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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEM WITH CONCAVE-CONVEX NONLINEARITIES[†]

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Abstract. In this paper, our main purpose is to establish the existence of weak solutions of a class of p-q-Laplacian system involving concave-convex nonlinearities:

$$\begin{cases} -\triangle_p u - \triangle_q u = \lambda V(x)|u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega \\ -\triangle_p v - \triangle_q v = \theta V(x)|v|^{r-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega \\ u = v = 0, \quad x \in \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $\lambda, \theta > 0$, and $1 < \alpha, \beta$, $\alpha + \beta = p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $\triangle_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian of u. when $1 < r < q < p < N$, we prove that there exist infinitely many weak solutions. We also obtain some results for the case $1 < q < p < r < p^*$. The existence results of solutions are obtained by variational methods.

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1. Introduction

In this paper, we are interested in finding multiple nontrivial weak solutions to the following nonlinear elliptic system of p-q-Laplacian type with concave-convex nonlinearities

$$\begin{cases} -\triangle_p u - \triangle_q u = \lambda V(x)|u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega \\ -\triangle_p v - \triangle_q v = \theta V(x)|v|^{r-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega \\ u = v = 0, \quad x \in \partial\Omega \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbf{R}^N$, $\lambda, \theta > 0$, and $1 < r < q < p < N$, $1 < \alpha, \beta$, $\alpha + \beta = p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $\triangle_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian of u.

When $u = v$, $\alpha = \beta$ and $\lambda = \theta$, System (1.1) reduce to the p-q-Laplacian equations:

$$\begin{cases} -\triangle_p u - \triangle_q u = \lambda V(x)|u|^{r-2}u + |u|^{p^*-2}u, & x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{cases} \tag{1.2}$$

Problem (1.2) comes, for example, from a general reaction-diffusion system

$$u_t = \operatorname{div}[H(u)\nabla u] + c(x, u) \tag{1.3}$$

where $H(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in physics and related science such as biophysics, plasma physics and chemical reaction design. Typically, in chemical and biological applications, the reaction term $c(x, u)$ has a polynomial form with respect to the concentration u.

Recently, the stationary solution of (1.3) was studied by many authors, that is many works considered the solutions of the following problem

$$-\operatorname{div}[H(u)\nabla u] = c(x, u). \tag{1.4}$$

for example,see [6.19-21,26].

If $p = q = 2$, (1.2) can be reduced to

$$\begin{cases} -\triangle u = \lambda V(x)|u|^{r-2}u + |u|^{2^*-2}u, & x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{cases} \tag{1.5}$$

which is a normal Schrodinger equation and has been widely studied, see[10-12,23].

The solutions of problem (1.5) corresponds to the critical points of the energy functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{r}\int_\Omega V(x)|u|^r dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx$$

defined on $W_0^{1,2}(\Omega)$. When $r = 2$, the pioneer result of Brezis-Nirenberg [8] studied problem (1.5) and shows that under some suitable conditions, probnlem (1.5) possesses a positive solution in $W_0^{1,2}(\Omega)$. For more results see [9,17] and reference therein.

The typically difficulty in dealing with problem (1.5) is that the corresponding functional $I(u)$ doesn't satisfy (PS) condition due to the lack of compactness of the embedding: $H_0^1 \hookrightarrow L^{2^*}(\Omega)$. Hence we couldn't use the standard variational methods.

However, if $1 < r < 2$, the situation is quite different, see [5,25]. The main essence is that when $1 < r < 2$, the functional $I(u)$ is sublinear, when λ is small enough, $I(u)$ satisfies $(PS)_c$ condition for $c < 0$, so we can look for critical points of negative critical values of $I(u)$.

For the general p-Laplacian problem

$$\begin{cases} -\triangle_p u = \lambda V(x)|u|^{r-2}u + |u|^{p^*-2}u, & x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{cases} \tag{1.6}$$

which is a special case of (1.2) when $p = q$. Problem (1.6) was also studied by many authors, many results valid for problem (1.5) has been extended to problem (1.6). For example, see [4,18,27]. The main difficulty in extending the results for problem (1.5) to the corresponding results for problem (1.6) is that $W_0^{1,p}(\Omega)$ is not a Hilbert space in general, then more analysis is needed.

