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ON GROUND STATE SOLUTIONS FOR SINGULAR QUASILINEAR ELLIPTIC EQUATIONS†

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Abstract. In this paper, our main purpose is to establish the existence of positive bounded entire solutions of second order quasilinear elliptic equation on $\mathbf{R}^N$. we obtained the results under different suitable conditions on the locally Hölder continuous nonlinearity $f(x,u)$, we needn't any monotonicity condition about the nonlinearity.

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1. Introduction

In this paper we consider some new results concerning the existence of solution for nonlinear quasilinear problem of the type

$$\begin{cases} -\triangle_p u = \lambda f(x,u), \quad x \in \mathbf{R}^N \\ u > 0, \quad x \in \mathbf{R}^N \\ u(x) \to 0, \quad |x| \to \infty \end{cases} \tag{1.1}$$

where $f : \mathbf{R}^N \times (0,\infty) \to R$ is a locally Hölder continuous function which maybe singular at $t = 0$ and $\lambda > 0$ is a real parameter, and $\triangle_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, where we needn't $f(x,u)$ always nonnegative.

A function u satisfy problem (1.1) is called ground state solution. A class of problem (1.1) appears in many nonlinear phenomena, such as in the theory of quasiregular and quasiconformal mappings or in the study of non-Newtonian fluids[1-3]. These kind of problems have been studied extensively for when $\mathbf{R}^N$ is replaced by a smooth bounded domain Ω with zero boundary Dirichlet conditions. We refer the reader to the papers [4,5,,13] and the reference therein.

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In recent years the study of ground state solutions have received a lot of interests and numerous existence results have been established. When $p = 2$, we cite the papers [6-10] and reference therein. In most of these investigations, the focus has been on separable nonlinearities with $f(x, s) = b(x)g(s)$. More specially consider the problem

$$\begin{cases} -\triangle u = b(x)g(u), \quad x \in \mathbf{R}^N \\ u > 0, \quad x \in \mathbf{R}^N \\ u(x) \to 0, \quad |x| \to \infty \end{cases}$$

Usually they need $b(x) > 0$ is a locally Hölder continuous function on $\mathbf{R}^N$ and $\int_1^\infty t\bar{b}(t)dt < \infty$, where $\bar{b}(t) = \max_{\|x\|=t} b(x)$ for $t > 0$, and concerning g, there are other conditions, such as g is non-increasing, g is bounded in a neighborhood of ∞,$\lim_{t\to 0^+} g(t) = \infty$, $\lim_{t\to 0^+} \frac{g(t)}{t} = \infty$, etc. We refer the readers to [6,11,12] for details. In a recent result A.Mohammed [6] improve all earlier results above the existence of the ground state solution for problems like (1.1) when $p = 2$, and C.A.Santos [7] completes the principal result of Mahammed in [6]. When $p > 2$, we often need the following condition

$$H_\infty = \int_0^\infty (s^{1-N} \int_0^s t^{N-1}\widetilde{b}(t)dt)^{\frac{1}{p-1}} ds < \infty \tag{1.2}$$

where $\widetilde{b}(t) = \max_{|x|=t} b(x)$, $t > 0$. More recently, in [4] studied the following boundary value problem

$$\begin{cases} -\triangle_p u = \lambda g(u), \quad x \in \Omega \\ u(x) = 0 \quad x \in \partial\Omega, \end{cases} \tag{1.3}$$

they established the existence result when $\lambda \gg 1$. Our purpose in this paper is to extend all the above results to the problem (1.1) where f is not necessarily separable and needn't nonnegative. The paper is organized as follows, In section 2, we recall some facts that will be needed in the paper. In section 3, we give the proofs of the main results in this paper.

