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OPTIMAL PARTIAL HEDGING USING COHERENT MEASURE OF RISK†

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ABSTRACT. We show how the dynamic optimization problem with the capital constraint can be reduced to the problem to find an optimal modified claim $\tilde{\psi}H$ where $\tilde{\psi}$ is a randomized test in the static problem. Coherent risk measure is used as risk measure in the L^∞ random variable spaces. The paper is written in expository style to some degree. We use an average risk of measure(AVaR), which is a special coherent risk measure, to see how to hedge the modified claim in a complete market model.

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1. Introduction

We consider an agent or an investor who sell a contingent claim and want to get rid of the associated shortfall risk by means of a dynamic hedging strategy. The shortfall risk is the difference between the payoff of the contingent claim and the value of the agent's or the investor's hedging strategy at maturity. It is known that there is a dynamic self-financing hedging strategy with arbitrage-free hedging price to super-replicate a contingent claim in complete or incomplete markets. The super-hedging price is the minimal initial capital that an agent or an investor has to invest to find a strategy which dominates the claim payoff with certainty [9]. The super-hedging price of a contingent claim is given by the supremum of the expected values over all equivalent martingale measures. If an agent or an investor sells the claim for the super-hedging price, then he/she could eliminate the shortfall risk completely by choosing a suitable hedging strategy. The corresponding value process is a supermartingale under equivalent martingale measures. The super-hedging strategy is determined by the optional decomposition [10]. The prices derived by super-replication are

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too high and not acceptable in practice. Then the claim should be sold for a price less than the super-hedging price. With the initial capital less than the super-hedging price, i.e., under the capital constraint an agent or an investor is unable to eliminate all exposed risk associated to the contingent claim completely and so wants to find optimal strategies which minimize the shortfall risk. The acceptable least price of the claim is given by the shortfall risk measure ρ with the investor's loss function and his/her threshold. In other words, he/she is seeking optimal partial hedging strategies with the initial capital less than the super-hedging price by taking some risks. Föllmer and Leukert [5] constructed a quantile hedging strategy which maximizes the probability of a successful hedge under the objective measure $\mathbb{P}$ under the capital constraint. In the quantile hedging approach, the size of the shortfall is not taken into account but only the probability of its occurrence. Föllmer and Leukert [6] also introduced optimal hedging strategies which minimize the shortfall risk under the capital constraint by using the expected loss functions as risk measures. In [6], the risk measure ρ is the form of $\rho(X) = \mathbb{E}^{\mathbb{P}}[\ell(X^+)]$, where X is a random variable on $(\Omega, \mathcal{F})$, $\mathbb{P}$ is a fixed probability measure on Ω, and $\ell : \mathbb{R} \to \mathbb{R}$ is a strictly convex function. Nakano [11] uses coherent risk measures as risk measures in the $L^1(\Omega, \mathcal{F}, \mathbb{P})$ random variable spaces instead of the loss function. Notice that the L^1 space is between $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $L^0(\Omega, \mathcal{F}, \mathbb{P})$. Arai [1] obtained robust representation results of shortfall risk measures on Orlicz hearts under the continuous time setting. The Orlicz hearts setting allows us to treat various loss functions and various claims in a unified framework. Coherent risk measure is introduced by Artzner et al. [2] as risk measures, and is extended to general probability spaces by Delbaen [3].

There is no explicit explanation of relations between the dynamic optimization problem and the static problem in the literature. In this paper, we show that the dynamic optimization problem with the capital constraint can be reduced to the problem to find an optimal modified claim $\tilde{\psi}H$ where $\tilde{\psi}$ is a randomized test in the static problem. Coherent risk measure is used as risk measure in the L^∞ random spaces. Average risk of measure(AVaR), which is a special coherent risk measure, is used to see how to hedge the modified claim in a complete market model.

This paper is constructed as follows. The definition of a superhedging price is given in section 2. It is shown how the dynamic hedging problem can be reduced to the static problem in section 3. Optimal solution is found when the risk measure is given by the average value at risk in section 4. Optimal partial hedging in a complete market model is given in section 5.

2. Mathematical settings and superhedging

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability space. Let $S = (S_t)_{0\le t\le T}$ be an adapted positive process which is a semimartingale. It is assumed that the riskless interest rate is zero for simplicity and $\mathcal{M} = \{\mathbb{Q} \,|\, \mathbb{Q} \sim \mathbb{P},\ S$ is a local martingale under $\mathbb{Q}\} \neq \emptyset$ to avoid the arbitrage opportunities [4].

Definition 2.1. A *self-financing* strategy (x, ξ) is defined as an initial capital $x \ge 0$ and a predictable process ξ_t such that the value process (value of the current holdings)

$$X_t = x + \int_0^t \xi_u dS_u, \quad t \in [0, T]$$

is $\mathbb{P}$-a.s. well defined.