We recall some results about problem (1.4) now. In [26], M.Wu and Z.Yang proved the existence of a nontrivial solution to problem (1.4) with

$$c(x,u) = a(x)|u|^{p-2}u + b(x)|u|^{q-2}u - f(x,u)$$

in the whole space $\mathbf{R}^N$, where $a(x), b(x)$ are positive functions, also when $a(x) \equiv m, b(x) \equiv n$ are positive constants, it was proved in [19] that problem (1.4) has a nontrivial solution. Recently in [20], G.Li and G.Zhang studied problem (1.4) involving critical exponent with

$$c(x,u) = |u|^{p^*-2}u + \theta|u|^{r-2}u$$

by using Lusternik-Schnirelman's theory(see also in [4]). Other results see [6,21] and reference therein.

At the same time, much attention has been paid to the existence of solutions for elliptic systems. especially for the following case

$$\begin{cases} -\triangle_p u = \lambda|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega \\ -\triangle_p v = \theta|v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega \\ u = v = 0 \quad x \in \partial\Omega \end{cases} \tag{1.7}$$

where $\alpha + \beta = p^*$. In fact system (1.7) is a special case of (1.1) when $p = q$. When $p = 2$ and $q = 2$, Alves et al [2] considered (1.7) and proved the existence of least energy solutions for any $\lambda, \theta \in (0, \lambda_1)$ and generalized the corresponding results of [8] to the case of system (1.7), here λ_1 denote the first eigenvalue of operator $-\triangle$. Subsequently, Han [14] considered the existence of multiple positive solutions for(1.7) and in [16] T.S.Hsu studied system (1.7) when $1 < q < p < N, \alpha + \beta = p^*$, more results see [15,24] etc..

However, as far as we know, there are few results on problem (1.1) with concave-convex nonlinearities. Motivated by [4,16,20], we shall extend the results of the above to problem (1.1). Let us denote the Banach space $H = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ in this paper, and for the functions $V(x)$, we add the following assumptions:

(V_0) Suppose $V(x) \in L^{\frac{p^*}{p^*-r}}(\Omega)$ and $V(x) > \sigma > 0$ in Ω.

Then we have the following results:

Theorem 1.1. *Assume $1 < r < q < p < N$, and (V_0) hold. Then there is a positive constant Λ^* such that for any $0 < (\lambda+\theta) \leq \Lambda^*$, problem (1.1) possesses infinitely many weak solutions in H.*

In the present parer, we also consider problem (1.1) for the case:$1 < q < p \leq r < p^*$, and obtain the following theorem:

Theorem 1.2. *If $1 < q < p \leq r < p^*$ and (V_0) hold, then there is a $\Lambda_* > 0$, such that for any $(\lambda + \theta) > \Lambda_*$, problem (1.1) has a nontrivial solution.*

Remark 1.3. In [4], J.G.Azvrero and I.P.Aloson obtained that there exist a nontrivial solution for (1.6) with $V(x) \equiv 1$ by the Mountain Pass Lemma. In fact, Theorem 1.2 is an extension of Theorem 3.2 in [4] to p-q-Laplacian system (1.1).

The present paper is organized as follows, in section 2, we give some preliminary results; in section 3, we will prove the main result, Theorem1.1.; and we will study (1.1) for the case $1 < q < p \leq r < p^*$, and prove Theorem 1.2 in section 4.

2. Preliminaries results

Let H' be dual of H, $\langle , \rangle$ the duality paring between H' and H, the norm on H is given by

$$\|z\|_p = \|(u,v)\|_p = (\|u\|_p^p + \|v\|_p^p)^{\frac{1}{p}}$$

and the norm on $L^p(\Omega) \times L^p(\Omega)$ is given by

$$|z|_p = |(u,v)|_p = (|u|_p^p + |v|_p^p)^{\frac{1}{p}}$$

where $z = (u,v) \in H$ and $\|\cdot\|_p$, $|\cdot|_p$ are the norm on $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$ respectively, that is,

$$\|u\|_p = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}, \quad |u|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}.$$

Throughout this paper, we denote weak converge by $\rightharpoonup$, and denote strong converge by $\to$, also we denote positive constants(possibly different) by C_i.

