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# RIGHT SEMIDIRECT SUMS IN NEAR-RINGS

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Abstract. In this paper, we begin with some basic concepts of substructures of near-rings, and then using some right substructures of near-rings, we may define the right semidirect sum of near-rings.

Next, we investigate that every near-ring can be decomposed with right semidirect sum of right ideal by right $R$-subgroup, and then give some examples.

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## 1. Introduction

Throughout this paper, a (left) near-ring $R$ is an algebraic system $(R, +, \cdot)$ with two binary operations, say $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b+c) = ab+ac$ for all $a, b, c$ in $R$. We note that obviously, $a0 = 0$ and $a(-b) = -ab$ for all $a, b$ in $R$, but in general, $0a \neq 0$ and $(-a)b \neq -ab$.

If a near-ring $R$ has a unity (or identity) 1, then $R$ is called *unitary*. An element $d$ in $R$ is called *distributive* if $(a + b)d = ad + bd$ for all $a$ and $b$ in $R$.

We consider the following substructures of near-rings: Given a near-ring $R$, $R_0 = \{a \in R \mid 0a = 0\}$ which is called the *zero symmetric part* of $R$,

$$R_c = \{a \in R \mid 0a = a\} = \{a \in R \mid ra = a,\ for\ all\ r \in R\} = \{0a \in R \mid a \in R\}$$

which is called the *constant part* of $R$, and $R_d = \{a \in R \mid a\ is\ distributive\}$ which is called the *distributive part* of $R$.

A non-empty subset $S$ of a near-ring $R$ is said to be a *subnear-ring* of $R$, if $S$ is a near-ring under the operations of $R$, equivalently, for all $a, b$ in $S$, $a - b \in S$ and $ab \in S$. Sometimes, we denote it by $S < R$.

We note that $R_0$ and $R_c$ are subnear-rings of $R$, but $R_d$ is not a subnear-ring of $R$. A near-ring $R$ with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is

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said to be *zero symmetric,* also, in case $R = R_c$, $R$ is called a *constant* near-ring, and in case $R = R_d$, $R$ is called a *distributive* near-ring.

Let $(G, +)$ be any group (not necessarily abelian). Then we may introduce simple example of near-rings as following:

First, if we define multiplication on $G$ as $xy = y$ for all $x, y$ in $G$, then $(G, +, \cdot)$ is a near-ring, because $(xy)z = z = x(yz)$ and $x(y+z) = y+z = xy+xz$, for all $x, y, z$ in $G$, but in general, $0x = 0$ and $(x+y)z = xz + yz$ are not true. These kinds of near-rings are constant near-rings.

Next, in the set

$$M(G) = \{f \mid f : G \longrightarrow G\}$$

of all the self maps of $G$, if we define the sum $f + g$ of any two mappings $f, g$ in $M(G)$ by the rule $x(f + g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* on the group $G$. Also, we can define the substructures of $(M(G), +, \cdot)$ as following: $M_0(G) = \{f \in M(G) \mid 0f = 0\}$ and $M_c(G) = \{f \in M(G) \mid f \textit{ is constant}\}$, then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

For the remainder basic concepts and results on near-rings, we can refer to G. Pilz [5].

## 2. Some results of right substructures in near-rings

An *ideal* of $R$ is a subset $I$ of $R$ such that (i) $(I,\ +)$ is a normal subgroup of $(R,\ +)$, (ii) $a(I + b) - ab \subset I$ for all $a, b \in R$, equivalently, $aI \subset I$ for all $a \in R$, (iii) $(I + a)b - ab \subset I$ for all $a, b \in R$. If $I$ satisfies (i) and (ii) then it is called a *left ideal* of $R$. If $I$ satisfies (i) and (iii) then it is called a *right ideal* of $R$.

On the other hand, an *R-subgroup* of $R$ is a subset $H$ of $R$ such that (i) $(H,\ +)$ is a subgroup of $(R,\ +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If $H$ satisfies (i) and (ii) then it is called a *left R-subgroup* of $R$. If $H$ satisfies (i) and (iii) then it is called a *right R-subgroup* of $R$. In case, $(H,\ +)$ is normal in above, we say that *normal R-subgroup*, *normal left R-subgroup* and *normal right R-subgroup* instead of $R$-subgroup, left R-subgroup and right R-subgroup, respectively.

Now we can define a new kind of definition as following:

**Definition 2.1.** A near-ring $R$ is a right semidirect sum of substructure $N$ by substructure $K$ of $R$ if (i) $R = N + K$, (ii) $N \cap K = 0$, (iii) $N$ is a right ideal, and (vi) $K$ is a right $R$-subgroup. One calls that $K$ is the complement of $N$. Sometimes, We write it as $R = N \uplus K$.