The self-financing strategy (x, ξ) is called *admissible* if the corresponding value process X_t satisfies

$$X_t = x + \int_0^t \xi_u \, dS_u \ge 0 \quad \forall t \in [0, T].$$

Define the admissible set $\mathcal{X}(\alpha)$ as

$$\mathcal{X}(\alpha) = \Big\{ (x, \xi) \,|\, (x, \xi) \text{ is an admissible strategy and } x \le \alpha \Big\}.$$

Definition 2.2. A contingent claim H is called *attainable* (or *replicable, redundant*) if there exists admissible strategy (x_0, ξ) such that

$$H = x_0 + \int_0^T \xi_u dS_u.$$

In discrete time,

$$H = x_0 + \sum_{t=1}^{T} \xi_t \cdot (S_t - S_{t-1}).$$

See the book [7] or the paper [4] for the following theorems (2.3), (2.4) and (2.5).

Theorem 2.3. *Any attainable claim H is integrable with respect to each equivalent martingale measure (or pricing measure),*

$$\mathbb{E}^{\mathbb{Q}}[H] < \infty \quad \forall \mathbb{Q} \in \mathcal{M}.$$

Moreover, $\forall \mathbb{Q} \in \mathcal{M}$

$$X_t = \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t] \quad \mathbb{Q} - a.s.$$

is a non-negative $\mathbb{Q}$-martingale.

Theorem 2.4. *The market model is arbitrage-free if and only if the $\mathcal{M}$ of all equivalent martingale measure is non-empty.*

Theorem 2.5. *Let $(X_t)_{t\geq 0}$ be a stochastic process. Then*

X is a value process of self-financing portfolio if and only if X is a local martingale with respect to all $\mathbb{Q} \in \mathcal{M}$.

X is a capital of wealth and consumption portfolio if and only if X is a supermartingale with respect to all $\mathbb{Q} \in \mathcal{M}$.

Lemma 2.6. *Let $H \geq 0$ be a $\mathcal{F}_T$-measurable contingent claim.Then there exists admissible strategy $(x_0, \xi) \in \mathcal{X}(\alpha)$ such that*

$$H \leq x_0 + \int_0^T \xi_u dS_u \quad \mathbb{P} - a.s. \tag{2.1}$$

if and only if

$$H \in \left\{ X \geq 0 \,\middle|\, X \text{ is } \mathcal{F}_T - measurable, \quad \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[X] \leq x_0 \right\} := W. \tag{2.2}$$

Proof. Suppose that $(x_0, \xi) \in \mathcal{X}(\alpha)$ is the admissible strategy satisfying (2.1). Let $\mathbb{Q} \in \mathcal{M}$. Since the value process $X_t = x_0 + \int_0^t \xi_u dS_u$ is $\mathbb{Q}$-martingale, we get

$$\mathbb{E}^{\mathbb{Q}}[H] \leq \mathbb{E}^{\mathbb{Q}}\Big[x_0 + \int_0^T \xi_u dS_u\Big] = x_0.$$

So $H \in W$.

Now let $H \in W$. Define X as

$$X_t := \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t].$$

Then $X_0 = \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[H] \leq x_0$ since $H \in W$. Since the process X_t is $\mathbb{Q}$-supermartingale, the optional decomposition theorem implies that there exists an admissible strategy (X_0, ξ) and an increasing optional process C_t with $C_0 = 0$ such that

$$X_t = X_0 + \int_0^t \xi_u dS_u - C_t \quad \mathbb{P} - a.s..$$

Hence we have

$$H = \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_T] = \mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{F}_T] \leq X_T = X_0 + \int_0^T \xi_u dS_u - C_T \leq X_0 + \int_0^T \xi_u dS_u.$$

The first equality holds since H is $\mathcal{F}_T$-measurable and the second one does by the definition of X_t. The first inequality holds since X_t is $\mathbb{Q}$-supermartingale. Hence we get the equation (2.1) by considering $X_0 \leq x_0$. □

Lemma (2.6) means that the pricing rule of H, i.e., $\mathbb{E}^{\mathbb{Q}}[H]$ is less than or equal to x_0 which is the initial capital of the admissible superhedging strategy (x_0, ξ) for H.

Definition 2.7. The superhedge price H_0 for H is defined as

$$H_0 = \inf \left\{ x \,\middle|\, \exists \text{ admissible strategy}(x,\xi) \text{ such that } H \leq x + \int_0^T \xi_u dS_u \quad \mathbb{P} - a.s. \right\}.$$

By the Lemma (2.6) we can see the superhedge price is $H_0 = \sup_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{Q}}[H]$. That is, H_0 is the smallest initial capital eliminating all shortfall risk. The seller of H can cover almost any possible obligation from the sale of H and thus eliminate completely the corresponding risk. The following example in the book [7] shows that the superhedge price of H is twice as the initial price of the underlying asset. So the hedging price of the seller is too high and can't be used in practice.

