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THE METHOD OF ASYMPTOTIC INNER BOUNDARY CONDITION FOR SINGULAR PERTURBATION PROBLEMS

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Abstract. The method of Asymptotic Inner Boundary Condition for Singularly Perturbed Two-Point Boundary value Problems is presented. By using a terminal point, the original second order problem is divided in to two problems namely inner region and outer region problems. The original problem is replaced by an asymptotically equivalent first order problem and using the stretching transformation, the asymptotic inner condition in implicit form at the terminal point is determined from the reduced equation of the original second order problem. The modified inner region problem, using the transformation with implicit boundary conditions is solved and produces a condition for the outer region problem. We used Chawla's fourth order method to solve both the inner and outer region problems. The proposed method is iterative on the terminal point. Some numerical examples are solved to demonstrate the applicability of the method.

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1. Introduction

The numerical solution of singular perturbation problems is currently a field in which active research is going on. Singular perturbation problems are of common occurrence in fluid mechanics (boundary layer theory) and other branches of Applied Mathematics. A wide variety of papers and books are available, describing varies techniques for solving singular perturbation problems, among these one can refer Bellman [2], Bender and Orsazag [3], Hinch [6], Kadalbajo and Reddy [7-8], Kevorkian and Cole [9], O'Malley [112], Nayfah [10-11] and Van Dyke [14]. Several authors published papers on solving SSP by dividing the domain of definition of the problem into non-overlapping subintervals called outer and inner regions, among these; we mention Awoke and Reddy [1], Chakravarthy and Reddy [4], Vigo-aguiar and Natesan [15] and Wang [16] .

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In the present paper, the method of an Asymptotic Inner Boundary Condition for Singularly Perturbed two point Boundary value Problems with the boundary layer at the left is presented. The method consists of the following steps: (1) The original second order problem is divided in to two problems, an inner region and an outer region problem using a terminal point. (2) The original second order problem is replaced by an asymptotically equivalent first order problem. (3) Using the stretching transformation, the asymptotic inner condition in mixed form at the terminal point is determined from the reduced equation of the original second order problem. (4) The modified inner region problem (using the transformation) with mixed boundary conditions is solved and produces a condition for the outer region problem. (5) The outer region problem is solved as a two point boundary value problem. Finally, we combine the solutions of both the inner region and outer region problems to get the approximate solution of the original problem. The present method is iterative on the terminal point. We repeat the process (numerical scheme) for various choices of the terminal point, until the solution profiles do not differ materially from iteration to iteration.

2. The method of asymptotic inner boundary condition

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + [a(x)y(x)]' = h(x), x \in [0, 1] \tag{1}$$

with

$$y(0) = \alpha, \tag{2a}$$

$$and, \quad y(1) = \beta; \tag{2b}$$

where ε is a small positive parameter ($0 < \varepsilon << 1$)and $a_i, i = 1, .., 4, \alpha, \beta$ are known constants. We assume that $a(x)$ *and* $h(x)$ are sufficiently continuously differentiable functions in [0,1]. Further more, we assume that $a(x) \geq M > 0$ throughout the interval [0,1], where M is some positive constant. Under this assumptions,(1) has a unique solution $y(x)$ which in general, displays a boundary layer of width O(ε) at $x = 0$ for small values of $\varepsilon^{\cdot}$

As mentioned the method consists of the following steps:

Step 1: Dividing the original problem in to two regions, an inner region and outer region problem. Let x_p be the terminal point or width or thickness of the boundary layer (inner region), then the inner and outer region problems are defined on $0 \leq x \leq x_p$ and $x_p \leq x \leq 1$ respectively.

Step 2: Replace the original second order problem (8.1) by an asymptotically equivalent first order problem as follows: Integrating (8.1), we get

$$\varepsilon y'(x) + a(x)y(x) = f(x) + K \tag{3}$$

Where $f(x) = \int h(x)dx$ and K is a constant to be determined.

