φ_2) $0 < \varphi(t) < t$, for each $t > 0$;
(φ_3) for any sequence $\{t_n\}$ of $(0, \infty)$, $\sum_{n=1}^{\infty} \varphi(t_n) < \infty$ implies $\sum_{n=1}^{\infty} t_n < \infty$.

2. Fixed point theorems

A map $f : X \to X$ is called *weakly q-contractive* if there exists a Q-function q on X such that for all $x, y \in X$

$$q(fx, fy) \leq q(x, y) - \varphi(q(x, y)).$$

A multivalued map $F : X \to 2^X$ is called *weakly q-contractive* if there exists a Q-function q on X such that for all $x, y \in X$

$$H_q(Fx, Fy) \leq q(x, y) - \varphi(q(x, y)).$$

A multivalued map $F : X \to 2^X$ is called *generalized weakly q-contractive* if there exists a Q-function q on X such that, for each $x, y \in X$ and $u \in Fx$, there exists $v \in Fy$ such that

$$q(u, v) \leq q(x, y) - \varphi(q(x, y)). \tag{1}$$

In this section, we give a new fixed point theorem for a generalized weakly q-contractive multivalued map in quasi-metric space with a Q-function. And then, we obtain generalizations of results in [2].

Theorem 2.1. *Let (X, d) be a left K-complete quasi-metric space. If $F : X \to C(X)$ is a generalized weakly q-contractive multivalued map, then F has a fixed point in X and $q(p, p) = 0$ for $p \in Fix(F)$, where $Fix(F)$ denotes the set of all fixed points of F.*

Proof. Let $x_0 \in X$ and $x_1 \in Fx_0$. By (1), there exists $x_2 \in Fx_1$ such that

$$q(x_1, x_2) \leq q(x_0, x_1) - \varphi(q(x_0, x_1)).$$

Again, by (1) there exists $x_3 \in Fx_2$ such that

$$q(x_2, x_3) \leq q(x_1, x_2) - \varphi(q(x_1, x_2)).$$

Continuing in this way, we have a sequence $\{x_n\}$ of points of X such that for all $n = 0, 1, 2 \cdots$

$$x_{n+1} \in Fx_n \ \ \text{and} \ \ q(x_{n+1}, x_{n+2}) \leq q(x_n, x_{n+1}) - \varphi(q(x_n, x_{n+1})).$$

Hence the sequence $\{q(x_n, x_{n+1})\}$ is nonincreasing. Thus there exists $l \geq 0$ such that $\lim_{n\to\infty} q(x_n, x_{n+1}) = l$.

We now show that $l = 0$.

Suppose $l > 0$. Then we have

$$q(x_n, x_{n+1}) \leq q(x_{n-1}, x_n) - \varphi(q(x_{n-1}, x_n)) \leq q(x_{n-1}, x_n) - \varphi(l),$$

and so

$$q(x_{n+N}, x_{n+N+1}) \leq q(x_{n-1}, x_n) - N\varphi(l),$$

which is a contradiction for N large enough. Thus we have

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0. \tag{2}$$

For $m \in \mathbb{N}$ with $m \geq 3$, we have

$$\begin{aligned} &q(x_{m-1}, x_m) \\ &\leq q(x_{m-2}, x_{m-1}) - \varphi(q(x_{m-2}, x_{m-1})) \cdots \\ &\leq q(x_1, x_2) - \varphi(q(x_1, x_2)) - \cdots - \varphi(q(x_{m-2}, x_{m-1})). \end{aligned}$$

Hence we have

$$\sum_{k=1}^{m-2} \varphi(q(x_k, x_{k+1})) \leq q(x_1, x_2) - q(x_{m-1}, x_m).$$