Example 2.8. Consider a single risky asset $S = S^1$ whose initial price is 1. Let the initial capital $S^0 \equiv 1$ and the riskless interest rate $r = 0$. Let S be a random variable of the distribution, i.e., S is $\mathbb{P} - a.s.$ integer-valued and

$$\mathbb{P}[S = k] = \frac{1}{2}\frac{e^{-2}2^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots.$$

If we take the risky asset price as $S_0 = 1$ at time 0, then $\mathbb{P}$ is a risk-neutral probability measure and there is no-arbitrage in the market model, since

$$\mathbb{E}^{\mathbb{P}}[S] = \frac{1}{2} \times 2 = 1,$$

and so

$$(1 + r)^T S_0 = \mathbb{E}^{\mathbb{P}}[S],$$

which shows that the risky asset price grows, on average, at the risk-free rate.

Let $H = (S - K)^+$ be a payoff function of a call option of underlying asset S. If we define

$$g_n(k) := 2\left(e^2 - \frac{2e^2}{n}\right) \cdot I_{\{0\}}(k) + (n-1)! \cdot 2^{2-n}e^2 \cdot I_{\{n\}}(k), \quad k = 0, 1, \ldots,$$

then $\mathbb{E}^{\mathbb{P}}[g_n(S)] = 1$ and the measure $\mathbb{Q}_n \in \mathcal{M}$ defined as

$$d\mathbb{Q}_n = g_n(S)d\mathbb{P}$$

satisfies

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_n}\left[(S - K)^+\right] &= 2\left(e^2 - \frac{2e^2}{n}\right)(0 - K)^+\mathbb{P}[S = 0] \\ &+ (n-1)! \cdot 2^{2-n}e^2 \cdot (n - K)^+ \cdot \mathbb{P}[S = n] = 2\left(1 - \frac{K}{n}\right)^+.\end{aligned}$$

Letting $n \to \infty$, we can see that the superhedge price is given by

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[H] = 2,$$

which is an arbitrage-free price of H, but is too expensive price which is twice higher than the initial price $S_0 = 1$.

When the seller is unwilling to invest the superhedge price in a hedging strategy, the seller looks for the optimal partial hedging strategy minimizing the problem

$$\min_{(x,\xi)\in\mathcal{X}(\alpha)} \left[\rho\Big(\Big(H - x - \int_0^\tau \xi_u dS_u\Big)^+\Big)\right] \tag{2.3}$$

with the initial capital constraint

$$0 < \alpha < H_0 = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[H]. \tag{2.4}$$

Here ρ in (2.3) is a coherent risk measure [2] defined in Definition (2.9). The traditional VaR does not satisfy the subadditivity, which means that if you add two portfolios together, then the total risk does not increase more than adding the two risks separately. Artzner et al. [2] introduced the risk measure coherent satisfying some axioms to cover some flaws in VaR.

Definition 2.9. A coherent risk measure $\rho : L^0(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \cup \{\infty\}$ is a mapping satisfying for X, $Y \in L^0(\Omega, \mathcal{F}, \mathbb{P})$

(1) $\rho(X+Y) \le \rho(X) + \rho(Y)$ (subadditivity),
(2) $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \ge 0$ (positive homogeneity),
(3) $\rho(X) \ge \rho(Y)$ if $X \le Y$ (monotonicity) ,
(4) $\rho(X+m) = \rho(X) - m$ for $m \in \mathbb{R}$ (cash invariance)

Artzner et al. [2] state axioms for acceptance set and define "the measure of risk of an unacceptable position, as the minimum extra capital, which, invested in the reference instrument, makes the future value of the modified position become acceptable." When $\rho(X)$ is positive, the number $\rho(X)$ can be thought of as the minimum extra cash the agent has to add to the risky position X, and invest in the reference instrument, to be allowed to proceed with his/her plans. When $\rho(X)$ is negative, the amount of cash $-\rho(X)$ can be withdrawn from the position or it can be received as restitution, as in the case of organized markets for financial futures [2].

The subadditivity and positive homogeneity can be relaxed to a weaker quantity, i.e., convexity

$$\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y) \quad \forall\lambda \in [0,1],$$

which means diversification should not increase the risk.

3. Reduction to the static problem

In this section, we show how the dynamic hedging problem can be reduced to the static problem. Assume that

$$0 < \alpha < H_0 = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[H].$$

Let ρ be a coherent measure of risk defined in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ throughout this paper.

