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OVERVIEWS ON LIMIT CONCEPTS OF A SEQUENCE OF FUZZY NUMBERS I

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Abstract. In this paper, we survey various notions and results related to statistical convergence of a sequence of fuzzy numbers, in which statistical convergence for fuzzy numbers was first introduced by Nuray and Savas in 1995. We will go over boundedness, convergence of sequences of fuzzy numbers, statistically convergence and statistically Cauchy sequences of fuzzy numbers, statistical limit and cluster point for sequences of fuzzy numbers, statistical monotonicity and boundedness of a sequence of fuzzy numbers and finally statistical limit inferior and limit inferior for the statistically bounded sequences of fuzzy numbers.

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1. Introduction

Fast [5] introduced an extension of the usual concept of sequential limits which is called statistical convergence and also independently by Steinhaus [20]. Schoenberg [19] gave some basic properties of statistical convergence and Fridy [6] introduced the concept of a statistically Cauchy sequence and proved that is equivalent to statistical convergence. The concepts of limit and cluster point of a real sequence have been extended in Fridy[6, 7] to a statistical limit and cluster point using the natural density of a set of positive integers. Fridy [7] was the first not only to introduce the set of all statistical cluster points and the set of all limit points but also to discuss their definitions and properties, as well as the specific relations among them in addition to their relations between the set of all ordinary limit points. In [7], Fridy investigated the statistical monotonicity for sequences of real numbers using classical techniques and established some basic results. Later the concept of statistical bounded and statistical monotonic sequences of

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real numbers were examined by Tripathy [21, 22]. As an application, these ideas were used in Turnpike theory by Pehlivan and Mammedov [16]. On the side of sequences of fuzzy numbers, the boundedness and convergence were first introduced by Matloka [12]. Nandra [13] has conclude that the set of bounded and convergent sequences of fuzzy numbers forms complete metric spaces. Nuray and Savas [15] introduced the concepts of statistically convergent and statistically Cauchy sequences of fuzzy numbers. In Aytar [1] he examined the concept of statistical limit and cluster point for sequences of fuzzy numbers. Aytar and Pehlivan [2] defined the concepts of statistical monotonicity and boundedness of a sequence of fuzzy numbers. Congxin and Cong [4] defined the concepts of the supremum and infimum for sets of bounded fuzzy numbers. Recently, Aytar and Mammadov [3] defined the notions of statistical limit inferior and limit inferior for the statistically bonded sequences of fuzzy numbers. In the study they showed that some results established for the sequences of real numbers are also valid for the sequences of fuzzy numbers. In this overview paper, we review various concepts and collect fundamental results related to statistically convergence and statistical limits of sequences of fuzzy numbers. We will go over boundedness, convergence of sequences of fuzzy numbers, statistically convergence and statistically Cauchy sequences of fuzzy numbers, statistical limit and cluster point for sequences of fuzzy numbers, statistical monotonicity and boundedness of a sequence of fuzzy numbers and finally statistical limit inferior and limit inferior for the statistically bonded sequences of fuzzy numbers. We based mainly on the materials of the following references([1,2,7,8, 12, 14, 16]).

2. Definitions and Preliminaries

Let D denote the set of all closed intervals $A = [\underline{A}, \overline{A}]$ on the real line $\mathbb{R}$. For $A, B \in D$, define

$$A \leq B \text{ if and only if} \underline{A} \leq \underline{B} \text{and} \overline{A} \leq \overline{B},$$

$$d(A,B) = max(|\underline{A} - \overline{B}|, |\overline{A} - \overline{B}|).$$

Then d defines a metric on D and (D, d) is a complete metric space. Also $\leq$is a partial order in D. A fuzzy number is a mapping $A : \mathbb{R} \to [0, 1]$. We will denote by A_α the α-level set of A(that is $A_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}$) for all $\alpha \in (0, 1]$ and by A_0 the closure of the support of A(that is $A_0 = cl\{x \in \mathbb{R} : A(x) > 0\}$). Let $L(\mathbb{R})$ be the class of the fuzzy sets A satisfying the following conditions;

(1) $A_1 \neq \emptyset$
(2) $A_0$is compact and
(3) A is upper semi-continuous
(4) $A_\alpha$is convex for all$\alpha \in [0, 1]$.

