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BOUNDARY BEHAVIOR OF LARGE SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS†

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Abstract. In this paper, our main purpose is to consider the quasilinear elliptic equation

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = (p-1)f(u)$$

on a bounded smooth domain $\Omega \subset \mathbf{R}^N$, where $p > 1$, $N > 1$ and f is a smooth, increasing function in $[0, \infty)$. We get some estimates of a solution u satisfying $u(x) \to \infty$ as $d(x, \partial\Omega) \to 0$ under different conditions on f.

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1. Introduction

In this paper, we will be concerned with the boundary behavior for solutions to quasi-linear problem of the form

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = (p-1)f(u) \text{ in } \Omega,\ u(x) \to \infty \ \text{ as } \ x \to \partial\Omega \qquad (1.1)$$

where $\Omega \subset \mathbf{R}^N, p > 1$ is a bounded smooth domain, and let $f(t)$ be a smooth, increasing function in $[0, \infty)$, which satisfies $f(0) = 0$. A local weak solution u of (1.1) is said to be a blow-up solution if u is continuous on Ω and $u(x) \to \infty$ as $d(x, \partial\Omega) \to 0$.

In [1], the author considered blow-up solutions to the question

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = g(x)f(u) \ \text{ for } \ x \in \Omega, \qquad (1.2)$$

and $u(x) \to \infty$ as $d(x, \partial\Omega) \to 0$. The following growth condition on f at infinity, first introduce by [2] and [3], is crucial in the investigation of the existence of

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blow-up solutions in this paper:

$$\int_1^\infty \frac{dt}{(pF(t))^{\frac{1}{p}}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)ds, \qquad (F-1)$$

where $p = 2$. Under some conditions on g, it is possible to show the existence of a non-negative blow-up solution. Meanwhile, some people also investigate asymptotic boundary estimates of such blow-up solutions and its main result can be listed as follows:

$$\lim_{\delta(x)\to 0} \frac{u(x)}{\phi(g(x)^{1/p}\delta(x))} = 1,$$

where ϕ is the function defined as $\int_{\phi(s)}^0 \frac{dt}{(2F(t))^{\frac{1}{2}}} = s$, $\delta(x)$ denotes the distance of x from $\partial\Omega$.

In the papers [2,3] the condition $(F-1)$ when $p = 2$ was shown to be necessary and sufficient condition for the equation

$$\triangle u = f(u) \qquad (1.3)$$

to admit a blow-up solution on a bounded domain Ω. The investigation in these papers led to several papers where important contributions were made to the question of existence, uniqueness, asymptotic boundary behavior, symmetry and convexity of blow-up solutions. We refer to the papers [4-10] and references therein for such results.

In [5], they considered a secondary effect in the asymptotic behavior of solutions of equation (1.3), namely, the behavior of

$$\frac{u}{\phi(\delta(x))} \to 1 \text{ as } \delta(x) \to 0.$$

They derived estimates for this expression under different conditions on f, which were valid for a large class of nonlinearities and extended a result of [9].

It was shown in [14] that problem

$$\text{div}(|\nabla u|^{m-2}\nabla u) + q(x)u^{-\gamma} = 0 \quad x \in \mathbf{R}^N$$

has a positive entire solution if $q \in C(\mathbf{R}^+),\ 0 \le \gamma < p-1,$ for any

$$0 < \epsilon < (N-p)(p-1-|\gamma|)/(p-1),$$

such that

$$\int_1^\infty r^{p+\epsilon-1} + [(N-p)|\gamma|/(p-1)]q(r)dr < \infty,$$

for $r \in (0,1),\ \delta < 1,\ q(r) = O(r^{-\delta})$.