Definition 3.1. An admissible strategy $(x^*, \xi^*) \in \mathcal{X}(\alpha)$ is called *robust-efficient* if it is the optimal solution:

$$(x^*, \xi^*) \in \arg\min_{(x,\xi)\in\mathcal{X}(\alpha)} \Big[\rho\Big(\Big(H - x - \int_0^\tau \xi_u dS_u\Big)^+\Big)\Big].$$

Definition 3.2. A $\mathcal{F}_\tau$-measurable random variable X^* is called *maxmin-optimal* if it is the optimal solution:

$$X^* \in \arg\min_{0\le X\le H, \mathbb{E}^{\mathbb{Q}}[X]\le\alpha, \mathbb{Q}\in\mathcal{M}} \rho(H - X). \tag{3.5}$$

Theorem 3.3. *Let $H \ge 0$ be a $\mathcal{F}_\tau$-measurable contingent claim. If the claim X^* with initial capital α is a maxmin-optimal solution, then the super-hedging strategy $(x^*, \xi^*) \in \mathcal{X}(\alpha)$ for the claim X^* is robust-efficient. Conversely, if $(\tilde{x}, \tilde{\xi})$ is a robust-efficient strategy, then the following claim*

$$\tilde{X} := \left(\tilde{x} + \int_0^\tau \tilde{\xi}_u dS_u\right) \wedge H \tag{3.6}$$

is maxmin-optimal.

Proof. Let the claim X^* with initial capital α be a maxmin-optimal solution. Then $0 \le X^* \le H$, $\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[X^*] \le \alpha$. The lemma (2.6) implies that there exist $(x^*, \xi^*) \in \mathcal{X}(\alpha)$ such that

$$X^* \le x^* + \int_0^\tau \xi_u^* dS_u, \quad \mathbb{P} - a.s.$$

For each $(x, \xi) \in \mathcal{X}(\alpha)$ define X as

$$X := \left(x + \int_0^\tau \xi_u dS_u\right) \wedge H.$$

Then X is $\mathcal{F}_\tau$-measurable, $0 \le X \le H$ and $\mathbb{E}^{\mathbb{Q}}[X] \le x \le \alpha$. Moreover, for each $(x, \xi) \in \mathcal{X}(\alpha)$ we have

$$\begin{aligned}\Big[\rho\Big(\Big(H - x - \int_0^\tau \xi_u dS_u\Big)^+\Big)\Big] \qquad &= \rho(H - X) \ge \rho(H - X^*) \\ &\ge \rho\Big(\Big(H - x^* - \int_0^\tau \xi_u^* dS_u\Big)^+\Big).\end{aligned}$$

Hence the admissible strategy $(x^*, \xi^*) \in \mathcal{X}(\alpha)$ is robust-efficient. Moreover, the minimal risk for the maxmin-optimal claim X^* with initial capital α is given by

$$\min_{(x,\xi)\in\mathcal{X}(\alpha)} \Big[\rho\Big(\Big(H - x - \int_0^\tau \xi_u dS_u\Big)^+\Big)\Big] = \rho(H - X^*).$$

Conversely, let an admissible strategy $(\tilde{x}, \tilde{\xi}) \in \mathcal{X}(\alpha)$ be robust-efficient. From the equation (3.6), we have

$$0 \le \tilde{X} \le H, \quad \mathbb{E}^{\mathbb{Q}}[\tilde{X}] \le \tilde{x}. \tag{3.7}$$

Since

$$\begin{aligned}\rho(H-\tilde{X}) &= \rho\Big(\Big(H-\tilde{x}-\int_0^\tau \tilde{\xi}_u dS_u\Big)^+\Big)\\ &\leq \rho\Big(\Big(H-x-\int_0^\tau \xi_u dS_u\Big)^+\Big), \quad (x,\xi)\in\mathcal{X}(\alpha),\end{aligned}$$

we have

$$\min_{(x,\xi)\in\mathcal{X}(\alpha)}\Big[\rho\Big(\Big(H-x-\int_0^\tau \xi_u dS_u\Big)^+\Big)\Big] = \rho(H-\tilde{X}). \tag{3.8}$$

The equations (3.7) and (3.8) imply that

$$\tilde{X}\in\arg\min_{0\leq X\leq H, \mathbb{E}^{\mathbb{Q}}[X]\leq\alpha, \mathbb{Q}\in\mathcal{M}}\rho(H-X).$$

Thus $\tilde{X}$ is maxmin-optimal. □

Notice that the Theorem(3.3) still holds if the risk measure ρ might be replaced by $\rho(X)=\sup_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}^{\mathbb{Q}}[\ell(X^+)]$, which is a convex measure of risk.

