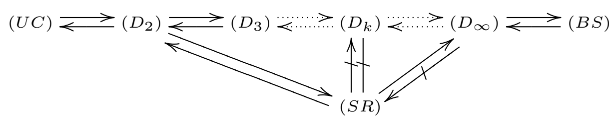
J. Appl. Math. & Informatics Vol. **29**(2011), No. 3 - 4, pp. 1001 - 1006
Website: http://www.kcam.biz

SUPERREFLEXIVITY AND PROPERTY (D_k) IN BANACH SPACES[†]

KYUGEUN CHO* AND CHONGSUNG LEE

Abstract. In this paper, we study the relation between superreflexivity (SR) and property (D_k) in Banach spaces and get the following diagrams.



AMS Mathematics Subject Classification : 46B20.
Key words and phrases : superreflexivity, property (D_k).

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By B_X, we denote the closed unit ball of X.

$(X, \|\cdot\|)$ is said to be uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x,\ y \in B_X$ with $\|x - y\| \geq \epsilon$,

$$\left\| \frac{1}{2}(x+y) \right\| \leq 1 - \delta.$$

A Banach space is said to have Banach-Saks property (BS) if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. S. Kakutani [5] showed that uniform convexity implies Banach-Saks property. T. Nishiura and D. Waterman [6] proved that Banach-Saks property implies reflexivity in Banach spaces.

For each positive integer $k \geq 2$, a Banach space $(X, \|\cdot\|)$ is said to have property (D_k), where $k \geq 2$ if it is reflexive and there exists a number α, $0 <$

Received September 13, 2010. Revised December 2, 2010. Accepted December 4, 2010. *Corresponding author. †This work was supported by the research grant of the Inha University.

$\alpha < 1$, such that for a weakly null sequence (x_n) in B_X, there exist $n_1 < n_2 < \cdots < n_k$ with

$$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$

We say that a Banach space X has property (D_∞) if it has (D_k) for some $k \in \mathbb{N}$. We can found the following strict implications in [3].

$$\text{(UC)} \Rightarrow (D_2) \Rightarrow (D_3) \Rightarrow \cdots \Rightarrow (D_\infty) \Rightarrow \text{(BS)}.$$

A Banach space Y is said to be finitely representable in a Banach space X if for every $\epsilon > 0$ and for every finite-dimensional subspace F of Y there exists an isomorphism T from F into X satisfying

$$(1-\epsilon)\|y\| \leq \|Ty\| \leq (1+\epsilon)\|y\|.$$

A Banach space X is said to be superreflexive (SR) if every Banach space Y finitely representable in X is reflexive. We shall say that a Banach space is uniformly convexifiable if it is isomorphic to a uniformly convex space, that is, if it can be endowed with an equivalent uniformly convex norm. It is well known that superreflexivity and uniform convexifiability are equivalent [1].

2. Main Results

The following is well known and found in [5].

Lemma 1. *If a Banach space X is uniformly convex, then there exists $0 < \theta < 1$ such that for a weakly null sequence (x_n) and $\|x_n\| \leq K$,*

$$\left\| \frac{x_{m_{2n-1}} + x_{m_{2n}}}{2} \right\| < \theta K,$$

for some subsequence (x_{m_n}) of (x_n).

Using Lemma 1, we get the following proposition.

Proposition 2. *Superreflexive Banach spaces have property (D_∞).*

Proof. Let $(X, \|\cdot\|)$ be a superreflexive Banach space. Then there exists uniformly convex norm $|\cdot|$ such that $m\|x\| \leq |x| \leq M\|x\|$, for all $x \in X$. Suppose that (x_n) is a weakly null sequence in $B_{(X,\|\cdot\|)}$. Then (x_n) is weakly null sequence in $(X, |\cdot|)$ and $|x_n| \leq M$. Since uniformly convexity implies property D_2 [2,3], there exists $0 < \alpha < 1$ such that for a weakly null sequence $\left(\frac{x_n}{M}\right)_{n \geq 1}$ in $B_{(X,|\cdot|)}$, there exist $n_1 < n_2$ with

$$\left| \frac{1}{2} \left(\frac{x_{n_1}}{M} - \frac{x_{n_2}}{M} \right) \right| \leq \alpha,$$

for a weakly null sequence $\left(\frac{x_n}{M}\right)_{n \geq n_2+1}$ in $B_{(X,|\cdot|)}$, there exist $n_3 < n_4$ with

$$\left| \frac{1}{2} \left(\frac{x_{n_3}}{M} - \frac{x_{n_4}}{M} \right) \right| \leq \alpha,$$

$$\vdots$$

Continuing this process, we get a subsequence (x_{n_i}) of (x_n) with

$$|x_{n_{2i-1}} - x_{n_{2i}}| \leq 2M\alpha, \quad \text{for all } i \in \mathbb{N}.$$

