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SELF-ADJOINT INTERPOLATION ON $Ax = y$ IN ALG$\mathcal{L}$†

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Abstract. Given vectors x and y in a Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator T such that $Tx = y$. An interpolating operator for n vectors satisfies the equations $Tx_i = y_i$, for $i = 1, 2, \cdots, n$. In this paper the following is proved : Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$ and let P_x be the projection onto $sp(x)$. If $P_xE = EP_x$ for each $E \in \mathcal{L}$, then the following are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, $Af = 0$ for all f in $sp(x)^{\perp}$ and $A = A^*$.

(2) $\sup\left\{\dfrac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} < \infty$, $y \in sp(x)$ and $< x, y > = < y, x >$.

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1. Introduction

Let $\mathcal{A}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. An *interpolation question* for $\mathcal{A}$ asks for which x and y is there a bounded operator $A \in \mathcal{A}$ such that $Ax = y$. A variation, the 'n-vector interpolation problem', asks for an operator A such that $Ax_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison[6]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance[7]: his result was extended by Hopenwasser[2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[8] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[3] extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors. We obtained conditions for interpolation in the case A

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is in Alg$\mathcal{L}$ when $\mathcal{L}$ is a CSL in [4]. Again we showed an interpolation condition to the case that $\mathcal{L}$ is a subspace lattice in [5].

We establish some notations and conventions. Let $\mathcal{H}$ be a Hilbert space. A *subspace lattice* $\mathcal{L}$ is a strongly closed lattice of orthogonal projections on $\mathcal{H}$. A *commutative subspace lattice* $\mathcal{L}$, or CSL $\mathcal{L}$ is a subspace lattice whose elements commute each other. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Then Alg$\mathcal{L}$ denotes the algebra of bounded operators on $\mathcal{H}$ that leave invariant every projection in $\mathcal{L}$; Alg$\mathcal{L}$ is a weakly closed subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Then $< x, y >$ means the inner product of vectors x and y. Let $x_1, x_2, \cdots, x_n$ be vectors of $\mathcal{H}$. Then $sp(\{x_1, x_2, \cdots, x_n\}) = \{ \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \mid \alpha_1, \cdots, \alpha_n \in \mathbb{C}\}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

2. Main results

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I. Let $\mathcal{M}$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{\mathcal{M}}$ means the closure of $\mathcal{M}$, $\mathcal{M}^\perp$ the orthogonal complement of $\mathcal{M}$ and $[\mathcal{M}]$ the closed subspace of $\mathcal{H}$ generated by $\mathcal{M}$.

Let x and y be vectors in $\mathcal{H}$.

Lemma 2.1. *Let A be an operator in Alg$\mathcal{L}$ such that $Ax = y$ and $Af = 0$ for all f in $sp(x)^\perp$. Then the following are equivalent.*

(1) $y \in sp(x)$.

(2) For all f in $sp(x)^\perp$, A^*f is a vector in $sp(x)^\perp$.

Proof. (1) $\Rightarrow$ (2). Let f be a vector in $sp(x)^\perp$. Then

$$\begin{aligned} < A^*f, x > &= < f, Ax > \\ &= < f, y > = 0. \end{aligned}$$

Hence A^*f is a vector in $sp(x)^\perp$.

(2) $\Rightarrow$ (1). Let f be a vector in $sp(x)^\perp$. Then

$$\begin{aligned} < y, f > &= < Ax, f > \\ &= < x, A^*f > = 0. \end{aligned}$$

Hence $y \in sp(x)$. □

Lemma 2.2. *Let A be an operator in Alg$\mathcal{L}$ such that $Ax = y$ and $Af = 0$ for all f in $sp(x)^\perp$. If $A = A^*$, then A^*f is a vector in $sp(x)^\perp$ for all $f \in sp(x)^\perp$.*

Proof. Let f be a vector in $sp(x)^{\perp}$ and $x = A^*x_1 + x_2$ for some x_1 in $\mathcal{H}$ and x_2 in $\overline{range\ A^*}^{\perp}$. Then

$$\begin{aligned} < A^*f, x > &=< A^*f, A^*x_1 + x_2 > \\ &=< A^*f, A^*x_1 > + < A^*f, x_2 > \\ &=< A^*f, A^*x_1 > \\ &=< Af, Ax_1 > \\ &= 0. \end{aligned}$$

So A^*f is a vector in $sp(x)^{\perp}$. □

Theorem 2.3. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$ and let P_x be the projection onto $sp(x)$. If $P_xE = EP_x$ for each $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, $Af = 0$ for all f in $sp(x)^{\perp}$ and $A = A^*$.