Define $\mathcal{R}$ and $\mathcal{R}_0$ as

$$\begin{aligned}\mathcal{R} &:= \{\psi \mid \psi:\Omega\to[0,1], \psi \text{ is } \mathcal{F}_T-\mathit{measurable}\},\\ \mathcal{R}_0 &:= \Big\{\psi\in\mathcal{R}\ \Big|\ \sup_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}^{\mathbb{Q}}[\psi H]\leq\alpha\Big\},\end{aligned}$$

respectively.

The Theorem (3.3) states that the optimal hedging strategy can be constructed as two steps. The first step is to find the maxmin-optimal solution X^* in the static problem (3.5) and the second step is to fit the terminal value X_T of an admissible strategy to the claim X^*.

Let X^* be a maxmin-optimal solution in the static problem (3.5) and $\tilde{X}:=H\wedge X^*$. Then we can conclude that $\tilde{X}$ is also the maxmin-optimal solution, since $0\leq\tilde{X}\leq H$, $\mathbb{E}^{\mathbb{Q}}[\tilde{X}]\leq\alpha$ and $H-\tilde{X}=H-H\wedge X^*=H-X^*$. So it may be assumed that $0\leq X^*\leq H$, or equivalently, that $X^*=H\psi^*$ for $\psi^*\in\mathcal{R}_0$. So the dynamic optimization problem (2.3) with the constraint (2.4) can be restated as two steps. The first one is to find an optimal modified claim $\tilde{\psi}H$ where $\tilde{\psi}$ is the solution of the static problem

$$\min_{\psi\in\mathcal{R}_0}\rho((1-\psi)H)=\rho((1-\tilde{\psi})H).$$

The second one is to find a superhedging strategy for the modified claim $\tilde{\psi}H$.

Lemma 3.4. *Let $(\xi_n)_{n\geq1}$ be a sequence in $L^0(\Omega,\mathcal{F},\mathbb{P})$ such that* $\sup_n|\xi_n|<+\infty$ *$\mathbb{P}$-a.s.. Then there exists a sequence of convex combinations*

$$\eta_n\in conv\{\xi_n,\xi_{n+1},\ldots\}$$

which converges $\mathbb{P}$-a.s. to some $\eta\in L^0(\Omega,\mathcal{F},\mathbb{P})$.

Definition 3.5. The risk measure $\rho : L^\infty \to \mathbb{R}$ is said to have the *Fatou property* if for any bounded sequence $(X_n)_{n\geq 0}$ which converges $\mathbb{P}$-a.s. to some X,

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

Proposition 3.6. *Let $\rho : L^\infty \to \mathbb{R}$ be a coherent risk measure which has the Fatou property. Then there exists a randomized test $\tilde{\psi} \in \mathcal{R}_0$ which is the optimal solution of the static problem:*

$$\min_{\psi\in\mathcal{R}_0} \rho((1-\psi)H) = \rho((1-\tilde{\psi})H). \tag{3.9}$$

Proof. The proof is similar to the one of Proposition 8.11 in [7]. Take $\psi_n \in \mathcal{R}_0$ such that

$$\rho((1-\psi_n)H) \searrow \inf_{\psi\in\mathcal{R}_0} \rho((1-\psi)H) \text{ as } n \to \infty.$$

By the lemma (3.4), there exist convex combinations $\tilde{\psi}_n \in conv\{\psi_n, \psi_{n+1}, \ldots\}$ and $\tilde{\psi} \in \mathcal{R}$ such that $\tilde{\psi}_n \to \tilde{\psi}$ $\mathbb{P}$-a.s as $n \to \infty$.

The sequence $\tilde{\psi}_n$ can be expressed as

$$\tilde{\psi}_n = \sum_{i=1}^m \lambda_{k_i}^n \psi_{k_i}, \qquad n \leq k_1 < \cdots < k_m, \quad \sum_{i=1}^m \lambda_{k_i}^n = 1, \quad \lambda_{k_i}^n \geq 0.$$

The Fatou property of ρ implies that

$$\rho((1-\tilde{\psi})H) \leq \liminf_{n\uparrow\infty} \rho((1-\tilde{\psi}_n)H). \tag{3.10}$$

The convexity of ρ implies that

$$\begin{aligned} \rho((1-\tilde{\psi}_n)H) &\leq \rho\Big(\sum_{i=1}^m \lambda_{k_i}^n (1-\psi_{k_i})H\Big) \leq \sum_{i=1}^m \lambda_{k_i}^n \rho((1-\psi_{k_i})H) \\ &\leq \sum_{i=1}^m \lambda_{k_i}^n \rho((1-\psi_n)H) = \rho((1-\psi_n)H). \end{aligned} \tag{3.11}$$