As usually, we also denote by

$$S_{\alpha+\beta} = \inf_{u \in W^{1,p}(\Omega)\backslash\{0\}} \frac{\|u\|^p}{(\int_\Omega |u|^{\alpha+\beta} dx))^{\frac{p}{\alpha+\beta}}} \tag{2.1}$$

and

$$S_{\alpha,\beta} = \inf_{z \in H\backslash\{0\}} \frac{\|z\|^p}{(\int_\Omega |u|^\alpha |v|^\beta dx))^{\frac{p}{\alpha+\beta}}}. \tag{2.2}$$

Easily, we have $\int_\Omega |u|^\alpha |v|^\beta dx \leq S_{\alpha,\beta}^{-\frac{\alpha+\beta}{p}} \|z\|^{\alpha+\beta}$ and

Lemma 2.1. *Assume $1 < \alpha, \beta$ and $\alpha + \beta \leq p^*$, $\Omega \in \mathbf{R}^N (N \geq 3)$ be a domain (not necessarily bounded). Then we have*

$$S_{\alpha,\beta} = [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}}] S_{\alpha+\beta}.$$

Proof. The proof of Lemma 2.1. is essentially given in [2] when $p = 2$, modifying the proof of [2], we can deduce our result. For the readers' convenience, we give a sketch here.

Suppose $\{w_n\}$ is a minimizing sequence for $S_{\alpha+\beta}$, let $u_n = sw_n, v_n = tw_n$, where $s, t > 0$ will be chosen later. Then from (2.2), we infer that

$$S_{\alpha,\beta} \leq \frac{s^p + t^p}{(s^\alpha t^\beta)^{\frac{p}{\alpha+\beta}}} \frac{\|w_n\|^p}{(\int_\Omega |w_n|^{\alpha+\beta} dx)^{\frac{p}{\alpha+\beta}}} = [(\frac{s}{t})^{\frac{p\beta}{\alpha+\beta}} + (\frac{s}{t})^{\frac{p\alpha}{\alpha+\beta}}] \frac{\|w_n\|^p}{(\int_\Omega |w_n|^{\alpha+\beta} dx)^{\frac{p}{\alpha+\beta}}} \quad (2.3)$$

Define the function

$$h(x) = x^{\frac{p\beta}{\alpha+\beta}} + x^{-\frac{p\alpha}{\alpha+\beta}}, x > 0.$$

By a direct calculation, the minimum of the function h is achieved at the point $x_0 = (\frac{\alpha}{\beta})^{\frac{1}{p}}$ with the minimum value

$$h(x_0) = (\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}}.$$

Thus, choosing $s, t > 0$ in (2.3) such that $\frac{s}{t} = (\frac{\alpha}{\beta})^{\frac{1}{p}}$, we obtain

$$S_{\alpha,\beta} \leq [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}}] S_{\alpha+\beta}.$$

To complete the proof, let $z_n = (u_n, v_n)$ be a minimizing sequence for $S_{\alpha,\beta}$. Define $\omega_n = t_n v_n$ for some $t_n > 0$ such that

$$\int_\Omega |u_n|^{\alpha+\beta} dx = \int_\Omega |\omega_n|^{\alpha+\beta} dx.$$

Then we have

$$\begin{aligned} \int_\Omega |u_n|^\alpha |\omega_n|^\beta dx &\leq \frac{\alpha}{\alpha+\beta} \int_\Omega |u_n|^{\alpha+\beta} dx + \frac{\beta}{\alpha+\beta} \int_\Omega |\omega_n|^{\alpha+\beta} dx \\ &= \int_\Omega |u_n|^{\alpha+\beta} dx = \int_\Omega |\omega_n|^{\alpha+\beta} dx. \end{aligned}$$

Therefore, we deduce from the above inequality that

$$\begin{aligned} \frac{\|z_n\|^p}{(\int_\Omega |u_n|^\alpha |v_n|^\beta dx))^{\frac{p}{\alpha+\beta}}} &= t_n^{\frac{p\beta}{\alpha+\beta}} \frac{\|z_n\|^p}{(\int_\Omega |u_n|^\alpha |\omega_n|^\beta dx))^{\frac{p}{\alpha+\beta}}} \\ &\geq t_n^{\frac{p\beta}{\alpha+\beta}} \frac{\|u_n\|^p}{(\int_\Omega |u_n|^{\alpha+\beta} dx))^{\frac{p}{\alpha+\beta}}} + t_n^{\frac{p\beta}{\alpha+\beta}-p} \frac{\|\omega_n\|^p}{(\int_\Omega |\omega_n|^{\alpha+\beta} dx))^{\frac{p}{\alpha+\beta}}} \\ &\geq h(t_n) S_{\alpha+\beta} \\ &\geq h(t_0) S_{\alpha+\beta}. \end{aligned}$$