The constant K can be determined by introducing the boundary condition y(1)

in to the reduced equation of (3).
i.e $a(1)y(1) = f(1) + K$

$$K = a(1)y(1) - f(1) \tag{4}$$

Note that, this choice of K ensures that the solution of the reduced problem of (1) satisfies the reduced problem of (3). Thus (3) can be written as:
$\varepsilon y'(x) + a(x)y(x) = f(x) + K, where K = a(1)y(1) - f(1)$ Step 3: Determining the asymptotic inner boundary condition
To determine the asymptotic inner condition, we take the transformation

$$t = x/\varepsilon \tag{5}$$

to produce a new differential equation. By using (5), we transform equations (1) and (3) with

$$y(x) = y(t\varepsilon) = Y(t) \tag{6a}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{6b}$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon} \tag{6c}$$

$$a(x) = a(t\varepsilon) = A(t) \tag{6d}$$

$$a'(x) = \frac{a'(t\varepsilon)}{\varepsilon} = \frac{A'(t)}{\varepsilon} \tag{6e}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{6f}$$

$$h(x) = h(t\varepsilon) = H(t) \tag{6g}$$

to obtain the new differential equations

$$Y''(t) + [(A(t)Y(t)]' = \varepsilon H(t) \tag{7}$$

and

$$Y'(t) + A(t)Y(t) = F(t) + K \tag{8}$$

At $t = t_p = x_p/\varepsilon$, equation (8) will take the form

$$c_1 Y(t_p) + Y'(t_p) = c_2 \tag{9a}$$

$$where\ c_1 = A(t_p)\ and\ c_2 = F(t_p) + K \tag{9b}$$

Equation (9) which is in implicit form is taken as an asymptotic inner condition at the terminal point $t = t_p = x_p/\varepsilon$.
Step 4: Solving the inner region problem: Now we solve the inner region problem:

$$Y''(t) + [(A(t)Y(t)]' = \varepsilon H(t) \tag{10}$$

Where

$$Y(0) = y(0) = \alpha \tag{11a}$$

$$and \quad c_1 Y(t_p) + Y'(t_p) = c_2 \tag{11b}$$

By solving (10) - (11), we get the value $Y(t_p) = y(x_p)$ and we take it as one boundary condition to solve the outer region problem.
Denote $Y(t_p) = y(x_p) = \delta$

Step 5: Solving the outer region problem: Since $x = x_p$ is a common point to both the inner and outer region, we set the outer region problem as:

$$\varepsilon y''(x) + [a(x)y(x)]' = h(x), x_p \leq x \leq 1 \tag{12}$$

with

$$Y(x_p) = \delta \tag{13a}$$
$$and\ y(1) = \beta \tag{13b}$$

Solution of the original problem:
To solve the two-point boundary value problems given in equations (10)-(11) [inner region problem] and (12)-(13) [outer region problem), we used Chawla's [5] fourth- order finite difference method. In fact, any standard analytic or numerical method can be used. Finally, we combine the solutions of both the inner region ($0 \leq x \leq x_p$) and outer region ($x_p \leq x \leq 1$) problems to get the approximate solution of the original problem. We repeat the process (numerical scheme) for various choices of $x + p$, until the solution profile do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$/Y^{m+1}(t) - Y^m(t))/ \leq \sigma,\ 0 \leq t \leq t_p \tag{14}$$

Where $y^m(x) =$ the solution for the m^{th} iterate of x_p and $\sigma=$ the prescribed tolerance bound.

3. Fourth-Order Difference Scheme

A finite difference scheme is often a convenient choice for the numerical solution of two point boundary value problems. We used Chawla's [5] fourth- order finite difference method to solve the inner and outer region problems.
Inner region problem: We can rewrite equations (10)-(11)in the form :

$$Y''(t) = \varepsilon H(t) - A(t)Y'(t) - A'(t)Y(t) = g(t, Y, Y') \tag{15}$$

With $Y(0) = y(0) = \alpha$
and $c_1 Y(t_p) + Y'(t_p) = c_2$
We divide the interval$0 \leq t \leq t_p$ into N equal parts with constant mesh length h. Let $0 = t_0, t_1, ..., t_N = t_p$ be the mesh points. Then we have $t_i = ih$; i=0, 1, 2,..., N.
Let us denote the exact solution $Y(t)$ at the grid points t_i by Y_i ; similarly, $Y'_i = Y'(t_i)$.
For i=1,2, ..., N-1, let

$$\overline{Y}'_i = \tfrac{Y_{i+1}-Y_{i-1}}{2h} \tag{16a}$$
$$\overline{Y}'_{i+1} = \tfrac{3Y_{i+1}-4Y_i+Y_{i-1}}{2h} \tag{16b}$$
$$\overline{Y}'_{i-1} = \tfrac{-Y_{i+1}+4Y_i-3Y_{i-1}}{2h} \tag{16c}$$
$$\overline{\overline{Y}}'_i = \overline{Y}'_i - \tfrac{h}{20}(\overline{g}_{i+1} - \overline{g}_{i-1}) \tag{16d}$$