(2) $\sup\left\{\dfrac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} < \infty$, *$y \in sp(x)$ and $< x, y >=< y, x >$.*

Proof. (1) $\Rightarrow$ (2). If we assume that (1) holds, then $\sup\left\{\dfrac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} < \infty$ by Theorem 3.4 [7]. Since $A = A^*$, $y \in sp(x)$ by Lemmas 2.1 and 2.2. And

$$\begin{aligned} < x, y > &=< x, Ax > \\ &=< x, A^*x > \\ &=< y, x > . \end{aligned}$$

(2) $\Rightarrow$ (1). If $\sup\left\{\dfrac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} < \infty$, then there exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$ and $Af = 0$ for all f in $sp(x)^{\perp}$ by Theorem 3.4 [7]. Since $< x, y >=< y, x >$, $< A^*x, x >=< x, Ax >=< Ax, x >$. Let f be a vector in $sp(x)^{\perp}$. Then A^*f is a vector in $sp(x)^{\perp}$ by Lemma 2.1. So $< A^*x, f >=< x, Af >= 0$ and $< Ax, f >=< x, A^*f >= 0$. Let $h = \alpha x + h_1$ be a vector in $\mathcal{H}$, where $h_1 \in sp(x)^{\perp}$. Then

$$\begin{aligned} < A^*f, h > &=< A^*f, \alpha x + h_1 > \\ &=< A^*f, \alpha x > + < A^*f, h_1 > \\ &=< f, Ah_1 > \\ &= 0. \end{aligned}$$

Hence $A^*f = 0$ for all f in $sp(x)^{\perp}$. So $A = A^*$. □

Lemma 2.4. *Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ be vectors in $\mathcal{H}$. Let A be an operator in Alg$\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \cdots, n)$ and $Ag = 0$ for all g in $sp(x_1, \cdots, x_n)^{\perp}$. Then the following are equivalent.*

(1) $y_k \in sp(x_1, \cdots, x_n)$ for all $k = 1, 2, \cdots, n$.
(2) If f is a vector in $sp(x_1, \cdots, x_n)^\perp$, A^*f is a vector in $sp(x_1, \cdots, x_n)^\perp$.

Proof. (1) $\Rightarrow$ (2). Let f be a vector in $sp(x_1, \cdots, x_n)^\perp$. Then for $k = 1, 2, \cdots, n$,

$$\begin{aligned} < A^*f, x_k > &=< f, Ax_k > \\ &=< f, y_k >= 0. \end{aligned}$$

So A^*f is a vector in $sp(x_1, \cdots, x_n)^\perp$.
(2) $\Rightarrow$ (1). Let f be a vector in $sp(x_1, \cdots, x_n)^\perp$. Then for all $k = 1, 2, \cdots, n$,

$$\begin{aligned} 0 =< A^*f, x_k > &=< f, Ax_k > \\ &=< f, y_k > . \end{aligned}$$

Hence $y_k \in sp(x_1, \cdots, x_n)$ for all $k = 1, 2, \cdots, n$. □

Lemma 2.5. *Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ be vectors in $\mathcal{H}$. Let A be an operator in $Alg\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \cdots, n)$, $Ag = 0$ for all g in $sp(x_1, \cdots, x_n)^\perp$ and $A = A^*$. Then A^*f is a vector in $sp(x_1, \cdots, x_n)^\perp$ for all f in $sp(x_1, \cdots, x_n)^\perp$.*

Proof. Let f be a vector in $sp(x_1, \cdots, x_n)^\perp$ and $x_k = A^*x_{k,1} + x_{k,2} (k = 1, 2, \cdots, n)$ for some $x_{k,1}$ in $\mathcal{H}$ and $x_{k,2} \in \overline{range\ A^*}^\perp$. Then for all $k = 1, 2, \cdots, n$,

$$\begin{aligned} < A^*f, x_k > &=< A^*f, A^*x_{k,1} + x_{k,2} > \\ &=< A^*f, A^*x_{k,1} > + < A^*f, x_{k,2} > \\ &=< A^*f, A^*x_{k,1} > \\ &=< Af, Ax_{k,1} > \\ &= 0. \end{aligned}$$

So A^*f is a vector in $sp(x_1, \cdots, x_n)^\perp$. □

Theorem 2.6. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and $x_1, \cdots, x_n, y_1, \cdots, y_n$ be vectors in $\mathcal{H}$. Let $\mathcal{M} = \left\{ \sum_{i=1}^{n} \alpha_i x_i \ : \ \alpha_i \in \mathbb{C} \right\}$ and $P_\mathcal{M}$ be the projection onto $\mathcal{M}$. If $P_\mathcal{M} E = E P_\mathcal{M}$ for each $E \in \mathcal{L}$, then the following are equivalent.*
(1) There is an operator A in $Alg\mathcal{L}$ such that $y_i = Ax_i (i = 1, 2, \cdots, n)$, $Ag = 0$ for all g in $\mathcal{M}^\perp$ and $A = A^*$.
(2) $\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n \alpha_i y_i)\|}{\|E^\perp(\sum_{i=1}^n \alpha_i x_i)\|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty$, $y_k \in \mathcal{M}$ and $< x_p, y_q >=< y_p, x_q >$ for all $k, p, q = 1, 2, \cdots, n$.