2. Preliminaries

First, we consider the following problem

$$\begin{cases} -\triangle_p u = \lambda f(x, u), \quad x \in \Omega \\ u(x) = 0 \quad x \in \partial\Omega \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary, $f : \mathbf{R}^N \times (0, \infty) \to R$ is a continuous function, we define $\underline{u} \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ to be a sub-solution of (2.1) of

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \le \lambda \int_\Omega f(x, \underline{u})h, \quad \forall h \in C_0^\infty(\Omega).\ h \ge 0$$

$$\underline{u} \le 0 \quad x \in \partial\Omega$$

and $\overline{u} \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ to be a super-solution of (2.1) of

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \geq \lambda \int_\Omega f(x, \overline{u}) h, \quad \forall h \in C_0^\infty(\Omega).\ h \geq 0$$

$$\overline{u} \geq 0 \quad x \in \partial\Omega.$$

Firstly, we give the following Lemmas

Lemma 2.1 (see[14]). Suppose there exist a sub-solution $\underline{u}$ and a super-solution $\overline{u}$ of (2.1) such that $\underline{u} \leq \overline{u}$, then there exists a solution u of (2.1) such that $\underline{u} \leq u \leq \overline{u}$.

We also consider the eigenvalue problem on a smooth bounded domain $\Omega \subseteq \mathbf{R}^N$:

$$\begin{cases} -\triangle_p u = \lambda \rho(x) |u|^{p-2} u, & x \in \Omega \\ u(x) = 0 \quad x \in \partial\Omega \end{cases} \tag{2.2}$$

where $\rho(x) \in C^\alpha(\overline{\Omega}, (0, \infty))$ for some $0 < \alpha < 1$. The first eigenvalue of the problem (2.2) will be denoted by λ_Ω. It is well known that the following result holds:

Lemma 2.2 (see[15]). Suppose that $\Omega_1 \subset \Omega_2$, and $\Omega_1 \neq \Omega_2$, then $\lambda_{\Omega_1} > \lambda_{\Omega_2}$ if both exist.

So there exists

$$\lambda_0 = \lim_{k \to \infty} \lambda_{B_k(0)} \in [0, \infty)$$

where $B_k(0)$ is the ball centered at the origin and radius $k = 1, 2, \ldots$.

Remark 2.3. By Picone's identity (see[15]) we can obtain λ_0 is positive if (1.2) is satisfied with ρ in place of b.

The non-linearity f in problem (1.1) is assumed to be a real function that satisfies the following conditions.

($f1$) $f(x, s)$ is locally Hölder continuous on $\mathbf{R}^N \times [0, \infty)$ and continuously differentiable in the variable s.

($f2$) there is a positive constant $r_0 > 0$ such that $f(x, s)(r_0 - s) > 0$, $s \neq r_0$ (a falling zero).

($f3$) $f(x, s) \geq a(x)h(s)$, $(x, s) \in \mathbf{R}^N \times [0, s_0)$ for some $0 < s_0 \leq r_0$. $a : \mathbf{R}^N \to (0, \infty)$ be continuous function, $h : [0, s_0) \to (0, \infty)$ continuous function and $\liminf_{s \to 0} \frac{h(s)}{s^{p-1}} = h_0 > 0$.

Secondly, we have the following result

Theorem 2.4. Suppose $\Omega \subseteq \mathbf{R}^N$ be a bounded domain and condition ($f1$), ($f2$), ($f3$) hold, then problem (2.1) has a positive solution $u \in C^1(\mathbf{R}^N)$ for any $\lambda \geq \frac{\lambda_\Omega}{h_0}$.