Passing to the limit in the above inequality, we obtain

$$S_{\alpha,\beta} \geq [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}}] S_{\alpha+\beta}.$$

That's end the proof of Lemma 2.1. □

Let's study the energy functional associated with problem (1.1) defined by

$$E(z)=E(u,v)=\frac{1}{p}\int_\Omega|\nabla u|^p+|\nabla v|^p dx+\frac{1}{q}\int_\Omega|\nabla u|^q+|\nabla v|^q dx$$

$$-\frac{1}{r}\int_\Omega\lambda V(x)|u|^r+\theta V(x)|u|^r dx-\frac{2}{\alpha+\beta}\int_\Omega|u|^\alpha|v|^\beta dx.$$

Obviously, $E(z)$ is even and it is well known that $E(z)\in C^1(H,R)$ and nontrival critical points of $E(z)$ are weak solutions of problem (1.1). By a weak solution of (1.1)we mean that $(u,v)\in H$ satisfying

$$\int_\Omega(|\nabla u|^{p-2}\nabla u\nabla\varphi+|\nabla v|^{p-2}\nabla v\nabla\psi)dx+\int_\Omega(|\nabla u|^{q-2}\nabla u\nabla\varphi+|\nabla v|^{q-2}\nabla v\nabla\psi)dx$$

$$-\lambda\int_\Omega V(x)|u|^r\varphi dx-\theta\int_\Omega V(x)|v|^r\psi dx$$

$$-\frac{2\alpha}{\alpha+\beta}\int_\Omega|u|^{\alpha-2}uv^\beta\varphi dx-\frac{2\beta}{\alpha+\beta}\int_\Omega|u|^\alpha v^{\beta-2}v\psi dx=0.$$

for all $(\varphi,\psi)\in E$.

Now, we define the Palais-Smale(PS)-sequence, (PS)-value, and (PS)-conditions in H for E as follows.

Definition 2.2. (I) For $c\in R$, a sequence $\{z_n\}\in H$ is a (PS)$_c$-sequence for E if $E(z_n)=c+o(1)$ and $E'(z_n)=o(1)$ strongly in H' as $n\to\infty$.

(II) $c\in R$ is a (PS)-value for E if there exists a (PS)$_c$-sequence in H for E.

(III) E satisfies the (PS)$_c$-condition in H for E if every (PS)$_c$-sequence in H for E contains a convergent sub-sequence.

Now we give some results for the proof of Theorem 1.1.

Lemma 2.3. *If $\{z_n\}\subset H$ is a (PS)$_c$ secquence for E, then $\{z_n\}$ is bounded in H.*

Proof. Modified the proof of Lemma 2.3 in [16], we can obtain the results. □

Lemma 2.4. *If $\{z_n\}\subset H$ is a (PS)$_c$ secquence for E, then there exists $z\in H$ and $M>0$ such that*

$$E(z)\geq -M(\lambda+\theta)^{\frac{q}{q-r}},$$

where M will be given later.

Proof. Similar to the proof of Lemma 2.2 in [16]. □

Lemma 2.5. *E satisfies the (PS)$_c$ condition with c satisfying*

$$c\leq\frac{2}{N}(\frac{S_{\alpha,\beta}}{2})^{\frac{N}{p}}-M(\lambda+\theta)^{\frac{q}{q-r}}.$$

Proof. Suppose $\{z_n\} \subset H$ is a $(\mathrm{PS})_c$ sequence of E, i.e.,

$$E(z_n) = c + o(1), E'(z_n) = o(1), \tag{2.4}$$

by Lemma 2.3, we may assume there exist a $z \in H$, $E'(z) = 0$, and extracting a subsequence such that $z_n \rightharpoonup z$ in H, Thus we have that

$$u_n \to u,\ v_n \to v \ \text{ in } \ L^s(\Omega),\ 1 \le s < p^*$$

and $u_n \to u, v_n \to v$ a.e. on Ω. Hence we have

$$\int_\Omega \lambda V(x)|u_n|^r + \theta V(x)|v_n|^r dx = \int_\Omega \lambda V(x)|u|^r + \theta V(x)|v|^r dx + o(1).$$

Let $\widetilde{v}_n = u_n - u, \widetilde{v}_n = v_n - v$ and $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$. Then by Brezis-Lieb's Lemma(see[7]), we deduce that

$$\|\widetilde{z}_n\|_p^p = \|z_n\|_p^p - \|z\|_p^p + o(1),\ \|\widetilde{z}_n\|_q^q = \|z_n\|_q^q - \|z\|_q^q + o(1). \tag{2.5}$$