From the equations (3.10) and (3.11), we have

$$\rho((1-\tilde{\psi})H) \leq \inf_{\psi\in\mathcal{R}_0} \rho((1-\psi)H).$$

Moreover, Fatou's lemma implies that

$$\mathbb{E}^{\mathbb{Q}}[\tilde{\psi}H] \leq \liminf_{n\uparrow\infty} \mathbb{E}^{\mathbb{Q}}[\tilde{\psi}_n H] \leq \alpha \quad \text{ for all } \mathbb{Q} \in \mathcal{M}.$$

Therefore, $\tilde{\psi} \in \mathcal{R}_0$ and the proof is done. □

4. Optimal solution of AVaR

In this section, we examine the Proposition (3.6) by taking the coherent risk measure $\rho(X) = AVaR_\lambda(X)$. For $\lambda \in (0,1)$, a λ-*quantile* of a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is a real number q such that

$$\mathbb{P}[X < q] \leq \lambda \leq \mathbb{P}[X \leq q].$$

The *upper* and the *lower* quantiles functions of X are defined as

$$\begin{aligned} q_X^+(\lambda) &= \inf\{x \in \mathbb{R} \,|\, \mathbb{P}[X \le x] > \lambda\} = \sup\{x \in \mathbb{R} \,|\, \mathbb{P}[X < x] \le \lambda\}, \\ q_X^-(\lambda) &= \sup\{x \in \mathbb{R} \,|\, \mathbb{P}[X < x] < \lambda\} = \inf\{x \in \mathbb{R} \,|\, \mathbb{P}[X \le x] \ge \lambda\}, \end{aligned}$$

respectively.

Definition 4.1. The Average Value at Risk at level $\lambda \in (0,1]$ of a random variable $X \in L^1$ is defined as

$$AVaR_\lambda(X) := -\frac{1}{\lambda}\int_0^\lambda q_X(t)\,dt,$$

where $q_X(t)$ is a quantile function of X.

Theorem 4.2 ([7]). *For $\lambda \in (0,1]$ and $X \in L^\infty$, $AVaR_\lambda$ is a coherent risk measure which has Fatou property and is represented as*

$$AVaR_\lambda(X) = \max_{\mathbb{Q}\in\mathcal{Q}_\lambda} \mathbb{E}^{\mathbb{Q}}[-X], \tag{4.12}$$

where $\mathcal{Q}_\lambda$ is defined as

$$\mathcal{Q}_\lambda = \Big\{\mathbb{Q} << \mathbb{P} \,\Big|\, \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\lambda} \quad \mathbb{P} - a.s.\Big\}.$$

Remark 4.3. For $\lambda \in (0,1)$, the maximum in (4.12) is attained by the measure $\mathbb{Q}_0 \in \mathcal{Q}_\lambda$, whose density is given by

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \frac{1}{\lambda}(I_{\{X<q\}} + \kappa I_{\{X=q\}}), \tag{4.13}$$

where q is a λ-quantile of X, and where κ is defined as

$$\kappa = \begin{cases} 0 & \text{if } \mathbb{P}[X=q]=0, \\ \frac{\lambda-\mathbb{P}[X<q]}{\mathbb{P}[X=q]} & o.w. \end{cases}. \tag{4.14}$$

If $\mathcal{Q}_\lambda$ is a singleton set, then the average value at risk can be shown that $AVaR_\lambda(X) = \frac{1}{\lambda}\mathbb{E}[-X]$ (see the Proposition (4.6)). When the risk measure is given by $\rho(X) = \mathbb{E}[X]$, the static problem (3.9) is to maximize the expectation

$$\mathbb{E}[\psi H]$$

with the constraint

$$\sup_{\mathbb{P}^*\in\mathcal{M}} \mathbb{E}^{\mathbb{P}^*}[\psi H] \le \alpha, \qquad \psi \in \mathcal{R}.$$

Assume that $\mathbb{E}[H] > 0$. Define the equivalent measures $\mathbb{Q}$ and $\mathbb{Q}^*$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{H}{\mathbb{E}[H]} \quad \text{and} \quad \frac{d\mathbb{Q}^*}{d\mathbb{P}^*} = \frac{H}{\mathbb{E}^{\mathbb{P}^*}[H]}.$$

Then the static problem can be written as the problem of maximizing

$$\mathbb{E}^{\mathbb{Q}}[\psi]$$

with the constraint

$$\mathbb{E}^{\mathbb{Q}^*}[\psi] \le \frac{\alpha}{\mathbb{E}^{\mathbb{P}^*}[H]} := \tilde{\alpha} \quad \forall \mathbb{P}^* \in \mathcal{M}.$$

Let $\tilde{a}$ be the lower $(1-\tilde{\alpha})$-quantile of $\varphi = \frac{d\mathbb{Q}}{d\mathbb{Q}^*}$, i.e.,

$$\tilde{a} := q_\varphi^-(1-\tilde{\alpha}) = \inf\{a \,|\, \mathbb{Q}^*[\varphi > a] \le \tilde{\alpha}\}.$$

Let $\tilde{A} = \left\{\frac{d\mathbb{Q}}{d\mathbb{Q}^*} > \tilde{a}\right\}$.