In the recent paper [12], with the aim to investigate the second order term of the expansion of the solution $u(x)$ of equation (1.3), the following condition on $f(t)$ is assumed.

$$\frac{2F(t)f(t)}{(f(t))^2} = 1 + [\alpha + o(1)](\log t)^{-1}, \quad F(t) = \int_0^t f(\tau)d\tau,$$

where $\alpha > 1$, $o(1) \to 0$ as $t \to \infty$. They show that this condition implies the following inequality:

$$Ct(\log t)^{\alpha-\varepsilon} < (F(t))^{\frac{1}{2}} < Ct(\log t)^{\alpha+\varepsilon},$$

for some $C > 0$, $0 < \varepsilon < \alpha - 1$. Under this assumption and some additional condition for f it is shown that

$$\phi(\delta)[1 + \frac{\alpha-1}{2(2\alpha-1)}(N-1)K\delta - \varepsilon\delta - C_\varepsilon\delta^2] < u(x)$$

$$< \phi(\delta)[1 + \frac{\alpha-1}{2(2\alpha-1)}(N-1)K\delta + \varepsilon\delta + C_\varepsilon\delta^2],$$

where ϕ is the function defined as $\int_{\phi(s)}^{0} \frac{dt}{(2F(t))^{\frac{1}{2}}} = s$, $\delta = \delta(x)$ denotes the distance of x from $\partial\Omega$ and $K = K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = constant\}$.

Motivated by the papers of [5] and [12], we further study the asymptotic behavior of large solutions of (1.1), the results of the semilinear equation are extended to the quasilinear ones. We can find the related results for $p = 2$ in [5, 12]. We need some conditions as follows.

$(F-2)$ $F(t)/t^p$ is monotone increasing for large t.

$(F-3)$ Let $G(t) = \int_0^t (F(s))^{1-\frac{1}{p}} ds$. There exist a, b, with $1 < a < b$, such that

$$aF/f \le G/G' \le bF/f,$$

for large t.

$(F-3)^*$ $\lim_{\alpha\to1,\delta\to0} \sup \phi'(\alpha\delta)/\phi'(\delta) < \infty$

$(F-4)$ $Ct(\log t)^{\alpha-\varepsilon} < (F(t))^{\frac{1}{p}} < Ct(\log t)^{\alpha+\varepsilon}$, $C > 0$, $0 < \varepsilon < \alpha - 1$, $\alpha > 1$.

2. An estimate in strip domains.

Let us first consider the 1-dimensional problem

$$(|\phi'|^{p-2}\phi')' = (p-1)f(\phi) \text{ with } \lim_{x\to0} \phi(x) = \infty.$$

All solutions are of the form $\phi_c = {\psi_c}^{-1}$, where $\psi_c(t) = \int_t^\infty \frac{ds}{(pF(s)+c)^{\frac{1}{p}}}$.

By a modification of the method given in [5], we obtain the following Lemma.

Lemma 2.1. Suppose that $f(t)$ is a smooth, increasing in $[0, \infty)$, satisfying $f(0) = 0$ as well as condition $(F-1)$, for any real numbers c_1 and c_2, $\lim_{x\to0}(\phi_{c_1}(x) - \phi_{c_2}(x)) = 0$.

Proof. Let $c_1 > c_2$. Then $\phi_{c_1}(x) < \phi_{c_2}(x)$. Fix x_0 in the domains of definition of ϕ_{c_i}, i=1,2. Let it be so close to 0 that $\phi_{c_i} > 0$. Define $L = \phi_{c_2}(x_0) - \phi_{c_1}(x_0)$ and $z = \phi_{c_2}(x + \varepsilon_0) - L$, where $\varepsilon_0 > 0$ is any small positive number such that $\phi_{c_2}(x_0 + \varepsilon_0) > 0$. It satisfies

$$(|z'|^{p-2}z')' = (p-1)f(\phi_{c_2}(x + \varepsilon_0)) \ge (p-1)f(z)$$

for $x < x_0, z(0) < \infty$ and $z(x_0) < \phi_{c_1}(x_0)$. By $(F-1)$ the difference $z - \phi_{c_1}$ cannot have a positive maximum in $(0, x_0)$. We thus have $z(x) < \phi_{c_1}(x)$. Since this inequality holds for any $\varepsilon < \varepsilon_0$ we conclude that

$$\phi_{c_1}(x) \le \phi_{c_2}(x) \le \phi_{c_1}(x) + L \text{ in } (0, x_0). \tag{2.1}$$