Define, a map

$$\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R} \text{ by } \overline{d}(X, Y) = sup_{\alpha \in [0,1]} d(X^\alpha, Y^\alpha).$$

Then $\overline{d}$ is a metric on $L(\mathbb{R})$. For $X, Y \in L(\mathbb{R})$, define

$$X \leq Y \text{ if and only if } X^\alpha \leq Y^\alpha \text{ for any } \alpha \in [0,1].$$

Then $\leq$ is a partial order in $L(\mathbb{R})$. The fuzzy numbers X and Y are said to be incomparable if neither $X \leq Y$ nor $Y \leq X$. We use the notation $X \not\sim Y$ in this case. We say $X < Y$, if $X \leq Y$ and there exists $\alpha_0 \in [0,1]$ such that $\underline{X_0^\alpha} < \underline{Y_0^\alpha}$ or $\overline{X}_0^\alpha < \overline{Y}_0^\alpha$. When $Y < X$ or X and Y are incomparable, then we write $X \not< Y$. A subset E of $L(\mathbb{R})$ is said to be bounded above if there exists a fuzzy number μ, called an upper bound of E, such that $X \leq \mu$ for every $X \in E$. μ is called the least upper bound(sup) of E if μ is an upper bound and is the smallest of all upper bounds. A lower bound and the greatest lower bound(inf) are defined similarly. E is said to be bounded if it is both bounded above and bounded below. If K is a subset of positive integers N, then K_n denote the set $\{k \in K : k \leq n\}$. The natural density of K is given by $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n. Clearly, finite subsets have zero natural density and $\delta(K^c) = 1 - \delta(K)$ where K^c is the complement of K.

3. Statistical convergence and limits of sequences of fuzzy numbers

Definition 3.1. A sequence $X = (X_k)$ of fuzzy sets is said to be convergent to the fuzzy set X_0, written as $\lim_k X_k = X_0$, if for every $\epsilon > 0$ there exists a positive integer n_0 such that $\overline{d}(X_k, X_0) < \epsilon$ for every $k > n_0$.

Definition 3.2. A sequence $X = (X_k)$ of fuzzy sets is said to be statistically convergent to the fuzzy set X_0, written as $st - \lim_k X_k = X_0$, if for every $\epsilon > 0$ $\delta(\{k \in N : \overline{d}(X_k, X_0)\}) = 0$. Clearly, since the natural density of the finite set is zero, a convergent sequence must be statistically convergent. But the converse does not true in general.

Theorem 3.3. Let $X = (X_k)$, $Y = (Y_k) \subset L(\mathbb{R})$.

(a) If $st - \lim_k X_k = X_0$ and $c \in \mathbb{R}$, then $st - \lim_k cX_k = cX_0$.

(b) If $st - \lim_k X_k = X_0$ and $st - \lim_k Y_k = Y_0$, then $st - \lim_k (X_k + Y_k) = X_0 + Y_0$.

Theorem 3.4. Let $X = (X_k)$ be a sequence of fuzzy numbers, Then the following are equivalent.

(a) X is statistically convergent to X_0.

(b) There exist $Y = (Y_k)$ and $Z = (Z_k)$ of fuzzy numbers such that $X = Y + Z$, $d(Y_k, X_0) \to 0$ as $k \to \infty$ and $\delta(suppZ) = 0$.

(c) There is a subsequence $K = \{k_n\}$ of N such that $\delta(K) = 1$ and $d(X_k, X_0) \to 0$ as $n \to \infty$.

Definition 3.5. The fuzzy set X_0 is called the limit of the sequence of fuzzy numbers $X = (X_k)$ if there is a subsequence of X that converges to X_0. Let L_x denote the set of limit fuzzy sets of X.

Definition 3.6. The fuzzy set X_0 is called statistical limit point(s.l.p) of sequence of fuzzy sets $X = (X_k)$ if there is a non thin subsequence of X that converge to X_0. Let Λ_X denote the set of s.l.p of the sequence X.

Definition 3.7. The fuzzy set X_0 is called statistical cluster point(s.c.p) of sequence of fuzzy sets $X = (X_k)$ if for every $\epsilon > 0$ $\overline{\delta}(\{k \in N : \overline{d}(X_k, X_0) < \epsilon\}) > 0$.