By an argument of Han [15,Lemma 2.1], we obtain

$$\int_\Omega |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = \int_\Omega |u_n|^\alpha |v_n|^\beta dx - \int_\Omega |u|^\alpha |v|^\beta dx + o(1). \tag{2.6}$$

Together with (2.4)-(2.6), we have that

$$\frac{1}{p}\|\widetilde{z}_n\|_p^p + \frac{1}{q}\|\widetilde{z}_n\|_q^q - \frac{1}{r}\int_\Omega (\lambda V(x)|u|^r + \theta V(x)|v|^r)dx - \frac{2}{p^*}\int_\Omega |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx$$
$$+\frac{1}{p}\|z\|_p^p + \frac{1}{q}\|z\|_q^q - \frac{2}{p^*}\int_\Omega |u|^\alpha |v|^\beta dx = c + o(1).$$

and

$$\|\widetilde{z}_n\|_p^p + \|\widetilde{z}_n\|_q^q - \int_\Omega (\lambda V(x)|u|^r + \theta V(x)|v|^r)dx - 2\int_\Omega |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx$$
$$+\|z\|_p^p + \|z\|_q^q - 2\int_\Omega |u|^\alpha |v|^\beta dx = o(1).$$

Or we have

$$\frac{1}{p}\|\widetilde{z}_n\|_p^p + \frac{1}{q}\|\widetilde{z}_n\|_q^q - \frac{2}{p^*}\int_\Omega |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = c - E(z) + o(1). \tag{2.7}$$

and

$$\|\widetilde{z}_n\|_p^p + \|\widetilde{z}_n\|_q^q - \frac{2}{p^*}\int_\Omega |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx = o(1). \tag{2.8}$$

Hence, we may suppose that

$$\|\widetilde{z}_n\|_p^p \to a,\ \|\widetilde{z}_n\|_q^q \to b,\ 2\int_\Omega |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta dx \to l,$$

if $a = 0$, then we have $z_n \to z$ in H, we complete the proof. On the contrary, we ssume $a > 0$, then from (2.2) and (2.8), we obtain

$$a \le l \le 2(S_{\alpha,\beta})^{-\frac{p^*}{p}} a^{\frac{p^*}{p}},$$

which implies that $a \ge 2(\frac{S_{\alpha,\beta}}{2})^{\frac{N}{p}}$.

On the other hand, from (2.7) we have that

$$\begin{aligned} c &= \frac{a}{p}+\frac{b}{q}-\frac{l}{p^*}+E(z) \\ &= (\frac{1}{p}-\frac{1}{p^*})a+(\frac{1}{q}-\frac{1}{p^*})b+E(z) \\ &> \frac{2}{N}(\frac{S_{\alpha,\beta}}{2})^{\frac{N}{p}}-M(\lambda+\theta)^{\frac{q}{q-r}} \end{aligned}$$

which contradicts $c \leq \frac{2}{N}(\frac{S_{\alpha,\beta}}{2})^{\frac{N}{p}}-M(\lambda+\theta)^{\frac{q}{q-r}}$. □

The following is the classical Deformation Lemma:

Lemma 2.6 *(see[1]). Let $f \in C^1(X,R)$ and satisfy (PS) condition. If $c \in R$ and N is any neighborhood of $K_c \doteq \{u \in X | f(u)=c, f'(u)=0\}$, there exists $\eta(t,x) \equiv \eta_t(x) \in C([0,1]\times X, X)$ and constants $\overline{\epsilon} > \epsilon > 0$ such that*

(1) $\eta_0(x)=x$ for all $x \in X$,

(2) $\eta_t(x)=x$ for all $x \overline{\in} f^{-1}[c-\overline{\epsilon}, c+\overline{\epsilon}]$,

(3) $\eta_t(x)$ is a homeomorphism of X onto X for all $t \in [0,1]$,

(4) $f(\eta_t(x)) \leq f(x)$ for all $x \in X, t \in [0,1]$,

(5) $\eta_1(A_{c+\epsilon}-N) \subset A_{c+\epsilon}$, where $A_c=\{x \in X | f(x) \leq c\}$ for any $c \in R$,

(6) if $K_c=\emptyset, \eta_1(A_{c+\epsilon}) \subset A_{c-\epsilon}$,

(7) if f is even, η_t is odd in x.