Let c be any positive number. By definition

$$-\phi_c{}' = \sqrt[p]{pF(\phi_c) + c} = \sqrt[p]{pF(\phi_c)}\sqrt[p]{1 + \frac{c}{pF(\phi_c)}},$$

whence

$$\int_{\phi_c(x)}^{\infty} \frac{ds}{(pF(s))^{\frac{1}{p}}} = \int_0^x \sqrt[p]{1 + \frac{c}{pF(\phi_c)}} d\xi = x + (1 + o(1))\frac{c}{p^2}\int_0^x \frac{d\xi}{F(\phi_c)}$$

as $x \to 0$. Hence,

$$\phi_c(x) = \phi(x + (1 + o(1))\frac{c}{p^2}\int_0^x \frac{d\xi}{F(\phi_c)}),$$

By mean value theorem, we have

$$\phi_c(x) = \phi(x) + \phi'(\tilde{x})(1 + o(1))\frac{c}{p^2}\int_0^x \frac{d\xi}{F(\phi_c)}, \tag{2.2}$$

where

$$x \le \tilde{x} \le x + (1 + o(1))(c/p^2)\int_0^x \frac{d\xi}{F(\phi_c)}.$$

Since $\phi' < 0$ and $(|\phi'|^{p-2}\phi')' \ge 0$,

$$(|\phi'|^{p-2}\phi')' = (-|\phi'|^{p-2}(-\phi'))' = (-|-\phi'|^{p-1})' \ge 0$$

which implies $|\phi'(x)|^{p-1}$ is decreasing, $|\phi'(x)| > |\phi'(\tilde{x})|$. This inequality and (2.2) imply that

$$0 \le |\phi(x) - \phi_c(x)| \le |\phi'(x)|(1 + o(1))\frac{c}{p^2}\int_0^x \frac{d\xi}{F(\phi_c)} = \eta(x). \tag{2.3}$$

In view of (2.1) there exists a constant L such that $\phi_c \ge \phi - L$. Hence

$$\int_0^x \frac{d\xi}{F(\phi_c)} \le \int_0^x \frac{d\xi}{F(\phi - L)} = \int_{\phi(x)}^{\infty} \frac{ds}{F(s-L)(pF(s))^{1/p}}.$$

Consequently,

$$\eta(x) \le (1 + o(1))(c/p^2)(pF(\phi))^{1/p}\int_{\phi}^{\infty} \frac{ds}{F(s-L)(pF(s))^{1/p}} \le const \int_{\phi}^{\infty} \frac{ds}{F(s-L)}.$$

The assertion now follows from (2.3) and $(F-2)$.

3. Estimates for radially symmetric solutions.

In this section we consider radially symmetric solutions in the annuli

$$A(\rho, R) = \{x : \rho < |x| < R\}.$$

Put

$$\Gamma(t) := \frac{1}{F(t)} \int_0^t (pF(s))^{1-\frac{1}{p}} ds. \tag{3.1}$$

Let us now introduce the following notation. Assume that f satisfies the Keller-Osserman condition (F-1). Then it is known (see Lemma 2.1 of [13]) that

$$\lim_{t\to\infty} \frac{(F(t))^{(p-1)/p}}{f(t)} = 0. \tag{3.2}$$

Hence, by the Bernoulli-1'Hospital rule

$$\lim_{t\to\infty} \Gamma(t) = 0. \tag{3.3}$$

By a modification of the method given in [5], we obtain the following Lemma.

Lemma 3.1. Suppose that $f(t)$ is a smooth, increasing in $[0, \infty)$, satisfying $f(0) = 0$ as well as condition $(F-1)$.