Proof. (1) $\Rightarrow$ (2). If we assume that (1) holds, then

$\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n}\alpha_i y_i)\|}{\|E^{\perp}(\sum_{i=1}^{n}\alpha_i x_i)\|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L}\right\} < \infty$ by Theorem 3.5 [7]. Let f be a vector in $\mathcal{M}^{\perp}$. Since $A = A^*$,

$$\begin{aligned} < y_k, f > &=< Ax_k, f > \\ &=< x_k, A^* f > \\ &=< x_k, Af > \\ &=< x_k, 0 >= 0 \text{ for } k = 1, 2, \cdots, n \end{aligned}$$

So $y_k \in \mathcal{M}$ for all $k = 1, 2, \cdots, n$. And

$$\begin{aligned} < x_p, y_q > &=< x_p, Ax_q > \\ &=< x_p, A^* x_q > \\ &=< y_p, x_q > \text{ for } p, q = 1, 2, \cdots, n. \end{aligned}$$

(2) $\Rightarrow$ (1). If $\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n}\alpha_i y_i)\|}{\|E^{\perp}(\sum_{i=1}^{n}\alpha_i x_i)\|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L}\right\} < \infty$, then there exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \cdots, n)$ and $Af = 0$ for all f in $\mathcal{M}^{\perp}$ by Theorem 3.5 [7]. Since $< x_p, y_q >=< y_p, x_q >$, $< x_p, Ax_q >=< Ax_p, x_q >$ for all $p, q = 1, 2, \cdots, n$. So $< Ax_p, h >=< A^* x_p, h >$ for all h in $\mathcal{M}$. Let f be a vector in $\mathcal{M}^{\perp}$. Then by Lemma 2.4, $A^* f$ is a vector in $\mathcal{M}^{\perp}$. Since $< Ax_p, f >=< x_p, A^* f >=< x_p, 0 >= 0$ and $< A^* x_p, f >=< x_p, Af >=< x_p, 0 >= 0$ for all $f \in {\mathcal{M}^{\perp}}^{\perp}$, $A^* x_p = Ax_p$ for $p = 1, 2, \cdots, n$. Let $h = \sum_{i=1}^{n}\alpha_i x_i + h_1$ be a vector in $\mathcal{H}$, where $h_1 \in \mathcal{M}^{\perp}$. Then

$$\begin{aligned} < A^* f, h > &=< A^* f, \sum_{i=1}^{n}\alpha_i x_i + h_1 > \\ &=< f, A(\sum_{i=1}^{n}\alpha_i x_i) > + < f, Ah_1 > \\ &=< f, A(\sum_{i=1}^{n}\alpha_i x_i) > \\ &=< f, \sum_{i=1}^{n}\alpha_i y_i > \\ &= 0. \end{aligned}$$

Hence $A^* f = 0$ for all f in $\mathcal{M}^{\perp}$. So $A = A^*$. $\square$

Let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. With the similar proof as Lemmas 2.4 amd 2.5, we can get the following lemmas and theorem.

Lemma 2.7. *Let A be an operator in Alg$\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \cdots)$ and $Ag = 0$ for all g in $[x_1, \cdots, x_n, \cdots]^{\perp}$. Then the following are equivalent.*

(1) $y_k \in [x_1, \cdots, x_n, \cdots]$ for all $k = 1, 2, \cdots$.

(2) If f is a vector in $[x_1, \cdots, x_n, \cdots]^{\perp}$, A^*f is a vector in $[x_1, \cdots, x_n, \cdots]^{\perp}$.

Lemma 2.8. *Let A be an operator in $Alg\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \cdots)$, $Ag = 0$ for all g in $[x_1, \cdots, x_n, \cdots]^{\perp}$ and $A = A^*$. Then A^*f is a vector in $[x_1, \cdots, x_n, \cdots]^{\perp}$ for all f in $[x_1, \cdots, x_n, \cdots]^{\perp}$.*

Theorem 2.9. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. Let $\mathcal{N} = \left\{ \sum_{i=1}^{n} \alpha_i x_i \ : \ \alpha_i \in \mathbb{C}, n \in \mathbb{N} \right\}$ and $P_{\overline{\mathcal{N}}}$ be the projection onto $\overline{\mathcal{N}}$. If $P_{\overline{\mathcal{N}}}E = EP_{\overline{\mathcal{N}}}$ for each $E \in \mathcal{L}$, then the following are equivalent.*

(1) There is an operator A in $Alg\mathcal{L}$ such that $y_i = Ax_i (i = 1, 2, \cdots)$, $Ag = 0$ for all g in $\mathcal{N}^{\perp}$ and $A = A^*$.

(2) $\sup \left\{ \dfrac{\|E^{\perp}(\sum_{i=1}^{n} \alpha_i y_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} \alpha_i x_i)\|} : n \in N, \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty,$

$y_k \in \mathcal{N}$ and $< x_p, y_q > = < y_p, x_q >$ for all $k, p, q = 1, 2, \cdots$.

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