Letting $m \to \infty$ in above inequality, we have

$$\sum_{n=1}^{\infty} \varphi(q(x_n, x_{n+1})) \leq q(x_1, x_2) < \infty$$

which implies

$$\sum_{n=1}^{\infty} q(x_n, x_{n+1}) < \infty \ \ \text{by} \ \ (\varphi_3).$$

Let $\alpha_n = \sum_{k=n}^{\infty} q(x_k, x_{k+1})$. Then for all $m > n$

$$q(x_n, x_m) \leq \sum_{k=n}^{m-1} q(x_k, x_{k+1}). \tag{3}$$

By Lemma 1.1 (iii), $\{x_n\}$ is a left K-Cauchy sequence in (X, d). Since (X, d) is left K-complete, there exists $p \in X$ such that $\lim_{m\to\infty} d(p, x_m) = 0$. By $(Q2)$ and (3), we have

$$q(x_n, p) \leq \alpha_n. \tag{4}$$

From (1) there exists $s_n \in Fp$ such that

$$q(x_n, s_n) \leq q(x_{n-1}, p) - \varphi(q(x_{n-1}, p)) \leq \alpha_{n-1}.$$

By Lemma 1.1 (ii), $\lim_{n\to\infty} d(p, s_n) = 0$. Hence $p \in Fp$ because $Fp \in C(X)$.

Next we show that $q(p, p) = 0$.

For $p \in Fp$, by (1) there exists $y_1 \in Fp$ such that $q(p, y_1) \leq q(p, p) - \varphi(q(p, p))$. For $p \in Fp$, there exists $y_2 \in Fy_1$ such that $q(p, y_2) \leq q(p, y_1) - \varphi(q(p, y_1))$.

Continuing in this process, we have a sequence $\{y_n\}$ of points of X such that $y_n \in Fy_{n-1}$ and $q(p, y_{n+1}) \leq q(p, y_n) - \varphi(q(p, y_n))$ for all $n \in \mathbb{N}$.

Thus we have

$$\begin{aligned} & q(p, y_{n+1}) \\ & \leq q(p, y_n) - \varphi(q(p, y_n)) \\ & \leq q(p, y_{n-1}) - \varphi(q(p, y_{n-1})) - \varphi(q(p, y_n)) \\ & \cdots \\ & \leq q(p, y_1) - \sum_{k=1}^{n} \varphi(q(p, y_k)). \end{aligned} \tag{5}$$

Since $\{q(p, y_n)\}$ is a nonincreasing sequence, as the proof of (2), we can show $\lim_{n\to\infty} q(p, y_n) = 0$.

Let $\beta_n = q(p, y_{n-1})$. Then $q(p, y_n) \leq \beta_n$. By Lemma 1.1 (iv), $\{y_n\}$ is a left K-Cauchy sequence in X. Let $\lim_{n\to\infty} d(y, y_n) = 0$. By $(Q2)$, $q(p, y) \leq \beta_n$ and so $q(p, y) = 0$.

From (4) and $(Q1)$ we have $q(x_n, y) \leq q(x_n, p) + q(p, y) = q(x_n, p) \leq \alpha_n$. By Lemma 1.1 (i) with (4), we have $y = p$. Therefore, $q(p, p) = 0$. □

Example. Let $X = \{\frac{1}{2^n} : n = 1, 2 \cdots\} \cup \{0\}$ and let $d(x, y) = \begin{cases} y - x & (y \geq x) \\ 2(x - y) & (x > y) \end{cases}$ for all $x, y \in X$. Let $q(x, y) = \mid x - y \mid$ for all $x, y \in X$. Then (X, d) is a left K-complete quasi-metric space and q is a Q-function on X. Let $\varphi(t) = \frac{1}{2}t$ for all $t \geq 0$ and let $F : X \to C(X)$ be multivalued map defined as

$$Fx = \begin{cases} \{\frac{1}{2^{n+1}}, \frac{1}{2}\} & (x = \frac{1}{2^n}, n = 1, 2, \cdots), \\ \{0, \frac{1}{2}\} & (x = 0). \end{cases}$$

We now show that F satisfies condition (1).

If $x = 0$ and $y = 0$, then F satisfies condition (1).

If $x = 0$ and $y = \frac{1}{2^n}(n = 1, 2, \cdots)$, then for $u = 0 \in Fx$ there exists $v = \frac{1}{2^{n+1}} \in Fy$ such that

$$q(u, v) = q(0, \frac{1}{2^{n+1}}) = \frac{1}{2^{n+1}} \leq q(x, y) - \varphi(q(x, y)).$$

For $u = \frac{1}{2} \in Fx$ there exists $v = \frac{1}{2} \in Fy$ such that

$$q(u, v) = 0 \leq q(x, y) - \varphi(q(x, y)).$$

Let $x = \frac{1}{2^n}$ and $y = \frac{1}{2^m}(m > n)$. Then for $u = \frac{1}{2} \in Fx$ there exists $v = \frac{1}{2} \in Fy$ such that

$$q(u, v) = 0 \leq q(x, y) - \varphi(q(x, y)).$$

For $u = \frac{1}{2^{n+1}} \in Fx$ there exists $v = \frac{1}{2^{m+1}} \in Fy$ such that

$$q(u, v) = \frac{1}{2^{n+1}} - \frac{1}{2^{m+1}} = \frac{2^m - 2^n}{2^{n+m+1}} \leq q(x, y) - \varphi(q(x, y)).$$

Thus F is a generalized weakly q-contractive multivalued map and $0 \in F0$.