(i) Let $v(r)$ be a radial solution of (1.1) in $A(\rho, R)$ such that $\lim_{r\to R} v(r) = \infty$.Then

$$\psi(v(r)) = R - r - \frac{N-1}{(p-1)pR}(1 + o(1)) \int_r^R \Gamma(v(s))ds \quad as \quad r \to R. \tag{3.4}$$

(ii) Let $w(r)$ be a radial solution of (1.1) in $A(\rho, R)$ such that $\lim_{r\to\rho} w(r) = \infty$. Then

$$\psi(w(r)) = r - \rho - \frac{N-1}{(p-1)p\rho}(1 + o(1)) \int_\rho^r \Gamma(w(s))ds \quad as \quad r \to \rho. \tag{3.5}$$

Proof. We first establish the result for the solution v. For $r \in (\rho, R)$ it satisfies the equation

$$(|v'|^{p-2}v')' + \frac{N-1}{r}|v'|^{p-2}v' = (p-1)f(v), \quad \lim_{r\to R} v(r) = \infty.$$

Multiplication by v' and integration yield

$$|v'(r)|^p - |v'(r_0)|^p + \frac{p}{p-1}(N-1) \int_{r_0}^r \frac{|v'|^p}{s} ds = p[F(v(r)) - F(v(r_0))]$$

or equivalently

$$|v'(r)|^p + \frac{p}{p-1}(N-1)I = pF(v)(1 + \frac{g(v(r_0))}{F(v)}), \tag{3.6}$$

where

$$I := \int_{r_0}^r \frac{|v'|^p}{s} ds \text{ and } g(v(r_0)) = \frac{|v'(r_0)|^p}{p} - F(v(r_0)).$$

For r sufficiently close to R, $v'(r) \geq 0$. Otherwise in every left neighborhood of R there would exist an interval (r_1, r_2) such that v' is positive at its end points but negative at some point inside. Since v blows up at R this contradicts the equation. Accordingly we choose $\bar{r}$ sufficiently close to R so that v is monotone increasing in $(\bar{r}, R)$ and assume that $\bar{r} \leq r_0 \leq R$. Then

$$|v'(r)|^p \leq pF(v)(1 + \frac{g(v(r_0))}{F(v(r_0))}),$$

and consequently

$$(|v'|^{p-2}v')' = (p-1)f(v) - \frac{N-1}{r}|v'|^{p-2}v' > (p-1)f(v) - c'(F(v))^{(p-1)/p}$$

for some constant c'. This inequality and (3.2) imply that $|v'|$ is monotone increasing in a left neighborhood of R. Hence

$$I \leq |v'|^p(r)\log\frac{r}{r_0} = \varepsilon|v'|^p(r) \text{ for } r \in (r_0, R).$$

By choosing r_0 sufficiently close to R, ε can be made arbitrarily small. Inserting this estimate into (3.6), we get

$$|v'|^p = pF(v)(1 + o(1)) \text{ as } r \to R. \tag{3.7}$$

Hence,

$$I = R^{-1}(1+o(1))\int_{r_0}^{r}|v'|^p ds = R^{-1}(1+o(1))\int_{r_0}^{r}(pF(v))^{(p-1)/p}v' ds$$

$$= R^{-1}(1+o(1))\int_{v(r_0)}^{v(r)}(pF(v))^{(p-1)/p}dv, \ \ as \ \ r_0, r \to R,$$

which implies that

$$I/F(v) = (1+o(1))\frac{\Gamma(v(r))}{R}, \ \ as \ \ r \to R.$$

Inserting this expression into (3.6) we obtain

$$v'(r) = (pF(v))^{\frac{1}{p}}(1 - \frac{(N-1)\Gamma(v)}{(p-1)R}(1+o(1)))^{\frac{1}{p}}.$$

Hence, by (3.3),

$$v'(r) = (pF(v))^{\frac{1}{p}}(1 - \frac{(N-1)\Gamma(v)}{(p-1)pR}(1+o(1))) \ \ as \ \ r \to R.$$

Dividing this expression by $(pF(v))^{1/p}$ and integrating, we get

$$\psi(v(r_0)) - \psi(v(r)) = r - r_0 - (1+o(1))\frac{N-1}{(p-1)pR}\int_{r_0}^{r}\Gamma(v(s))ds.$$

Now let $r \to R$, we obtain (3.4).