Then for each t_i,$i = 1, 2, ..., N-1$ (15) can be described as:

$$\frac{1}{h^2}\delta^2 Y_i = \frac{1}{12}(\overline{g}_{i+1} + 10\overline{\overline{g}}_i + \overline{g}_{i-1}) \tag{17}$$

Where

$$\overline{\overline{g}}_i = g(x_i, Y_i, \overline{\overline{Y}}'_i) \tag{18a}$$
$$and\ \overline{g}_{i\pm1} = g(x_{i\pm1}, Y_{i\pm1}, \overline{Y}'_{i\pm1}) \tag{18b}$$

Using (16) and (18), terms of the right hand side expressions of (17) can be simplified as:

$$\tfrac{1}{12}\overline{g}_{i+1} = \tfrac{\varepsilon H_{i+1}}{12} - (\tfrac{A_{i+1}}{8h} + \tfrac{A'_{i+1}}{12})Y_{i+1} + \tfrac{A_{i+1}}{6h}Y_i - \tfrac{A_{i+1}}{24h}Y_{i-1} \tag{19a}$$
$$\tfrac{10}{12}\overline{\overline{g}}_i = K_iY_{i-1} + L_iY_i + M_iY_{i+1} + \tfrac{10}{12}\varepsilon H_i + \tfrac{hA_i}{24}\varepsilon H_{i+1} - \tfrac{hA_i}{24}\varepsilon H_{i-1} \tag{19b}$$
$$\tfrac{1}{12}\overline{g}_{i-1} = \tfrac{\varepsilon H_{i-1}}{12} + \tfrac{A_{i-1}}{24h}Y_{i+1} - \tfrac{A_{i-1}}{6h}Y_i + (\tfrac{A_{i-1}}{8h} - \tfrac{A'_{i-1}}{12})Y_{i-1} \tag{19c}$$

Where

$$K_i = \tfrac{10A_i}{24h} - \tfrac{A_iA_{i+1}}{48} - \tfrac{A_iA_{i-1}}{16} + \tfrac{hA_iA'_{i-1}}{24}$$
$$L_i = \tfrac{A_i(A_{i+1}+A_{i-1})-10A'_i}{12}$$
$$M_i = -\tfrac{10A_i}{24h} - \tfrac{A_iA_{i+1}}{24h} - \tfrac{hA_iA'_{i+1}}{24} - \tfrac{A_iA_{i-1}}{48}$$

Now substituting (19) in (17) we get:

$$\frac{1}{h^2}(Y_{i-1} - 2Y_i + Y_{i+1}) = C_iY_{i-1} + D_iY_i + P_iY_{i+1} + S_i \tag{20}$$

Where

$$C_i = \tfrac{-A_{i+1}+10A_i}{24h} - \tfrac{A_iA_{i+1}}{48} - \tfrac{A_iA_{i-1}}{16} + \tfrac{hA_iA'_{i-1}}{24} + \tfrac{A_{i-1}}{8h} - \tfrac{A'_{i-1}}{12}$$
$$D_i = \tfrac{A_{i+1}}{6h} + \tfrac{A_i(A_{i+1}+A_{i-1})-10A'_i}{12} - \tfrac{A_{i-1}}{6h}$$
$$P_i = \tfrac{-A_{i+1}}{8h} - \tfrac{A'_{i+1}}{12} - \tfrac{10A_i}{24h} - \tfrac{A_iA_{i+1}}{16} - \tfrac{hA_iA'_{i+1}}{24} - \tfrac{A_iA_{i-1}}{48} + \tfrac{A_{i-1}}{24h}$$
$$S_i = \tfrac{\varepsilon(H_{i+1}+10H_i+H_{i-1})}{12} + \tfrac{\varepsilon hA_i(H_{i+1}-H_{i-1})}{20}$$