**Lemma 2.2.** *([5] 1.13) Let $R$ be a near-ring. Then we have that* $(R,\ +) = (R_0,\ +) \oplus (R_c,\ +)$ *as additive subgroups.*

An element $e$ of a near-ring $R$ is called an *idempotent* if $e^2 = e$

For an element $x$ of a near-ring $R$, the (right) annihilator $x$ is of the form

$$Ann(x) = \{a \in R \mid xa = 0\}$$

Also, for any nonempty subset $X$ of a near-ring $R$, the (right) annihilator of $X$ is of the form

$$Ann(X) = \{a \in R \mid xa = 0, \forall x \in X\} = \cap_{x \in X} Ann(x)$$

**Theorem 2.3.** *For any element $x$ of a near-ring $R$, $Ann(x)$ is a right ideal of $R$. Moreover, if $X$ is a nonempty subset of a near-ring $R$, then $Ann(X)$ is a right ideal of $R$.*

*On the other hand, if $X$ is a right $R$-subgroup of $R$, then $Ann(X)$ is an ideal of $R$.*

*Proof.* Certainly, $0 \in Ann(x)$, because $x0 = 0$. Let $a,\ b \in Ann(x)$. Then $xa = 0$ and $xb = 0$, so $x(a - b) = xa - xb = 0 - 0 = 0$. Hence, $a - b \in Ann(x)$, and so $(Ann(x),\ +)$ is a subgroup of $(R,\ +)$.

Next, let $a \in Ann(x)$ and $r \in R$. Then since $xa = 0$,

$$x(r + a - r) = xr + xa - xr = xr + 0 - xr = 0$$

so $r + a - r \in Ann(x)$, and $(Ann(x),\ +)$ is a normal subgroup of the group $(R,\ +)$.

Finally, let $a \in Ann(x)$ and $r,\ s \in R$. From $xa = 0$, we obtain that

$$x[(a + r)s - rs] = (xa + xr)s - xrs = 0 + (xr)s - x(rs) = 0.$$

Consequently, $Ann(x)$ is a right ideal of $R$.

Moreover, from the definition of $Ann(X)$ and the fact that the intersection of a family of right ideals of $R$ is again a right ideal of $R$, we have that $Ann(X)$ is a right ideal of $R$.

On the other hand, for any $a \in Ann(x)$ and $r \in R$, since $X$ is right $R$-subgroup, $\forall x \in X,\ xa \in X$, thus we have that

$$x(ra) = (xr)a = x'a = 0.$$

Where $x' \in X$. Therefore, $Ann(X)$ is an ideal of $R$. □

We have the following property which is useful in the sequel.

**Theorem 2.4.** *If $e$ is any idempotent element of a near-ring $R$, then $eR = \{ea \mid a \in R\}$ is a right $R$-subgroup of $R$.*

*Proof.* Clearly, $eR$ is nonempty, because $0 = e0 \in eR$. For any $ea,\ eb \in eR$, $ea - eb = e(a - b) \in eR$, so $eR$ is a subgroup of $(R,\ +)$. Also, clearly $eR = \{ea \mid a \in R\}$ is a right $R$-subgroup of $R$. □

**Theorem 2.5.** *Let $e$ be an idempotent element of a near-ring $R$. Then the near-ring $R$ is a right semidirect sum of a right ideal $Ann(e)$ by a right $R$-subgroup $eR$.*

*Proof.* Certainly, $Ann(e) + eR \subset R$. Let $r \in R$. Consider that $r = r - er + er$. Then we see that $r - er \in Ann(e)$ and $er \in eR$. Hence, $Ann(e) + eR = R$.

Next, let $x \in Ann(e) \cap eR$. Then $ex = 0$ and $x = ea$ for some $a$ in $R$, and hence

$$x = ea = eea = ex = 0.$$

Consequently, we obtain that $Ann(e) \cap eR = \{0\}$.

Now, applying the theorems 2.3 and 2.4, our proof is complete. □

**Corollary 2.6.** *Let $R$ be a near-ring. Then the near-ring $R$ is a right semidirect sum of a substructure $R_0$ by a substructure $R_c$. That is, $R = R_0 \uplus R_c$.*

*Proof.* Since 0 is an idempotent in $R$, let $e = 0$, an idempotent. Then we can deduce easily that $Ann(e) = R_0$ and $eR = R_c$ by the definition of $R_c$. □

Since every element of a constant near-ring is a left identity, it is also an idempotent. Thus we have the following:

**Theorem 2.7.** *For any near-ring $(R, +, \cdot)$, every element of $R_c$ is an idempotent.*

**Definition 2.8.** From Corollary 2.6 and theorem 2.7, there are lots of examples of right semidirect sums of substructures of arbitrary near-ring $R$, since every element $a$ of $R_c$ is an idempotent, from theorem 2.5, each idempotent $a$ gives us a decomposition $R = Ann(a) \uplus aR$. Also, we have $M(G) = M_0(G) \uplus M_c(G)$.

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