The proof of the second assertion is very similar. We omit the details.

Put

$$\omega = \frac{N-1}{(p-1)pR}(1+o(1))\int_r^R \Gamma(v(s))ds,$$

$$\tilde{\omega} = \frac{N-1}{(p-1)p\rho}(1+o(1))\int_\rho^r \Gamma(w(s))ds. \tag{3.8}$$

Then, by Lemma 3.1

$$v(r) = \phi(\delta-\omega) = \phi(\delta) - \phi'(\delta')\omega, \text{ where } \delta = R-r \geq \delta' \geq \delta-\omega,$$

$$w(r) = \phi(\delta+\tilde{\omega}) = \phi(\delta) + \phi'(\tilde{\delta}')\tilde{\omega}, \text{ where } \delta = r-\rho \leq \tilde{\delta}' \leq \delta+\tilde{\omega}. \tag{3.9}$$

This will be the key for the estimates concerning the behavior of the large radial solutions.

By a modification of the method given in [5, 12], we obtain the following main results.

Theorem 3.2. Let v and w be solutions of problem (1.1) in $A(\rho, R)$, as in the previous lemmas, and assume that $f(t)$ is smooth, increasing in $[0,\infty)$ and satisfies conditions $(F-1)$ and $(F-2)$. Then the following statements hold.

(i) Put $\delta = R-r$ and let $o(1)$ denote a quantity which tends to zero as $\delta \to 0$. Then

$$\phi(\delta) \leq v(r) \leq \phi(\delta)[1+(1+o(1))\frac{N-1}{(p-1)R}\delta]. \tag{3.10}$$

(ii) Put $\delta = r-\rho$ and let $o(1)$ denote a quantity which tends to zero as $\delta \to 0$. If $(F-3)$ holds, then there exists a constant c_1 such that

$$\phi(\delta) \geq w(r) \geq \phi(\delta) + c_1\frac{1+o(1)}{(p-1)\rho}\delta^2\phi'(\delta). \tag{3.11}$$

Note that $0 < -\delta\phi'(\delta) \leq p\phi(\delta/p)$. Alternatively, if $(F-3)^*$ holds, then there exists a constant ${c_1}^*$ such that

$$\phi(\delta) \geq w(r) \geq \phi(\delta) - {c_1}^*\frac{1+o(1)}{(p-1)\rho}\delta\phi(\delta). \tag{3.12}$$

(iii) If f satisfies condition $(F-4)$, then for some $\sigma \in (0,1)$ and a suitable $C(p)$ we have

$$v(r) < \phi(R-r)[1+C(p)(R-r)^{1-\sigma}] \text{ for } r \text{ near } R \tag{3.13}$$

and

$$w(r) > \phi(r-\rho)[1-C(p)(r-\rho)^{1-\sigma}] \text{ for } r \text{ near } \rho. \tag{3.14}$$

Proof. (i) By the definition of $\Gamma(s)$ and the monotonicity of F

$$\Gamma(v) \leq \frac{p^{1-\frac{1}{p}}v}{(F(v))^{1/p}}. \tag{3.15}$$

Let r_0 be sufficiently close to R so that $v(r)$ is increasing for $r > r_0$. Then by $(F-2)$ and (3.15)

$$\int_r^R \Gamma(v(s))ds \leq \frac{p^{1-\frac{1}{p}} v(r)}{(F(v(r)))^{1/p}} \delta. \tag{3.16}$$

By (3.9) and the fact that $\phi(\delta)$ is decreasing, we have

$$-\phi'(\delta') = (pF(\phi(\delta')))^{\frac{1}{p}} \leq (pF(\phi(\delta - \omega)))^{\frac{1}{p}} = (pF(v(r)))^{\frac{1}{p}}.$$

Inserting this inequality in the first part of (3.9) and using (3.8) and (3.16) we obtain

$$v(r) \leq \phi(\delta) + (pF(v(r)))^{\frac{1}{p}} \omega \leq \phi(\delta) + \frac{N-1}{(p-1)R}(1+o(1))v(r)\delta. \tag{3.17}$$

The left inequality in (i) is an immediate consequence of (3.9). To verify the right inequality in (i) we observe that, by (3.17),

$$v(r)/\phi(\delta) \leq (1 - \frac{N-1}{(p-1)R}(1+o(1))\delta)^{-1}.$$

Hence

$$v(r)/\phi(\delta) \leq 1 + (1+o(1))\frac{N-1}{(p-1)R}\delta.$$

This proves (i).