From equation (20) we get the recurrence relation of the form:

$$E_iY_{i-1} - F_iY_i + G_iY_{i+1} = R_i; i = 1, 2.., N \tag{21}$$

Where

$$E_i = \frac{1}{h^2} + \frac{(A_{i+1}-10A_i)}{24h} + \frac{A_iA_{i+1}}{48} + \frac{A_iA_{i-1}}{16} - \frac{hA_iA'_{i-1}}{24} - \frac{A_{i-1}}{8h} + \frac{A'_{i-1}}{12} \tag{22a}$$

$$F_i = \frac{2}{h^2} + \frac{(A_{i+1}-A_{i-1})}{6h} + \frac{A_i(A_{i+1}+A_{i-1})-10A'_i}{12} \tag{22b}$$

$$G_i = \frac{1}{h^2} + \frac{A_{i+1}}{8h} + \frac{A'_{i+1}}{12} + \frac{(10A_i-A_{i-1})}{24h} + \frac{hA_iA'_{i+1}}{24} + \frac{A_iA_{i+1}}{16} + \frac{A_iA_{i-1}}{48} \tag{22c}$$

$$R_i = \frac{\varepsilon(H_{i+1}+10H_i+H_{i-1})}{12} + \frac{\varepsilon hA_i(H_{i+1}-H_{i-1})}{24} \tag{22d}$$

Equation (21) gives a system of N equations with N+1 unknown's Y_1 to Y_N and the unwanted unknown Y_{N+1}. To eliminate the unknown Y_{N+1}, we make use of the equation (11b) given as boundary conditions in implicit form.
By employing the second order central difference approximation in (11b), we get

$$c_1Y_N + \frac{Y_{N+1} - Y_{N-1}}{2h} = c_2 \; and \; Y_{N+1} = Y_{N-1} - 2hc_1Y_N + 2hc_2 \tag{23}$$

Where c_1 and c_2 are defined in (9b). Making use of (23) in the last equation of the recurrence relation (21) at $i = N$, we get

$$(E_N + G_N)Y_{N-1} - (F_N + 2hc_1G_N)Y_N = R_N - 2hc_2G_N \tag{24}$$

Now, equations (21) and (24) give an N by N tri-diagonal system which can be solved by using Thomas Algorithm.
The outer region Problem: A similar approach to outer region problem (12)-(13) produces the recurrence relation

$$E_iY_{i-1} - F_iY_i + G_iY_{i+1} = R_i; i = 1, 2.., N-1 \tag{25}$$

Where

$$E_i = \frac{\varepsilon}{h^2} + \frac{(a_{i+1}-10a_i)}{24h} + \frac{a_ia_{i+1}}{48} + \frac{a_ia_{i-1}}{16} - \frac{ha_ia'_{i-1}}{24} - \frac{a_{i-1}}{8h} + \frac{a'_{i-1}}{12} \tag{26a}$$

$$F_i = \frac{2\varepsilon}{h^2} + \frac{(a_{i+1}-a_{i-1})}{6h} + \frac{a_i(a_{i+1}+a_{i-1})-10a'_i}{12} \tag{26b}$$

$$G_i = \frac{\varepsilon}{h^2} + \frac{a_{i+1}}{8h} + \frac{a'_{i+1}}{12} + \frac{(10a_i-a_{i-1})}{24h} + \frac{ha_ia'_{i+1}}{24} + \frac{a_ia_{i+1}}{16} + \frac{a_ia_{i-1}}{48} \tag{26c}$$

$$R_i = \frac{(h_{i+1}+10h_i+h_{i-1})}{12} + \frac{ha_i(h_{i+1}-h_{i-1})}{24} \tag{26d}$$

Where the interval $x_p \leq x \leq 1$ is subdivided in to N subintervals of equal mesh, $h = \frac{1-x_p}{N}$. To solve the tri diagonal system (25), we used Thomas Algorithm.