Let $\Gamma_X$denote the set of s.c.p of the sequence X.

Theorem 3.8. If $X = (X_k)$ and $Y = (Y_k)$ are sequences of fuzzy numbers such that $\delta(\{k \in N : X_k \neq Y_k\}) = 0$, then $\Lambda_X = \Lambda_Y$ and $\Gamma_X = \Gamma_Y$.

Theorem 3.9. If $X = (X_k)$ is a sequence of fuzzy sets, then $\Lambda_X \subset \Gamma_X$ and $\Gamma_X \subset L_X$.

Proof. Assume that $\mu \in \Gamma_X$, say $\overline{\delta}(\{k \in N : \overline{d}(X_k, \mu) < \epsilon\}) > 0$ for every$\epsilon > 0$. We set $\{X\}_k$ a non thin subsequence of X such that $K := \{k(j) \in N : \overline{d}(X_k(j), \mu < 0) < \epsilon\}$, for every $\epsilon > 0$ and $\overline{\delta} > 0$. Since there are infinitely many elements in K, we have $\mu \in L_X$. □

The converse of Theorem 3.9 does not hold in general.

Theorem 3.10. For any sequence of fuzzy sets $X = (X_k)$, $\Lambda_X \subset \Gamma_X$.

The following is statistical analogue of result given for real number sequences in the classical analysis.

Theorem 3.11. If $X_k \leq Y_k \leq Z_k$, for all $k \in K \subset N$, with $\delta(K) = 1$ and $\mu = st - \lim X_k = st - \lim Z_k$, then $st - \lim Y_k = \mu$.

4. Statistical monotonicity and boundedness of a sequence of fuzzy numbers

Definition 4.1. The sequence $X = (X_k)$ is said to be statistically bounded from above if there is a fuzzy number μ (called statistical upper bound) such that $\delta(\{k \in N : X_k > \mu\} \cup \{k \in N : X_k \not\sim \mu\}) = 0$. Similarly, $X = (X_k)$ is said to be statistically bounded from below provided that there is a fuzzy number ν(called the statistical lower bound) such that $\delta(\{k \in N : X_k > \nu\} \cup \{k \in N : X_k \not\sim \mu\}) = 0$. If the sequence $X = (X_k)$ is both statistically bounded above and statistically bounded below then it is called statistically bounded. Generally, if a sequence $X = (X_k)$ is bounded, it is also statistically bounded but not vice versa.

Definition 4.2. A sequence is said to be statistically monotonic increasing if there is a subset $K = \{k_i : k_1 < k_2 < \cdots\} \subset N$ such that $\delta(K) = 1$ and (X_{k_n}) is monotonically increasing. Similarly, statistically monotonic decreasing can be defined. The fuzzy set sequence $X = (X_k)$ is called statistically monotonic sequence if it is statistically monotonic increasing or statistically monotonic decreasing.

Theorem 4.3. If $X = (X_k)$ is statistically bounded sequence of fuzzy numbers, then X can be written as $X = Y + Z$ where $Y = (Y_k)$ is bounded and Z is statistically 0 sequence(that is $\delta(\{k \in N : Z_k \neq 0\}) = 0$).

Theorem 4.4. A fuzzy number sequence $X = (X_k)$ is statistically bounded if and only if there is a set $K = \{k_i : k_1 < k_2 < \cdots\} \subset N$ such that $\delta(K) = 1$ and (X_{k_n}) is bounded.

Theorem 4.5. A statistically monotonic sequence of fuzzy numbers is statistically convergent if and only if it is statistically bounded.

Theorem 4.6. If $X = (X_k)$ is statistically monotonic increasing(decreasing) sequence, then $\delta(\{k \in N : X_k \not\preceq X_{k+1}\}) = 0$ ($\delta(\{k \in N : X_k \not\succeq X_{k+1}\}) = 0$). Now we give a Decomposition theorem for statistically bounded sequences of fuzzy numbers

Theorem 4.7. If $X = (X_k)$ is statistically bounded sequence of fuzzy numbers, then we can write $X = Y + Z$ where $Y = (Y_k)$ is bounded and $\delta(\{k \in N : Z_k \neq 0\}) = 0$.