Proof. since $\lambda \geq \frac{\lambda_\Omega}{h_0}$ and ($f3$) hold, there is some $0 < s_0 \leq r_0$, such that

$$\lambda h(s) \geq \lambda_\Omega s^{p-1}, \ \ o < s \leq s_0 \tag{2.3}$$

where $r_0 > 0$ is given by ($f2$). we denote the corresponding eigenfunction by $\varphi_\Omega > 0$ to the first eigenvalue λ_Ω for problem (2.2) with $\rho(x)$ be replaced by

$a(x)$, normalized such that $\|\varphi_\Omega\|_\infty \le s_0$, then we have

$$-\triangle_p \varphi_\Omega = \lambda_\Omega a(x)\varphi_\Omega^{p-1} \le \lambda a(x)h(\varphi_\Omega) \le \lambda f(x,\varphi_\Omega)$$

thus φ_Ω is a sub-solution of (2.1), obviously r_0 is a super-solution of (2.1) and $\varphi_\Omega \le r_0$ in Ω, by Lemma 2.1, there is a positive solution u such that $\varphi_\Omega \le u \le r_0$. □

Remark 2.5. Under assumption $(f2)$, every constant $r > r_0$ is a supper-solution of (2.1).

3. Main Result

We give the proof of our main results in this section.

Theorem 3.1. Suppose condition $(f1),(f2),(f3)$ and (1.2) holds with $b(x)$ replaced by $a(x)$, then problem (1.1) has a positive solution $u \in C^1(\mathbf{R}^N)$ for any $\lambda > \frac{\lambda_0}{h_0}$.

To prove theorem 3.1 we need another result:

Lemma 3.2 (see [17]) (Diaz-Saa's Inequality). Let $\Omega \in \mathbf{R}^N$ be an open set, for $i=1,2$, let $\omega_i \in L^\infty(\Omega)\cap W^{1,p}(\Omega)$ such that $\omega_i \ge 0$ a.e in Ω, $\triangle_p \omega_i^{\frac{1}{p}} \in L^\infty(\Omega)$ and $\omega_1 = \omega_2$ on $\partial\Omega$. Then

$$\int_\Omega \Big(\frac{-\triangle_p \omega_1^{\frac{1}{p}}}{\omega_1^{\frac{p-1}{p}}} - \frac{-\triangle_p \omega_2^{\frac{1}{p}}}{\omega_2^{\frac{p-1}{p}}}\Big)(\omega_1-\omega_2) \ge 0$$

if $(\frac{\omega_i}{\omega_j}) \in L^\infty(\Omega)$ for $i \ne j, i,j=1,2$. Additionally, the above equality occurs if and only if $\omega_1 = t\omega_2$ for some $t \in (0,\infty)$.

Proof of Theorem 3.1. Since $\lambda > \frac{\lambda_0}{h_0}$, we can take $k_0 > 0$ large enough such that $\lambda \ge \frac{\lambda_{B_{k_0}(0)}}{h_0}$, where $B_{k_0}(0)$ is the ball centered at the origin and radius k_0, form Theorem 2.4, we may assume u_k be the solutions of problem (2.1) with $\Omega = B_k(0), k = k_0+1, k_0+2, \dots,$

We claim that:

$$\varphi_{B_j} \le u_k \le r_0,\ x \in B_j(0),\ \forall k > j \ge k_0$$

where $\varphi_{B_j} = \varphi_{B_j(0)}$ with $\|\varphi_{B_j}\| \le s_0$ is given by $(f3)$. In fact the second inequality is always hold. suppose, by contradiction, there exist $j_1, k_1 \ge k_0$ with $j_1 < k_1$ and $x_0 \in B_{j_1}(0)$ such that $u_{k_1}(x_0) < \varphi_{Bj_1}(x_0)$. Taking

$$\emptyset \ne \Omega_{k_1} = \{x \in B_{j_1}(0) : u_{k_1}(x) < \varphi_{Bj_1}(x)\} \subset\subset B_{j_1}(0)$$

and applying Lemma 3.2, we get

$$\begin{aligned} 0 &\le \int_{\Omega_{k_1}} \Big(\frac{\triangle_p \varphi_{B_{j_1}}}{\varphi_{B_{j_1}}^{p-1}} - \frac{\triangle_p u_{k_1}}{u_{k_1}^{p-1}}\Big)(u_{k_1}^p - \varphi_{Bj_1}^p) \\ &= \int_{\Omega_{k_1}} \Big(\lambda \frac{f(x,u_{k_1})}{u_{k_1}^{p-1}} - \lambda_{B_{j_1}} a(x)\Big)(u_{k_1}^p - \varphi_{Bj_1}^p) \end{aligned}$$