4. Numrical Examples

Example 4.1: Consider the following singular perturbation problem from Kevorkian and Cole [[9] Page 33 equations 2.3.26 and 2.3.27 with $\alpha = 0$].

$$\varepsilon y''(x) + y'(x) = 0, 0 \leq x \leq 1 \,; with\ y(0) = 0\ and\ y(1) = 1 \tag{27}$$

Integrating (27) we get $\varepsilon y'(x) + y(x) = K$. Using (4), the value of K is K=1. Using the transformation $t = \frac{x}{\varepsilon}$ The Inner region problem is: $Y''(t) + Y'(t) = 0, 0 \leq t \leq t_p$ with $Y(0) = 0$ and $Y(t_p) + Y'(t_p) = 1$. The outer region problem is: $\varepsilon y"(x) + y'(x) = 0, x_p \leq x \leq 1$, with $y(x_p = Y(t_p)$ and $y(1) = 1$. The exact solution is given by: $y(x) = \frac{1-exp(-x/\varepsilon)}{1-exp(-1/\varepsilon)}$. Numerical results are presented in tables 1a and 1b for ε =10^{-3} and ε=10^{-4} respectively.

Example 4.2: Consider the following singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt [13, Example 2].

$$\varepsilon y''(x) + y'(x) = 1 + 2x; x \in [0,1]\ with\ y(0) = 0\ and\ y(1) = 1 \tag{28}$$

Integrating (28) we get $\varepsilon y'(x) + y(x) = x + x^2 + K$. Using (4), the value of K is $K = -1$. Using the transformation $t = \frac{x}{\varepsilon}$ $c_1 = 1$ and $c_2 = \varepsilon t_p + \varepsilon^2 t_p^2 - 1$. The Inner region problem is: $Y''(t) + Y'(t) = \varepsilon(1 + 2\varepsilon t), 0 \leq t \leq t_p$ with $Y(0) = 0$ and $Y(t_p) + Y'(t_p) = \varepsilon t_p + \varepsilon^2 t_p^2 - 1$ The outer region problem is: $\varepsilon y"(x) + y'(x) = 1 + 2x, x_p \leq x \leq 1$, with $y(x_p = Y(t_p)$ and $y(1) = 1$. The exact solution is given by $y(x) = x(x+1-2\varepsilon) + (2\varepsilon - 1)(\frac{1-\exp(-x/\varepsilon)}{1-\exp(1/\varepsilon})$. The numerical results are given in tables 2(a), 2(b) for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively.

Example 4.3: Consider the following singular perturbation problem from Kevorkian and Cole [9, Page 33 equations 2.3.26 and 2.3.27 with $\alpha = -1/2$].

$$\varepsilon y''(x) + (1 - \frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0; x \in [0,1]\ with\ y(0) = 0\ and\ y(1) = 1 \tag{29}$$

Equation (29) can be rewritten in the form of (1) as:

$$\varepsilon y''(x) + [(1 - \frac{x}{2})y(x)]' = 0 \tag{30}$$

Integrating (30) we get $\varepsilon y'(x) - (1 - \frac{x}{2})y(x) = K$. Using (4), the value of K is $K = (1 - 1/2)y(1) = 1/2$. Using the transformation $t = \frac{x}{\varepsilon}$, $A(t) = 1 - \frac{\varepsilon t_p}{2}$ and $F(t) = 0$, $c_1 = A(t) = 1 - \frac{\varepsilon t_p}{2}$ and $c_2 = K = \frac{1}{2}$. The Inner region problem is $Y''(t) + (1 - \frac{\varepsilon t}{2})Y'(t) - \frac{\varepsilon}{2} = 0, 0 \leq t \leq t_p$ with $Y(0) = 0$ and $(1 - \frac{\varepsilon t_p}{2})Y(t_p) + Y'(t_p) = 0$. The outer region problem is $\varepsilon y''(x) + (1 - \frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0, x_p \leq x \leq 1$, with $y(x_p = Y(t_p)$ and $y(1) = 1$. The exact solution is given by $y(x) = \frac{1}{2-x} - \frac{1}{2}\exp(-\frac{(x - \frac{x^2}{4})}{\varepsilon})$. The numerical results are given in tables 3(a), 3(b) for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively.