Proof. Let $X = (X_k)$ be statistically bounded sequence of fuzzy numbers. For $M > 0$ large enough, $K = \{k \in N : \overline{d}(X_k, 0) > M\}$ satisfies $\delta(K) = 0$. Define the sequences $Y = (Y_k)$ and $Z = (Z_k)$ as follows ;

$$Y_k = \begin{cases} X_k \text{ if } k \in K^c \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

$$Z_k = \begin{cases} X_k \text{ if } k \in K^c \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

Then, obviously, we have that $(X_k) = (Y_k) + (Z_k)$ for all $k \in N$, $Y = (Y_k)$ is a bounded sequence and $\delta(\{k \in N : Z_k \neq 0\}) = 0$. □

Following Savas [16], we have the following;

Theorem 4.8. $X = (X_k)$ is statistically bounded if and only if there is $K = \{k_i : k_1 < k_2 < \cdots\}$ such that $\delta(K) = 1$ and $\{X_{k_n}\}$ is bounded.

Theorem 4.9. A statistically monotonic sequence $X = (X_k)$ is statistically convergent if and only if it is statistically bounded.

5. Statistical Limit Inferior and Limit Superior

In this section we introduce the concepts of statistical limit superior and limit inferior for statistically bounded sequences of fuzzy numbers. Given a sequence $X = (X_k)$ we define the following sets;

$$A_X = \{\mu \in L(\mathbb{R}) : \delta(k \in N : X_k < \mu\}) \neq 0\}$$
$$\overline{A}_X = \{\mu \in L(\mathbb{R}) : \delta(k \in N : X_k > \mu\}) = 1\}$$
$$B_X = \{\mu \in L(\mathbb{R}) : \delta(k \in N : X_k > \mu\}) \neq 0\}$$

$$\overline{B}_X = \{\mu \in L(\mathbb{R}) : \delta(k \in N : X_k < \mu\}) = 1\}$$

The set $\overline{A}_X$ and $\overline{B}_X$ are the sets of statistical lower bounds and statistical upper bounds, respectively. It is clear that if the sequence $X = (X_k)$ is statistically bounded then these sets are non-empty. It is also clear that the sets A_X and $\overline{B}_X$ have a lower bound and the sets $\overline{A}_X$ and B_X have an upper bound.

Definition 5.1. If $X = (X_k)$ is statistically bounded sequence of fuzzy numbers, then the statistical limit inferior of $X = (X_k)$ is given by $st{-}liminf X = inf A_X$. Also, the statistical limit superior of $X = (X_k)$ is given by $st - limsupX = supB_X$.

Theorem 5.2. If $X = (X_k)$ is statistically bounded, then $infA_X = sup\overline{A}_X$ and $supB_X = inf\overline{B}_X$.

Theorem 5.3. Let $X = (X_k)$ be a statistically bounded sequence of fuzzy numbers. If $\nu = st - liminf X$, then

$$\delta(\{k \in N : X_k < \nu - \epsilon\}) = 0,$$

$$\delta(\{k \in N : X_k < \nu + \epsilon\} \cup \{k \in N : X_k \not\sim \nu + \epsilon\}) \neq 0$$

for every $\epsilon > 0$.

Proof. We suppose that there is some $\epsilon > 0$ such that $\delta(\{k \in N : X_k < \nu - \epsilon\}) \neq 0$. This means that $\nu - \epsilon \in A_X$. By definition of A_X, we have $\nu \leq \nu - \epsilon$ which is a contradiction. For the second one, assume that it is not true. That is, there is $\epsilon > 0$ such that $\delta(\{k \in N : X_k < \nu + \epsilon\}) = 0$ and $\delta(\{k \in N : X_k \not\sim \nu + \epsilon\}) = 0$. For each $k \in N$, only the following three cases are possible;

$$X_k < \nu + \epsilon, X_k \not\sim \nu + \epsilon \text{ and } X_k \geq \nu + \epsilon.$$

Then we obtain

$$X_k < \nu + \epsilon \cup X_k \not\sim \nu + \epsilon \cup X_k \geq \nu + \epsilon = N$$

Thus we have

$$\delta(\{k \in N : X_k \geq \nu + \epsilon\}) = 1.$$

This means that $\nu + \epsilon \in \overline{A}_X$. Hence we can write $\nu + \epsilon \leq sup\overline{A}_X = infA_X = \nu$. This is impossible. □

It is known that the converse of theorem 5.3 is valid for the sequences of real numbers but for sequences of fuzzy numbers it may not be true.