Corollary 2.2. *Let (X, d) be a left K-complete quasi-metric space. Suppose that $F : X \to C(X)$ is a multivalued map satisfying*

$$D_q(Fx, Fy) \leq q(x, y) - \varphi(q(x, y)),$$

for each $x, y \in X$, where q is a Q-function on X.

Then F has a fixed point in X and $q(p, p) = 0$ for $p \in Fix(F)$.

Corollary 2.3. *Let (X, d) be a left K-complete quasi-metric space. If $F : X \to C(X)$ is a weakly q-contractive multivalued map, then F has a fixed point in X and $q(p, p) = 0$ for $p \in Fix(F)$.*

Corollary 2.4. *Let (X, d) be a left K-complete quasi-metric space. If $f : X \to X$ is a weakly q-contractive map, then f has a unique fixed point in X and $q(p, p) = 0$ for $p \in Fix(f)$.*

Proof. From Corollary 2.3 there exists a point $p \in X$ such that $p = fp$ and $q(p, p) = 0$. We show the uniqueness of the fixed point p of f.

Let $z \in X$ be such that $z = fz$. If $q(p, z) \neq 0$, then $q(p, z) = q(fp, fz) \leq q(p, z) - \varphi(q(p, z)) < q(p, z)$ which is a contradiction. Hence $q(p, z) = 0$. By Lemma 1.2 (i), $p = z$. □

Remark 2.1. *In Theorem 2.1 and Corollary 2.2, if the map is single valued then it has a unique fixed point.*

Remark 2.2. *Corollary 2.3 is an extension of theorem 3.1 in* [3] *to quasi-metric space, and Corollary 2.4 is an extension of theorem 1 in* [12] *to quasi-metric space.*

From now on, let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function satisfying the following conditions:
(ϕ_1) $\phi(0) = 0$ and $0 < \phi(t) < t$ for each $t > 0$,
(ϕ_2) $t \le s$ implies $\phi(s) - \phi(t) \le s - t$,
(ϕ_3) for any sequence $\{t_n\}$ of $(0, \infty)$, $\sum_{n=1}^{\infty}(t_n - \phi(t_n)) < \infty$ implies $\sum_{n=1}^{\infty} t_n < \infty$.

Let $\varphi(t) = t - \phi(t)$. Then $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying (φ_1) $\sim$ (φ_3). From Theorem 2.1, Corollary 2.2, Corollary 2.3 and Corollary 2.4 we have the following results.

Corollary 2.5. *Let (X, d) be a left K-complete quasi-metric space. Suppose that $F : X \to C(X)$ is a multivalued map satisfying the following condition:*

there exists a Q-function q on X such that, for each $x, y \in X$ and $u \in Fx$, there exists $v \in Fy$ satisfying

$$q(u, v) \le \phi(q(x, y)).$$

Then F has a fixed point in X and $q(p, p) = 0$ for $p \in Fix(F)$.

Corollary 2.6. *Let (X, d) be a left K-complete quasi-metric space. Suppose that $F : X \to C(X)$ is a multivalued map satisfying*

$$D_q(Fx, Fy) \le \phi(q(x, y)),$$

for each $x, y \in X$, where q is a Q-function on X.

Then F has a fixed point in X and $q(p, p) = 0$ for $p \in Fix(F)$.

Corollary 2.7. *Let (X, d) be a left K-complete quasi-metric space. If $F : X \to C(X)$ is a multivalued map satisfying*

$$H_q(Fx, Fy) \le \phi(q(x, y)),$$

for each $x, y \in X$, where q is a Q-function on X.

Then F has a fixed point in X and $q(p, p) = 0$ for $p \in Fix(F)$.