Remark 2.7. Lemma 2.6 is also true if f satisfies $(PS)_c$ condition for $c < c_0$ for some $c_0 \in R$.

At the end of this section, we recall some concepts in minimax theory. Let X be a Banach space, and

$$\Sigma=\{A \subset X \setminus \{0\} | A \text{ is closed}, \ -A=A\},$$

and

$$\Sigma_k=\{A \in \Sigma | \gamma(A) \geq k\},$$

where $\gamma(A)$ is the Z_2 genus of A, that is

$$\gamma(A)=\begin{cases} \inf\{n: \text{there exist odd, continuous } h: A \to R^n \setminus \{0\}\}, \\ +\infty, \ \text{if it doesn't exist odd, continuous } h: A \to R^n \setminus \{0\}, \forall n \in Z_+, \\ 0, \ \text{if } A=\emptyset. \end{cases}$$

The main properties of genus are contained in the following lemma.

Lemma 2.8 *(see[22]). Let $A, B \in \Sigma$. Then*

(1) If there exists $f \in C(A,B)$, odd, then $\gamma(A) \leq \gamma(B)$.

(2) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

(3) If there exists an odd homeomorphism between A and B, then $\gamma(A)=\gamma(B)$.

(4) If S^{N-1} is the sphere in R^N, then $\gamma(S^{N-1})=N$.

(5) $\gamma(A \cup B) \leq \gamma(A)+\gamma(B)$.

(6) If $\gamma(A) < \infty$, then $\gamma(\overline{A-B}) \geq \gamma(A)-\gamma(B)$.

(7) If A is compact, then $\gamma(A) < \infty$, and there exists $\delta > 0$ such that $\gamma(A)=\gamma(N_\delta(A))$, where $N_\delta(A)=\{x \in X | d(x,A) \leq \delta\}$.

(8) If X_0 is a subspace of X with codimension k, and $\gamma(A) > k$, then $A \cap X_0 \neq \emptyset$.

3. Proof of Theorem 1.1

We will prove the existence of infinitely many solutions for system (1.1) in this section. We try to use Lusternik-Schnirelman's theory for Z_2-invariant functional (see [22]). But since the functional $E(z)$ defined in section 2 is not bounded from below, so we following [4](or see [20]) to consider a truncated functional $E_\infty(z)$ which will be constructed later.

At first, let's consider the functional $E(z)$, using the Sobolev's inequality with the hypothesis $1 < r < q < p < N$, we obtain

$$\begin{aligned} E(z) &\geq \frac{1}{p}\|z\|_p^p - \frac{1}{r}\int_\Omega \lambda V(x)|u|^r + \theta V(x)|u|^r dx - \frac{2}{p^*}\int_\Omega |u|^\alpha |v|^\beta dx \\ &\geq \frac{1}{p}\|z\|_p^p - \frac{2}{p^* S_{\alpha,\beta}^{\frac{p^*}{p}}}\|z\|_p^{p^*} - \frac{1}{r}S_p^{-\frac{r}{p}}|V(x)|_{\frac{p^*}{p^*-r}}(\lambda+\theta)\|z\|_p^r \\ &= C_3\|z\|_p^p - C_4\|z\|_p^{p^*} - C_5(\lambda+\theta)\|z\|_p^r \end{aligned}$$

where $C_3 = \frac{1}{p}, C_4 = \frac{2}{p^* S_{\alpha,\beta}^{\frac{p^*}{p}}}, C_5 = \frac{1}{r}S_p^{-\frac{r}{p}}|V(x)|_{\frac{p^*}{p^*-r}}$ are all positive constants.

We now consider function

$$h(x) = C_3x^p - C_4x^{p^*} - C_5(\lambda+\theta)x^r,\ x > 0$$

by the hypothesis $1 < r < p < p^*$, we easily know that there exists a $\Lambda^* > 0$ such that for any $0 < (\lambda+\theta) \leq \Lambda^*$, we have the following results hold:

(a) $h(x)$ reaches its positive maximum;

(b) $\frac{2}{N}(\frac{S_{\alpha,\beta}}{2})^{\frac{N}{p}} - M(\lambda+\theta)^{\frac{q}{q-r}} \geq 0$, where M is given in Lemma 2.4.