Let (x_m^1) be the sequence defined by

$$2x_m^1 = x_{n_{2m-1}} - x_{n_{2m}}.$$

Then (x_m^1) is weakly null and $|x_m^1| \leq M\alpha$. By Lemma 1, there exists a subsequence of (x_m^1) (which we still call (x_m^1)) such that

$$|x_{2m-1}^1 + x_{2m}^1| \leq 2\theta M\alpha,$$

for some $0 < \theta < 1$ (which is not dependent on (x_m^1)).

Let (x_m^2) be the sequence given by

$$2x_m^2 = x_{2m-1}^1 + x_{2m}^1.$$

Then (x_m^2) is weakly null and $|x_n^2| \leq \theta M\alpha$. By Lemma 1, there exists a subsequence of (x_m^2) (which we still call (x_m^2)) such that

$$|x_{2m-1}^2 + x_{2m}^2| \leq 2\theta^2 M\alpha,$$

for some $0 < \theta < 1$ (which is not dependent on (x_m^2)). Continue this process, we get a subsequence (x_m^k) of (x_n) such that

$$|x_{2m-1}^k + x_{2m}^k| \leq 2\theta^k M\alpha,$$

for all $k \in \mathbb{N}$. For a sufficiently large $N \in \mathbb{N}$, choose $\delta > 0$ such that

$$2\theta^N \frac{M}{m}\alpha < 1 - \delta.$$

Since

$$|x_1^N + x_2^N| \leq 2\theta^N M\alpha < 2m(1-\delta)$$

and

$$\begin{aligned}
x_1^N + x_2^N &= \frac{1}{2}(x_1^{N-1} + x_2^{N-1}) + \frac{1}{2}(x_3^{N-1} + x_4^{N-1}) \\
&= \frac{1}{4}(x_1^{N-2} + x_2^{N-2} + \cdots + x_8^{N-2}) \\
&= \frac{1}{2^{N-1}}(x_1^1 + x_2^1 + \cdots + x_{2^N}^1) \\
&\ \vdots \\
&= \frac{1}{2^N}(x_{n_1} - x_{n_2} + x_{n_3} - x_{n_4} + \cdots + x_{2^{N+1}-1} - x_{2^{N+1}}),
\end{aligned}$$

$$\left| \frac{1}{2^{N+1}} \sum_{k=1}^{2^{N+1}} (-1)^{k+1} x_{n_k} \right| < (1-\delta)m$$

and

$$\left\| \frac{1}{2^{N+1}} \sum_{k=1}^{2^{N+1}} (-1)^{k+1} x_{n_k} \right\| < 1 - \delta.$$

Since N and δ depend only on X, it follows that X has property $(D_{2^{N+1}})$, hence (D_∞). □

The following is the example with property (D_{k+1}) which dose not have property (D_k) [3].

Example 3. For $x = (a_n) \in l_2$, we define a norm $\|x\|_{(k)}$ by

$$\|x\|_{(k)} = \left(\sup_{n_1 < n_2 < \cdots < n_k} \left(\sum_{i=1}^{k} |a_{n_i}| \right)^2 + \sum_{n \neq n_1, n_2 \cdots, n_k} |a_n|^2 \right)^{\frac{1}{2}}.$$

Then $\|x\|_2 \leq \|x\|_{(k)} \leq \sqrt{k}\|x\|_2$. Let $X_k = (l_2, \|\cdot\|_{(k)})$.

Since X_k is isomorphic to l_2, it is clear that X_k is superreflexive. Since X_k has no property (D_k), we get the following proposition.

Proposition 4. *Superreflexivity dose not implies property* (D_k).

It is natural to consider the converse of Proposition 2 and 4. We investigate the question whether property D_k or D_∞ are superreflexive or not.

Proposition 5. *Let Y be a Banach space with basis (e_n) and with norm such that for $0 \leq |a_n| \leq |b_n|$,*

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| \leq \left\| \sum_{n=1}^{\infty} b_n e_n \right\|.$$

Let (X_n) be a family of finite dimensional spaces, let

$$Z = \left\{ x = (x_n) \in \prod_{n=1}^{\infty} X_n : \sum_{n=1}^{\infty} \|x_n\| e_n \in Y \right\},$$

and let Z have the norm

$$\|x\| = \left\| \sum_{n=1}^{\infty} \|x_n\| e_n \right\|.$$

If Y is uniformly convex, then Z has property (D_2).