5. More General Class of Problems

The present method is extended to more general class of singular perturbation problems of the form:

$$\varepsilon y''(x) + [a(x)y(x)]' + b(x)y(x) = h(x); for x\varepsilon[0,1] \tag{31}$$

with

$$y(0) = \alpha, \tag{32a}$$

$$and, \quad y(1) = \beta \tag{32b}$$

where ε is a small positive parameter$(0 < \varepsilon << 1)$ and $\alpha and \beta$ are known constants and $a(x), b(x)$ *and* $h(x)$ are assumed to be sufficiently continuously differentiable functions in [0,1] and $a(x) \geq M > 0$throughout the interval [0,1], where M is some positive constant.
In these types of problems, we can't apply the integration process in Step 2 of the present method due to the presence of the term $b(x)y(x)$. So, we need to modify equation (31) to the form of equation (1) and then apply the present method. We treated as follows:
Let y_0 be the solution of the reduced problem of (31)-(32); that is

$$[[a(x)y_0(x)]' + b(x)y_0(x) = h(x) \tag{33}$$

with

$$y_0(1) = \beta \tag{34}$$

Next, set up the approximate equation to the given equation (31) as:

$$\varepsilon y''(x) + [a(x)y(x)]' + b(x)y_0(x) = h(x) \tag{35}$$

Where we have simply replaced the y(x) term by y0(x)(the solution of the reduced problem (31-32)). Then equation (35) can be rewritten in the form of equation (1) as:

$$\varepsilon y''(x) + [a(x)y(x)]' = s(x) \tag{36}$$

Where $s(x) = h(x) - b(x)y_0$
Finally, we apply the present method to solve the modified problem (36)-(32) and justified by solving the following two problems.

Example 5.1: Consider the following singular perturbation problem from Bender and Orszag [[1], page 480, Problem 9.17 with α=0].

$$\varepsilon y''(x) + y'(x) - y(x) = 0; for x\varepsilon[0,1] \ with \ y(0) = 1 \ and \ y(1) = 1 \tag{37}$$

The modified form of equation (37) is :

$$\varepsilon y''(x) + y'(x) - y_0(x) = 0 \tag{38}$$

Where $y_0(x) = \exp(x - 1)$ is the solution of the reduced problem of (37)
That is'

$$y_0'(x) - y_0(x) = 0 \ with \ y_0(0) = 1 \tag{39}$$

Then we rewrite equation (37) in the form of (1) as:

$$\varepsilon y''(x) + y'(x) = \exp(x-1) \tag{40}$$

By integrating (40), we get

$$\varepsilon y'(x) + y(x) = \exp(x-1) + K \tag{41}$$

Using equation (4) we determine K as $K = a(1)y(1) - f(1) = (1)(1) - \exp(1-1) = 0$

Inner region problem: using the transformation $t = \frac{x}{\varepsilon}$, we get the inner region problem:

$$Y''(t) + Y'(t) = \exp(\varepsilon t - 1),\ \ for\ 0 \le t \le t_p \tag{42}$$

with

$$y(0) = 1, \tag{43a}$$

$$Y(t_p) + Y'(t_p) = \exp(\varepsilon t_p - 1) \tag{43b}$$

Outer region problem:

$$\varepsilon y''(x) + y'(x) = \exp(x-1),\ \ for\ x_p \le x \le 1 \tag{44}$$

with $y(x_p = Y(t_p)\ and\ y(1) = 1$

The exact solution is given by: $y(x) = \frac{(\exp(m_2-1)\exp(m_1 x)) + (1-\exp(m_1))\exp(m_2 x)}{\exp(m_2) - \exp(m_1)}$

where $m_1 = \frac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon}$ and $m_2 = \frac{-1-\sqrt{1+4\varepsilon}}{2\varepsilon}$

The numerical results are given in tables 4(a), 4(b) for $\varepsilon = 10^{-3} and\ \varepsilon = 10^{-4}$ respectively.