From the structure of $h(x)$, we see that there are two positive solutions $R_1 < R_2$ of $h(x) = 0$. Then we can easily know that

$$h(x)\begin{cases} < 0, \ x \in (0,R_1)\cup(R_2,\infty) \\ > 0, \ x \in (R_1,R_2) \end{cases} \tag{3.1}$$

We let $\tau : R^+ \to [0,1]$ be C^∞ and nonincreasing function such that

$$\tau(x) = 1, \ \text{ if } x \in (0,R_1)$$
$$\tau(x) = 0, \ \text{ if } x \in (R_2,\infty).$$

Let $\varphi(u) = \tau(\|u\|_p)$, we consider the truncated functional

$$\begin{aligned} E_\infty(z) = \frac{1}{p}\|z\|_p^p + \frac{1}{q}\|z\|_q^q - \frac{1}{r}\int_\Omega \lambda V(x)|u|^r + \theta V(x)|v|^r dx \\ -\frac{2}{p^*}\int_\Omega |u|^\alpha|v|^\beta\varphi(u)dx. \end{aligned}$$

similar as above, we consider the function

$$\overline{h}(x) = C_3x^p - C_4x^{p^*}\tau(x) - C_5(\lambda+\theta)x^r,$$

and have that

$$E_\infty(z) \geq \overline{h}(\|z\|_p) \qquad (3.2).$$

By further analysis, we can see $\overline{h}(x) \geq h(x)$, for all $x \in (0, \infty)$; and $\overline{h}(x) = h(x)$, for $x \in (0, R_1]$; and $\overline{h}(x) \geq 0$, for $x \in [R_2, \infty)$. So we have that $E(z) = E_\infty(z)$ when $\|z\|_p \in (0, R_1]$, and since $\tau \in C^\infty$, we get $E_\infty(z) \in C^1(H, R)$. Also we obtain the following results.

Lemma 3.1. *(1) If $E_\infty(z) < 0$, then $\|z\|_p \in (0, R_1)$, and $E(w) = E_\infty(w)$ for all w in a small enough neighborhood of z.*

(2) There exists a $\Lambda^* > 0$, such that when $0 < (\lambda + \theta) \leq \Lambda^*$, $E_\infty(z)$ satisfies the $(PS)_c$ condition for $c < 0$.

Proof. We prove (1) by contradiction, assume $E_\infty(z) < 0$ and $\|z\|_p \in [R_1, \infty)$. Then if $\|z\|_p \in [R_1, R_2]$, by (3.1),(3.2), we see that

$$E_\infty(z) \geq \overline{h}(\|z\|_p) \geq h(\|z\|_p) \geq 0.$$

If $\|z\|_p \in (R_2, \infty)$, by (3.2) and above analysis, we also have that

$$E_\infty(z) \geq \overline{h}(\|z\|_p) \geq 0.$$

Thus $\|z\|_p \in (0, R_1)$, (1) holds.

Now, we prove (2), let Λ^* as above. If $c < 0$ and $\{z_n\} \subset H$ is a $(PS)_c$ sequence of E_∞, then we may assume that $E_\infty(z_n) < 0$ and $E'_\infty(z_n) = o(1)$, by (1),$\|z_n\|_p \in (0, R_1)$, hence $E(z_n) = E_\infty(z_n)$ and $E'(z_n) = E'_\infty(z_n)$. Since (b) hold when $0 < (\lambda + \theta) \leq \Lambda^*$, By Lemma 2.5, $E(z)$ satisfies the $(PS)_c$ condition for $c < 0$. Thus $E_\infty(z)$ satisfies the $(PS)_c$ condition for $c < 0$, (2) holds. □

Now we prove our main result via genus.

Proof of Theorem 1.1. Let $\Sigma_k = \{A \subset H - \{(0,0)\}, A \text{ is closed}, \ A = -A, \gamma(A) \geq k\}$, $c_k = \inf_{A \in \Sigma_k} \sup_{z \in A} E_\infty(z)$, $K_c = \{z \in H | \ E_\infty(z) = c, E'_\infty(z) = 0\}$, and suppose that $0 < (\lambda + \theta) \leq \Lambda^*$, Λ^* is as above.

We claim that if $k, l \in N$ are such that $c = c_k = c_{k+1} = \cdots = c_{k+l}$, then $\gamma(K_c) \geq l + 1$.