from $(f3)$ and (2.3) we obtain

$$\int_{\Omega_{k_1}} \left(\frac{\triangle_p \varphi_{B_{j_1}}}{\varphi_{B_{j_1}}^{p-1}} - \frac{\triangle_p u_{k_1}}{u_{k_1}^{p-1}}\right)(u_{k_1}^p - \varphi_{Bj_1}^p) = 0$$

then by Lemma 3.2 we have $u_{k_1} = t\varphi_{Bj_1}$, $x \in \Omega_{k_1} \subset\subset B_{j_1}(0)$, for some $t \in (0, \infty)$. But this is impossible because $u_{k_1} = \varphi_{Bj_1}$, on $\partial\Omega_{k_1}$, thus proves the claim.

Following the standard arguments (see [16] for details) we obtain a subsequence of $\{u_k\}_{k=k_0}^{\infty}$ that converges uniformly. By a diagonalization process that $\{u_k\}$ has a subsequence that converges uniformly on open bounded subsets of $\mathbf{R}^N$ to $u \in C^{1,\alpha}_{loc}(\mathbf{R}^N)$, that's u is a solution of (1.1). □

If we haven't the condition $\liminf_{s\to 0} \frac{h(s)}{s^{p-1}} = h_0 > 0$ in $(f3)$, that's $(f3)$ turn to be:

$(f'3)$ $f(x, s) \geq a(x)h(s)$, $(x, s) \in \mathbf{R}^N \times [0, s_0)$ for some $0 < s_0 \leq r_0$. $a : \mathbf{R}^N \to (0, \infty)$ be continuous function, $h : [0, s_0) \to (0, \infty)$ continuous function.

Then we have the following result:

Theorem 3.3. Suppose condition $(f1), (f2), (f'3)$ and (1.2) holds with $b(x)$ replaced by $a(x)$, then problem (1.1) has a positive solution $u \in C^1(\mathbf{R}^N)$ for any $\lambda > 0$ large enough.

Proof. under the condition (1.2) the following problem has a positive ground state solution.

$$\begin{cases} -\triangle_p u = a(x), & x \in \mathbf{R}^N \\ u > 0, & x \in \mathbf{R}^N \\ u(x) \to 0, & |x| \to \infty \end{cases} \tag{3.1}$$

In fact we can set

$$V(x) = \int_{|x|}^{\infty} \left(\frac{1}{s^{N-1}} \int_0^s \sigma^{N-1}\psi(\sigma)d\sigma\right)^{\frac{1}{p-1}} ds$$

which is a solution for the $-\mathrm{div}(|\nabla V|^{p-2}\nabla V) = a(r)$ in $\mathbf{R}^N$ and $\lim_{|x|\to\infty} V(x) = 0$, so V is a upper solution for (3.1). On the other hand, 0 is a lower solution for (3.1), then (3.1) exists bounded entire solution φ such that $0 < \varphi \leq V(x)$.

Set $\underline{u} = \sigma\varphi$, $\sigma \in (0, \frac{r_0}{\|\varphi\|_\infty})$, then we have

$$-\triangle_p \underline{u} == \sigma^{p-1} a(x) \leq \lambda a(x) \min_{[0,r_0]} h(s)$$

for $\lambda > 0$ large enough. Then

$$-\triangle_p \underline{u} \leq \lambda a(x) \min_{[0,r_0]} h(\sigma\varphi(x)) \leq \lambda f(x, \sigma\varphi(x))$$

then φ is a sub-solution of (1.1), obviously r_0 is a super-solution of (1.1) and $\varphi \leq r_0$ in $\mathbf{R}^N$, by Lemma 2.1(or see [16] theorem 1), there is a positive solution u such that $\varphi \leq u \leq r_0$. □

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