(ii) We turn to the proof of the assertion assuming that $(F-3)^*$ holds. Since $\tilde{w} \to 0$ as $\delta \to 0$, it follows that

$$\lim_{\delta \to 0} \sup \frac{\phi'(\tilde{\delta}')}{\phi'(\delta + \tilde{\omega})} \leq c < \infty.$$

Consequently, by the second part of (3.9),

$$w(r) - \phi(\delta) \geq c\tilde{\omega}\phi'(\delta + \tilde{\omega}) = -c\tilde{\omega}(pF(w))^{1/p}.$$

Further,by $(F-2)$,(3.17) and (3.10),

$$\tilde{\omega}(pF(w))^{1/p} \leq \frac{N-1}{(p-1)\rho}(1+o(1))w(r)\delta.$$

The last two inequalities and (3.9) imply (3.14).

Next we prove the assertion assuming that $(F-3)$ holds. Let $\gamma := \Gamma \circ w$. Then, by (3.9) and (3.1)

$$\gamma'(r) = \Gamma'(w(r))w'(r) = -\Gamma'(w(r))(pF(w(r)))^{1/p}(1+o(1)) = p(1+o(1))(\frac{fG}{F^{2-\frac{1}{p}}} - 1),$$

where $o(1)$ is a quantity which tends to zero as $r \to \rho$. Hence, by $(F-3)$,

$$p(a-1) \leq \frac{\gamma'(r)}{1+o(1)} \leq p(b-1). \tag{3.18}$$

Further, by (3.3), $\gamma(r) \to 0$ as $r \to \rho$. Therefore, (3.18) implies that

$$\Gamma(w(r)) \leq (1+o(1))p(b-1)(r-\rho). \tag{3.19}$$

Hence, by (3.8),

$$\tilde{\omega}(r) \leq c_1 \frac{1+o(1)}{(p-1)\rho}\delta^2, \quad \text{where} \quad c_1 = (b-1)(N-1). \tag{3.20}$$

Finally, by (3.9) and (3.20),

$$\phi(\delta) \geq w(r) \geq \phi(\delta) + \phi'(\delta)\tilde{\omega(r)} \geq \phi(\delta) + c_1 \frac{1+o(1)}{(p-1)\rho}\delta^2 \phi'(\delta),$$

which proves (3.11).

(iii) According to Lemma 3.1, since ϕ is the inverse function of ψ, by (3.4) we get

$$v(r) = \phi(R-r) - \phi'(\omega)\frac{N-1}{(p-1)pR}(1+o(1))\int_r^R \Gamma(v(s))ds, \tag{3.21}$$

with

$$R - r > \omega > R - r - \frac{N-1}{(p-1)pR}(1+o(1))\int_r^R \Gamma(v(s))ds. \tag{3.22}$$

Since F is increasing and ϕ is decreasing, by (3.22) and (3.4) we find

$$-\phi'(\omega) = (pF(\phi(\omega)))^{1/p} \leq (pF(v))^{1/p}.$$

Insertion of the latter estimate into (3.21) yields

$$v(r) < \phi(R-r) + C_1 \frac{(pF(v))^{1/p}}{p(p-1)}\int_r^R \Gamma(v(s))ds. \tag{3.23}$$

We denote by C_i suitable positive constants.