Proposition 4.4 (Neyman-Pearson lemma)**.** *If $A \in \mathcal{F}$ is such that $\mathbb{Q}^*(A) \le \mathbb{Q}^*(\tilde{A})$, then $\mathbb{Q}(A) \le \mathbb{Q}(\tilde{A})$.*

Let $\tilde{\psi} = I_{\{\varphi > \tilde{a}\}}$. The Neyman-Pearson lemma says that if $\mathbb{E}^{\mathbb{Q}^*}[\psi] \le \mathbb{E}^{\mathbb{Q}^*}[\tilde{\psi}]$ for all $\psi \in \mathcal{R}$, then $\mathbb{E}^{\mathbb{Q}}[\psi] \le \mathbb{E}^{\mathbb{Q}}[\tilde{\psi}]$. I.e., $\max_{\psi\in\mathcal{R}} \mathbb{E}^{\mathbb{Q}}[\psi] = \mathbb{E}^{\mathbb{Q}}[\tilde{\psi}]$ under the constraint

$$\mathbb{E}^{\mathbb{Q}^*}[\psi] \le \tilde{\alpha}.$$

Thus $\tilde{\psi} = I_{\{\varphi > \tilde{a}\}}$ is the maximal solution of $\mathbb{E}^{\mathbb{Q}}[\psi]$.

So in the complete market case, i.e., $\mathcal{M} = \{\mathbb{P}^*\}$, by the Neyman-Pearson lemma, we can solve the problem explicitly. The following Proposition (4.5) can be attained by changing the expressions in terms of probability measures $\mathbb{Q}$ and $\mathbb{Q}^*$ into ones in terms of probability measures $\mathbb{P}$ and $\mathbb{P}^*$.

Proposition 4.5 ([6])**.** *Let $\varphi = \frac{d\mathbb{P}}{d\mathbb{P}^*}$. When the risk measure is given by $\rho(X) = \mathbb{E}[X]$ in the complete market model, the optimal randomized test $\tilde{\psi}_1 \in \mathcal{R}$ is given by*

$$\tilde{\psi}_1 = I_{\{\varphi > \tilde{a}\}} + \gamma I_{\{\varphi = \tilde{a}\}} \tag{4.15}$$

where

$$\tilde{a} = \inf\left\{a \,\middle|\, \mathbb{E}^{\mathbb{P}^*}[HI_{\{\varphi > a\}}] \le \alpha\right\}$$

and

$$\gamma = \begin{cases} \frac{\alpha - \mathbb{E}^{\mathbb{P}^*}[HI_{\{\varphi > \tilde{a}\}}]}{\mathbb{E}^{\mathbb{P}^*}[HI_{\{\varphi > \tilde{a}\}}]} & \text{if } \mathbb{P}^*[\{\varphi = \tilde{a}\} \cap \{H > 0\}] > 0, \\ c \in [0,1] \text{ arbitrarily} & \text{if } \mathbb{P}^*[\{\varphi = \tilde{a}\} \cap \{H > 0\}] = 0. \end{cases}$$

If $\mathbb{P}^*[\{\varphi = \tilde{a}\} \cap \{H > 0\}] = 0$, then $\tilde{\psi}_1$ becomes the function $I_{\{\varphi > \tilde{a}\}}$.

Proposition 4.6. *For $\lambda \in (0,1]$, let $X > 0$ be a contingent claim such that*

$$\mathbb{P}(X > 0) \le \lambda.$$

Then we have

$$AVaR_\lambda(-X) = \frac{1}{\lambda}\mathbb{E}[X] \quad \text{for } X \ge 0, \tag{4.16}$$

and the optimal solution $\tilde{\psi}$ to the problem

$$\inf_{\psi\in\mathcal{R}_0} AVaR_\lambda(-(1-\psi)H) = AVaR_\lambda(-(1-\tilde{\psi})H) = \frac{1}{\lambda}\mathbb{E}[(1-\tilde{\psi})H] \tag{4.17}$$

is of the form as in the equation (4.15).