Corollary 2.8. *Let (X, d) be a left K-complete quasi-metric space. If $f : X \to X$ is a map satisfying*

$$q(fx, fy) \le \phi(q(x, y)), \tag{6}$$

for each $x, y \in X$, where q is a Q-function on X, then f has a unique fixed point in X and $q(p, p) = 0$ for $p \in Fix(f)$.

Remark 2.3. *If $\phi(t) = kt$, for somr $k \in [0, 1)$ in Corollary 2.5(Corollary 2.8), then we have theorem 6.1(corollary 6.2) in* [2].

In Corollary 2.7, if we have $\phi(t) = kt$, for somr $k \in [0, 1)$ then we obtain the following result which is a quasi-metric space of Nadler's fixed point theorem.

Corollary 2.9. *Let (X, d) be a left K-complete quasi-metric space. If $F : X \to C(X)$ is a multivalued map satisfying*

$$H_q(Fx, Fy) \leq kq(x, y),$$

for each $x, y \in X$, where $k \in [0, 1)$ and q is a Q-function on X, then F has a fixed point in X and $q(p, p) = 0$ for $p \in Fix(F)$.

In Corollary 2.8, if we have $\phi(t) = kt$, for somr $k \in [0, 1)$ then we obtain the following corollary which is a quasi-metric space version of Banach's fixed point theorem.

Corollary 2.10. *Let (X, d) be a left K-complete quasi-metric space. If $f : X \to X$ is a map satisfying*

$$q(fx, fy) \leq kq(x, y), \tag{7}$$

for each $x, y \in X$, where $k \in [0, 1)$ and q is a Q-function on X, then f has a unique fixed point in X and $q(p, p) = 0$ for $p \in Fix(F)$.

Theorem 2.11. *Let (X, d) be a left K-complete quasi-metric space and $F : X \to C(X)$ be a weakly q-contractive multivalued map. If $p \in Fp$, then there exists a sequence $\{y_n\}$ of points of X with $y_{n+1} \in Fy_n$ such that $\lim_{n\to\infty} q(p, y_n) = 0$, with the following error estimate:*

$$q(p, y_{n+1}) \leq q(p, y_1) - \sum_{k=1}^{n} \varphi(q(p, y_k)). \tag{8}$$

Proof. Since $p \in Fp$, as in proof of Theorem 2.1, we can construct a sequence $\{y_n\}$ of points of X satisfying

$y_{n+1} \in Fy_n$, $q(p, y_{n+1}) \leq q(p, y_n) - \varphi(q(p, y_n))$ for all $n \in \mathbb{N}$ and (5).

From (5) we obtain (8). As in proof of Theorem 2.1, we can show $\lim_{n\to\infty} q(p, y_n) = 0$. □

3. Periodic point theorems

A self map f of a nonempty set has *property P* [7] if it satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbb{N}$.

Theorem 3.1. *Let (X, d) be a left K-complete quasi-metric space. If $f : X \to X$ is a weakly q-contractive map, then f has property P.*

Proof. Let $u \in F(f^n)$. We show that $q(fu, u) = 0$.

If $q(fu, u) \neq 0$, then $q(f^2u, fu) \leq q(fu, u) - \varphi(q(fu, u)) < q(fu, u)$. Also, we have $q(f^3u, f^2u) \leq q(f^2u, fu) - \varphi(q(f^2u, fu)) \leq q(f^2u, fu) < q(fu, u)$.

Continuing in this way, we obtain

$$q(fu, u) = q(f^{n+1}u, f^n u) \leq q(f^n u, f^{n-1}u) \leq \cdots \leq q(f^2u, fu) < q(fu, u)$$

which is a contradiction. Thus we have $q(fu, u) = 0$.

Similary, we can show that $q(u, fu) = 0$. Hence we have $q(u, u) \leq q(u, fu) + q(fu, u) = 0$. By Lemma 1.2 (i), $u = fu$ and f has property P. □

In (6), if $\phi(t) = t - \varphi(t)$ for all $t \geq 0$ then from Theorem 3.1 we have the following corollary.

Corollary 3.2. *Let (X, d) be a left K-complete quasi-metric space. If a map $f : X \to X$ is satisfying (6), then f has property P.*

Corollary 3.3. *Let (X, d) be a left K-complete quasi-metric space. If a map $f : X \to X$ is satisfying (7), then f has property P.*

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