Example 5.2: Consider the following singular perturbation problem from which has earlier been solved by Reinhardt [13].

$$\varepsilon y''(x) + y'(x) + y(x) = 0; for\ x\varepsilon[0,1]\ with\ y(0) = 1\ and\ y(1) = 2 \tag{45}$$

The modified form of equation (45) is :

$$\varepsilon y''(x) + y'(x) + y_0(x) = 0 \tag{46}$$

Where $y_0(x) = 2\exp(1-x)$ is the solution of the reduced problem of(46)
That is'

$$y_0'(x) + y_0(x) = 0\ with\ y_0(1) = 2 \tag{47}$$

Then we rewrite equation (45) in the form of (1) as:

$$\varepsilon y''(x) + y'(x) = -2\exp(1-x) \tag{48}$$

By integrating (48), we get

$$\varepsilon y'(x) + y(x) = 2\exp(1-x) + K \tag{49}$$

Using equation (4) we determine K as $K = y(1) - 2\exp(1-1) = 0$
Inner region problem: using the transformation $t = \frac{x}{\varepsilon}$, we get the inner region problem:

$$Y''(t) + Y'(t) = \varepsilon(-2\exp(1-\varepsilon t)),\ for\ 0 \le t \le t_p \tag{50}$$

with

$$y(0) = 1, \tag{51a}$$

$$and,\ \ Y(t_p) + Y'(t_p) = 2\exp(1-\varepsilon t_p) \tag{51b}$$

Outer region problem:

$$\varepsilon y''(x) + y'(x) = -2\exp(1-x),\ for\ x_p \le x \le 1 \tag{52}$$

with $y(x_p = Y(t_p)\ and\ y(1) = 2$
The exact solution is given by: $y(x) = \frac{(2-\exp(m_2)\exp(m_1 x))+(1-\exp(m_1-2))\exp(m_2 x)}{\exp(m_1)-\exp(m_2)}$
where $m_1 = \frac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon}$ and $m_2 = \frac{-1-\sqrt{1+4\varepsilon}}{2\varepsilon}$
The numerical results are given in tables 5(a), 5(b) for $\varepsilon = 10^{-3} and\ \varepsilon = 10^{-4}$ respectively.

6. Conclusions

We have described the method of asymptotic inner boundary condition for the numerical solution of a class of singular perturbation problems. As mentioned the method is iterative on the terminal point and the process is to be repeated for different values of (the terminal point which is not unique), until the solution profile stabilizes in both the inner and outer region. We have implemented the present method first on three problems with left boundary layer and extended to two more general classes of problems, by taking different values of .We have tabulated error tables for the numerical results. From the table, it is observed that the present method approximates the exact solution very well.

Table 1. Maximum Errors for Example 4.1

	$t_p = 1$		$t_p = 5$		$t_p = 10$	
ε	Inner	Outer	Inner	Outer	Inner	Outer
10^{-3}	1.01E-05	5.36E-05	2.70E-05	5.36E-05	1.19E-03	5.30E-05
10^{-4}	1.01E-05	5.36E-04	2.70E-05	5.36E-04	1.19E-03	5.36E-04

Table 2. Maximum Errors for Example 4.2

	$t_p = 1$		$t_p = 5$		$t_p = 10$	
ε	Inner	Outer	Inner	Outer	Inner	Outer
10^{-3}	1.88E-03	5.38E-04	2.96E-03	5.38E-04	2.56E-03	5.38E-04
10^{-4}	6.68E-04	9.91E-05	1.63E-03	1.08E-04	2.39E-03	1.08E-04

Table 3. Maximum Errors for Example 4.3

	$t_p = 1$		$t_p = 5$		$t_p = 10$	
ε	Inner	Outer	Inner	Outer	Inner	Outer
10^{-3}	8.98E-04	2.19E-03	3.87E-04	2.19E-03	3.87E-04	2.19E-03
10^{-4}	5.63E-04	1.25E-04	6.58E-04	1.25E-04	1.74E-03	1.25E-04

Table 4. Maximum Errors for Example 5.1

	$t_p = 1$		$t_p = 5$		$t_p = 10$	
ε	Inner	Outer	Inner	Outer	Inner	Outer
10^{-3}	2.38E-04	1.54E-03	7.16E-04	1.54E-03	1.16E-03	1.54E-03
10^{-4}	2.84E-05	3.30E-04	7.61E-05	3.30E-04	5.11E-04	3.30E-04

Table 5. Maximum Errors for Example 5.2

	$t_p = 1$		$t_p = 5$		$t_p = 10$	
ε	Inner	Outer	Inner	Outer	Inner	Outer
10^{-3}	1.58E-03	3.69E-03	1.60E-03	3.69E-03	6.28E-03	3.69E-03
10^{-4}	1.07E-04	1.89E-03	1.21E-04	1.89E-03	6.47E-03	1.89E-03

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