By (3.1) and the left-hand side of $(F-4)$ we get

$$\Gamma(t) \leq \frac{pt}{(pF)^{1/p}} < C_2 \frac{p^{1-\frac{1}{p}}}{(\log t)^{\alpha-\epsilon}} \tag{3.24}$$

By (3.23), using the right-hand side of $(F-4)$ and (3.24) we find

$$v(r) < \phi(R-r) + \frac{C_3}{p-1}v(\log(v))^{\alpha+\epsilon}\int_r^R \frac{1}{(\log(v))^{\alpha-\epsilon}}ds,$$

and

$$v(r) < \phi(R-r) + \frac{C_4}{p-1}v(r)(\log(v(r)))^{2\epsilon}(R-r) \quad \text{for } r \text{ near } R. \tag{3.25}$$

On the other side by (3.4) we find, for r near to R,

$$\psi(v(r)) > (1 - \frac{1}{p(p-1)})(R-r),$$

whence

$$v(r) < \phi[(1 - \frac{1}{p(p-1)})(R-r)]. \tag{3.26}$$

By (1.2) and the left-hand side of $(F-4)$ we find

$$\log\phi(s) < C_5 p^{1/p}(\frac{1}{s})^{\frac{1}{\alpha-1-\epsilon}}. \tag{3.27}$$

Therefore, (3.26) implies

$$\log(v(r)) < C_6 p^{1/p}(\frac{p(p-1)}{(p^2-p-1)(R-r)})^{\frac{1}{\alpha-1-\epsilon}}.$$

Insertion of the latter estimate into (3.24) yields

$$v(r) < \phi(R-r) + C(p)(R-r)^{1-\sigma}v(r),$$

where $C(p) = C_6\frac{p^{\frac{1}{p}}}{p-1}(\frac{p^2-p}{p^2-p-1})^{\frac{2\epsilon}{\alpha-1-\epsilon}}$ and $\sigma = \frac{2\epsilon}{\alpha-1-\epsilon}$, from which we find

$$v(r)[1 - C(p)(R-r)^{1-\sigma}] \le \phi(R-r).$$

Let us prove (3.14). By (3.5) we get

$$v(r) = \phi(r-\rho) + \phi'(\tilde{\omega})\frac{N-1}{(p-1)p\rho}(1+o(1))\int_\rho^r \Gamma(v(s))ds, \tag{3.28}$$

with

$$r-\rho < \tilde{\omega} < r-\rho + \frac{N-1}{(p-1)p\rho}(1+o(1))\int_\rho^r \Gamma(v(s))ds.$$

Since $\phi'(s)$ is increasing, we have $\phi'(\tilde{\omega}) > \phi'(r-\rho)$, and (3.28) implies

$$v(r) > \phi(r-\rho) + \frac{C_1}{p(p-1)}\phi'(r-\rho)\int_\rho^r \Gamma(v(s))ds.$$

By using (3.24), the last estimate yields

$$v(r) > \phi(r-\rho) - \frac{C_2}{p-1}(F(\phi(r-\rho)))^{1/p}\frac{1}{(\log v)^{\alpha-\epsilon}}(r-\rho) \quad \text{for } r \text{ near } \rho. \tag{3.29}$$

On the other side by (3.5) we find, for r near ρ

$$\psi(v(r)) < 2(r-\rho),$$

whence

$$v(r) > \phi(2(r-\rho)). \tag{3.30}$$

By (1.2) and the right-hand side of $(F-4)$ we find

$$\frac{1}{\log\phi(s)} < C_3 p^{1/p} s^{\frac{1}{\alpha-1+\epsilon}}. \tag{3.31}$$

By the latter estimate and (3.30) we find

$$\frac{1}{\log v(r)} < C_3 p^{1/p}(2(r-\rho))^{\frac{1}{\alpha-1+\epsilon}}. \tag{3.32}$$

Furthermore, using again the right-hand side of $(F-4)$ and (3.27) we have

$$(F(\phi(r-\rho)))^{1/p} < C_4\phi(r-\rho)(\log(\phi(r-\rho)))^{\alpha+\epsilon} < C_5\phi(r-\rho)(p^{\frac{1}{p}})^{\alpha+\epsilon}(\frac{1}{r-\rho})^{\frac{\alpha+\epsilon}{\alpha-1-\epsilon}}.$$