Theorem 5.4. Let $X = (X_k)$ be a statistically bounded sequence of fuzzy numbers. If $\nu = st - limsupX$, then

$$\delta(\{k \in N : X_k \geq \nu + \epsilon\}) = 0$$

$$\delta(\{k \in N : X_k \geq \nu - \epsilon\} \cup \{k \in N : X_k \not\sim \nu - \epsilon\}) \neq 0$$

for every $\epsilon > 0$.

Theorem 5.5. Let $X = (X_k)$ be a statistically bounded sequence of fuzzy numbers. Then we have

$$st - liminfX \leq st - limsupX.$$

Theorem 5.6. If $X = (X_k)$ is a statistically convergent to X_0. Then we have

$$st - liminfX = st - limsupX.$$

Proof. Let $\epsilon > 0$ be given. Assume that is statistically convergent to X_0, then we have

$$\delta(\{k \in N : \overline{d}(X_k, X_0) \geq \epsilon\}) = 0,$$
$$\delta(\{k \in N : \overline{d}(X_k, X_0) < \epsilon\}) = 1,$$
$$\delta(\{k \in N : sup_{\alpha \in [0,1]} d({X^\alpha}_k, {X^\alpha}_0) < \epsilon\}) = 1.$$

In other words, for almost all k,

$$sup_{\alpha \in [0,1]} d({X^\alpha}_k, {X^\alpha}_0) < \epsilon \text{ or } X_0 - \epsilon < X_k < X_0 + \epsilon.$$

Therefore we have the following;

1) $\delta(\{k \in N : X_k < X_0 + \epsilon\}) = 1$. It means that $X_0 - \epsilon \in \overline{B}_X$ and that $st - limsupX = inf\overline{B}_X = \mu \leq X_0) + \epsilon$.

2) $\delta(\{k \in N : X_k > X_0 - \epsilon\}) = 1$. It means that $X_0 - \epsilon \in \overline{A}_X$ and that $st - liminfX = sup\overline{A}_X = \nu \geq X_0) - \epsilon$.

From 1) and 2), we have $X_0 - \epsilon \leq X_k \leq X_0 + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\nu = \mu = X_0$. □

Theorem 5.7. Let $X = (X_k)$ be a sequence of fuzzy numbers and $st - liminfX_k = st - limsupX_k = \mu$. Suppose that there is an ϵ^0 such that for each $\epsilon \in (0, \epsilon^0)$,

$$\delta(\{k \in N : X_k \not> \nu + \epsilon\}) = 0 \text{ and } \delta(\{k \in N : X_k \not< \nu - \epsilon\}) = 0.$$

Then we have $st - limX_k = \mu$.

Proof. Let $\epsilon \in (0, \epsilon^0)$ be given. Then $\delta(\{k \in N : X_k < \mu - \epsilon\}) = 0$ because $st - liminfX_k = \mu$. Similarly we have $\delta(\{k \in N : X_k > \mu + \epsilon\}) = 0$ since $st - limsupX_k = \mu$. From these two equalities and from the hypothesis

$$\delta(\{k \in N : X_k \not> \nu + \epsilon\}) = 0 \text{ and } \delta(\{k \in N : X_k \not< \nu - \epsilon\}) = 0.$$

We have

$$\delta(K_1(\epsilon)) = 1 \text{ and } \delta(K_2(\epsilon)) = 1$$

with

$$K_1 = \{k \in N : X_k \geq \mu - \epsilon\} \text{and} K_2 = \{k \in N : X_k \leq \mu + \epsilon\}.$$

Now

$$\begin{aligned} K_1(\epsilon) \cap K_2(\epsilon) &= \{k \in N : \mu - \epsilon \leq X_k \leq \mu + \epsilon\} \\ &= \{k \in N : \overline{d}(X_k, \mu) \leq \epsilon\}. \end{aligned}$$

Therefore we have $\delta(\{k \in N : \overline{d}(X_k, \mu) \le \epsilon\}) = 1$ and hence

$$\delta(\{k \in N : \overline{d}(X_k, \mu) > \epsilon\}) = 0.$$

Since ϵ is arbitrary, we have $st - lim X_k = \mu$. □

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