Proof. Let $(z^{(i)}) = ((z_n^{(i)}))$ be a weakly null sequence in B_Z. Then $(z_n^{(i)})$ is weakly null in X_n as $i \to \infty$, for each $n \in \mathbb{N}$. Since X_n is finite dimensional, $(z_n^{(i)})$ is norm null in X_n as $i \to \infty$, for each $n \in \mathbb{N}$.

Let $x_i = \sum_{n=1}^{\infty} \|z_n^{(i)}\| e_n$. Then $\|x_i\| = \|z^{(i)}\| \le 1$. Since uniform convexity implies reflexivity, there exists a weakly convergent subsequence of (x_i) (which we still call (x_i)), say $x_i \to x = \sum a_n e_n$ weakly in Y.

For $n \in \mathbb{N}$, $a_n = e_n^*(x) = \lim_{i\to\infty} e_n^*(x_i) = \lim_{i\to\infty} \|z_n^{(i)}\| = 0$. This implies that (x_i) is weakly null in Y. By Lemma 1, there exists $\alpha < 1$ such that

$$2\alpha \ge \|x_{i_{m_1}} + x_{i_{m_2}}\| = \left\| \sum_{n=1}^{\infty} \left(\left\| z_n^{(i_{m_1})} \right\| + \left\| z_n^{(i_{m_2})} \right\| \right) e_n \right\|$$

for some $i_{m_1} < i_{m_2}$. Thus,

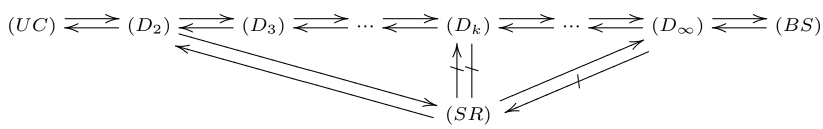
$$\begin{aligned} \|z^{(i_{m_1})} - z^{(i_{m_2})}\| &= \left\| \sum_{n=1}^{\infty} \left\| z_n^{(i_{m_1})} - z_n^{(i_{m_2})} \right\| e_n \right\| \\ &\le \left\| \sum_{n=1}^{\infty} \left(\left\| z_n^{(i_{m_1})} \right\| + \left\| z_n^{(i_{m_2})} \right\| \right) e_n \right\| \\ &\le 2\alpha. \end{aligned}$$

This means that Z has property (D_2). □

It is well known that $\left(\prod_{n\ge1} l_\infty^n\right)_{l_2}$ is not superreflexive but reflexive [1]. $\left(\prod_{n\ge1} l_\infty^n\right)_{l_2}$ has property (D_2), by Proposition 5. We then get the following corollary.

Corollary 6. *Property (D_2) does not imply superreflexivity.*

By Proposition 2, Proposition 4, Corollary 6 and [3], we get the following diagrams;



References

1. B. Beauzamy, *Introduction to Banach spaces and their geometry*, North-Holland mathematics studies 86, 1985.
2. K. G. Cho and C. S. Lee, *Some convex properties in Banach spaces*, J. Appl. Math. & Informatics **26** (2008), 407–416.

3. K. G. Cho and C. S. Lee, *Property (D_k) in Banach spaces*, J. Appl. Math. & Informatics **28** (2010), 1519–1525.
4. S. Guerre-Delabrière, *Classical Sequences in Banach spaces*, Monographs and textbooks in pure and applied mathematics 166, 1992.
5. S. Kakutani, *Weak convergence in uniformly convex spaces*, Tôhoku Math. J. **45** (1938), 188–193.
6. T. Nishiura and D. Waterman, *Reflexivity and summability*, Studia Math. **23** (1963), 53–57.

Kyugeun Cho received his MS and Ph.D in Mathematics from Inha University under the direction of Professor Chongsung Lee. He is currently a Professor of Bangmok College of Basic Studies, Myong Ji University. His research interests focus on Geometry of Banach spaces and their related topics.

Bangmok College of Basic Studies, Myong Ji University, Yong-In 449-728, Korea.
e-mail: kgjo@mju.ac.kr

Chongsung Lee received his MS in Mathematics from Seoul National University. Also he received Ph.D in Mathematics from University of Illinois at Urbana-Champaign. He is a Professor of Department of Mathematics education, Inha University. His research interests focus on Measure theory, Geometry of Banach spaces.

Department of Mathematics education, Inha University, Inchon 402-751, Korea.
e-mail: cslee@inha.ac.kr