Inserting the latter estimate and (3.32) into (3.29) we find

$$v(r) > \phi(r-\rho) - C_6\phi(r-\rho)(p^{\frac{1}{p}})^{2\alpha}(\frac{1}{r-\rho})^{\frac{\alpha+\epsilon}{\alpha-1-\epsilon}}(r-\rho)^{\frac{\alpha-\epsilon}{\alpha-1+\epsilon}}(r-\rho).$$

Inequality (3.14) follows with $\sigma = \frac{2\epsilon(2\alpha-1)}{(\alpha-1)^2-\epsilon^2}$ and a suitable M. Since

$$\frac{2\epsilon}{\alpha-1-\epsilon} < \frac{2\epsilon(2\alpha-1)}{(\alpha-1)^2-\epsilon^2},$$

(3.13) and (3.14) hold both with $\sigma = \frac{2\epsilon(2\alpha-1)}{(\alpha-1)^2-\epsilon^2}$. The Lemma is proved.

4. General domains.

In this section we consider bounded domains $\Omega \subset \mathbf{R}^N$ with a smooth boundary $\partial\Omega$. For our purposes, the boundary $\partial\Omega$ is smooth if it is of class C^4. Furthermore, if Ω is smooth, then $\partial\Omega$ satisfies a uniform interior and exterior condition. Using Theorem 3 we derive an estimate for large solutions in arbitrary smooth domains. The proof employs a weak comparison method.

By a modification of the method given in [5, 12], we obtain the following results.

Theorem 4.1. Let $\Omega \subset \mathbf{R}^N$, $N \geq 2$ be a bounded domain with a smooth boundary $\partial\Omega$, let $f(t)$ be smooth, increasing in $[0,\infty)$ with $f(0)=0$, satisfying $(F-1)$ and $(F-2)$. If $u(x)$ is a solution to problem (1.1), then the following statements hold.

(i) If f satisfies condition $(F-3)^*$, there exists a constant c^* such that every u of problem (1.1) satisfies

$$|\frac{u(x)}{\phi(\delta(x))} - 1| \leq \frac{c^*}{p-1}\delta. \tag{4.1}$$

(ii) If f satisfies condition $(F-3)$, there exists a constant c such that

$$\frac{c}{p-1}\frac{\delta^2\phi'(\delta)}{\phi(\delta)} \leq \frac{u(x)}{\phi(\delta(x))} - 1 \leq c\delta. \tag{4.2}$$

(iii) If f satisfies condition $(F-4)$, then for some $\sigma \in (0,1)$ we have

$$\phi(\delta)[1 - C(p)\delta^{1-\sigma}] < u(x) < \phi(\delta)[1 + C(p)\delta^{1-\sigma}] \text{ for } x \text{ near } \partial\Omega, \tag{4.3}$$

where ϕ is the function defined as $\int_{\phi(s)}^{0} \frac{dt}{(2F(t))^{\frac{1}{2}}} = s$, $\delta = \delta(x)$ denotes the distance from x to $\partial\Omega$ and $C(p)$ is a suitable positive constant related with p.

Proof. We can make use of Theorem 3.2. If $x_0 \in \partial\Omega$, let $A(R_1, R)$ be a small annulus contained in Ω and such that $\bar{\Omega} \bigcap \bar{A}((R_1, R)) = x_0$. If $v(x) > 0$ is a radial solution of problem (1.1) in $\bar{A}((R_1, R))$, by the comparison principle for elliptic equations we have

$$u(x) \leq v(x).$$

Now consider the annulus $A(\rho, R_2)$, whose inner boundary Γ_ρ is contained in $\mathbf{R}^N \backslash \Omega$, $\Gamma_\rho \bigcap \partial\Omega = \{x_0\}$, and moreover $R_2 >$ diam Ω. Of course, $\bar{\Omega} \subset \bar{A}(\rho, R_2)$. If $z(x)$ is a positive radial solution of problem (1.1) in $\bar{A}(\rho, R_2)$, then by a comparison argument we have

$$z(x) \leq u(x).$$

Then according to Theorem 3.2, the theorem is proved.

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