Proof. Let $c = q_{-X}(\lambda)$. Then by the Remark (4.3) the maximum in (4.12) is attained by the measure $\mathbb{Q}_0 \in \mathcal{Q}_\lambda$, whose density is given by

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \frac{1}{\lambda}(I_{\{-X<c\}} + \kappa I_{\{-X=c\}}),$$

κ is the same as (4.14) in which X is replaced by $-X$. Since $\mathbb{P}(X > 0) \leq \lambda$, the upper and the lower λ-quantiles of $-X$ are

$$\begin{aligned} q^+_{-X}(\lambda) &= -\inf\{x \in \mathbb{R} \,|\, \mathbb{P}[X > x] \leq \lambda\} = 0, \\ q^-_{-X}(\lambda) &= -\inf\{x \in \mathbb{R} \,|\, \mathbb{P}[X > x] < \lambda\} = 0, \end{aligned}$$

respectively. Since $c = q_{-X}(\lambda)$ is contained in the interval $[q^-_{-X}(\lambda), q^+_{-X}(\lambda)]$, we get $c = 0$. For $-X$ with $X \geq 0$ the density becomes

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \frac{1}{\lambda}(I_{\{X>0\}} + \kappa I_{\{X=0\}}).$$

and hence we have

$$AVaR_\lambda(-X) = \mathbb{E}^{\mathbb{Q}_0}[X] = \mathbb{E}\Big[X \cdot \frac{1}{\lambda}(I_{\{X>0\}} + \kappa I_{\{X=0\}})\Big] = \frac{1}{\lambda}\mathbb{E}[X].$$

By the Proposition (4.5), $\tilde{\psi}$ is of the form (4.15). □

5. Optimal partial hedging in a complete market

In this section, we will see how to do optimal partial hedging by using the risk measure $\rho(X) = AVaR_\lambda(X)$ in a complete market. Let $W_t, 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{P}^*$ be a unique equivalent martingale measure on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a generalized geometric Brownian motion of stock price process whose differential is given by

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_t, \quad 0 \leq t \leq T. \tag{5.18}$$

This equation can be equivalently written as

$$S_t = S_0 \exp\Big\{\int_0^t \sigma_s \, dW_s + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) \, ds\Big\}.$$

Assume that the interest rate is zero, $\mu_t = \mu(> 0)$, $\sigma_t = \sigma(> 0)$ are constants for simplicity. The Girsanov's Theorem implies that the equivalent martingale measure $\mathbb{P}^*$ is given by

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}} &= \exp\Big\{-\int_0^t \Theta_t \, dW_t - \frac{1}{2}\int_0^t \|\Theta_u\|^2 \, du\Big\} \\ &= \exp\Big(-\frac{\mu}{\sigma}W_T - \frac{1}{2}\Big(\frac{\mu}{\sigma}\Big)^2 T\Big) = const \cdot S_T^{-\mu/\sigma^2}, \end{aligned}$$

where Θ_t is the market price of risk, i.e, $\Theta_t = \frac{\mu_t}{\sigma_t}$ [8](also see [6]). The process W^* defined as

$$W^*_t = W_t + \int_0^t \Theta_u \, du = W_t + \frac{\mu}{\sigma}t$$

is a Brownian motion under $\mathbb{P}^*$. Consider a European call option $H = (S_T - K)^+$. Then the claim H can be replicated completely with the initial capital, i.e., superhedging price

$$H_0 = \mathbb{E}^{\mathbb{P}^*}[H] = x_0 N(d_+) - KN(d_-),$$

where $x_0 = S_0$ and N denotes the standard normal distribution function, and

$$d_\pm = \frac{\ln(x_0/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$

For the optimal partial hedging, let $\alpha < \mathbb{E}^{\mathbb{P}^*}[H]$, i.e., α be smaller than the Black-Scholes price $H_0 = \mathbb{E}^{\mathbb{P}^*}[H]$.

Assume that

$$\mathbb{P}(H > 0) = N\Big(\frac{\mu}{\sigma}\sqrt{T} + d_-\Big) \le \lambda.$$

Then $\tilde{\psi} = I_{\{S_T > c\}}$ is the solution to the problem (4.17) by the Proposition (4.6) and c is determined by

$$\begin{aligned} \alpha &= \mathbb{E}^{\mathbb{P}^*}[HI_{\{S_T>c\}}] \\ &= S_0 N\Big(\frac{\ln(S_0/c)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\Big) - KN\Big(\frac{\ln(S_0/c)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\Big). \end{aligned}$$

Thus the modified claim $\tilde{\psi}H = HI_{\{S_T>c\}} = (S_T - c)^+ + (c-K)I_{\{S_T>c\}}$ should be hedged, and the price of the modified claim at time t is given by

$$\mathbb{E}^{\mathbb{P}^*}[\tilde{\psi}H \mid \mathcal{F}_t] = S_t N\Big(\frac{\ln(S_t/c)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\Big) - KN\Big(\frac{\ln(S_t/c)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\Big).$$

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