In fact, we assume

$$E_\infty^{-\varepsilon} = \{z \in H | \ E_\infty(z) \leq -\varepsilon\},$$

we will show for any $k \in N$, there exist an $\varepsilon = \varepsilon(k) > 0$, such that

$$\gamma(E_\infty^{-\varepsilon}(z)) \geq k.$$

Fix $k \in N$, denote H_k be an k-dimensional subspace of H, choose $z = (u, v) \in H_k$, with $\|z\|_p = 1$, for $0 < \rho < R_1$, we have

$$E(\rho z) = E_\infty(\rho z) = \frac{1}{p}\rho^p + \frac{\rho^q}{q}\|z\|_q^q - \frac{\rho^r}{r}\int_\Omega \lambda V(x)|u|^r$$
$$+\theta V(x)|v|^r dx - \frac{2\rho^{p^*}}{p^*}\int_\Omega |u|^\alpha |v|^\beta dx. \qquad (3.3)$$

For H_k is a finite dimension space, all the norms in H_k are equivalent. So we can define

$$\alpha_k = \sup\{\|z\|_q^q |\ u \in H_k, \|z\|_p = 1\} < \infty, \tag{3.4}$$

$$\beta_k = \inf\{|z|_r^r |\ z \in H_k, \|z\|_p = 1\} > 0, \tag{3.5}$$

from (3.3)-(3.5), we have

$$E_\infty(\rho z) \le \frac{1}{p}\rho^p + \alpha_k \frac{\rho^q}{q} - \sigma\beta_k \frac{\min\{\lambda, \theta\}\rho^r}{r}.$$

For any $\varepsilon > 0$ and an $0 < \rho < R_1$ such that $E_\infty(\rho z) \le -\varepsilon$ for $z \in H_k$, $\|z\|_p = 1$, let $S_\rho = \{z \in H |\ \|z\|_p = \rho\}$, then $S_\rho \cap H_k \subset E_\infty^{-\varepsilon}$. By Lemma 2.8, we obtain that

$$\gamma(E_\infty^{-\varepsilon}(z)) \ge \gamma(S_\rho \cap H_k) = k. \tag{3.6}$$

Since E_∞ is continuous and even, with (3.6), we have $E_\infty^{-\varepsilon} \in \Sigma_k$ and $c = c_k \le -\varepsilon < 0$. As E_∞ is bounded from below, we see that $c = c_k > -\infty$(This is the main reason that we consider E_∞ instead of E). Then by Lemma 3.1 E_∞ satisfies $(PS)_c$ condition and it is easy to see that K_c is a compact set.

Now we prove our claim by contradiction, suppose on the contrary $\gamma(K_c) \le l$. By Lemma 2.8, there is a closed and symmetric set U with $K_c \subset U$ and $\gamma(U) \le l$. Since $c < 0$, we also can assume that the closed set $U \subset E_\infty^0$. By Lemma 2.6, there exists an odd homeomorphism

$$\eta :\ H \to H$$

such that $\eta(E_\infty^{c+\delta} - U) \subset E_\infty^{c-\delta}$ for some $0 < \delta < -c$.

From the definition of $c = c_{k+l}$, we know that there is an $A \in \Sigma_{k+l}$ such that

$$\sup_{z \in A} E_\infty(z) < c + \delta$$

i.e., $A \subset E_\infty^{c+\delta}$, and

$$\eta(A - U) \subset \eta(E_\infty^{c+\delta} - U) \subset E_\infty^{c-\delta},$$

that's meaning

$$\sup_{z \in \eta(A-U)} E_\infty(z) \le c - \delta. \tag{3.7}$$

Again by Lemma 2.8, we have

$$\gamma(\eta(\overline{A-U})) \ge \gamma(\overline{A-U}) \ge \gamma(A) - \gamma(U) \ge k.$$

Thus we have $\eta(\overline{A-U}) \in \Sigma_k$ and $\sup_{z \in \eta(\overline{A-U})} E_\infty(z) \ge c_k = c$, which contradicts to (3.7). So we have proved our claim.

Now let's complete the proof of Theorem 1.1. If for all $k \in N$, we have $\Sigma_{k+1} \subset \Sigma_k$, $c_k \le c_{k+1} < 0$. If all c_k are distinct, then $\gamma(K_{c_k}) \ge 1$, and we see that $\{c_k\}$ is a sequence of distinct negative critical values of E_∞; if for some k_0, there is a $l \ge 1$ such that $c = c_{k_0} = c_{k_0+1} = \cdots = c_{k_)+l}$, then by the claim, we have

$$\gamma(K_c) \ge l + 1,$$

which shows that K_c contains infinitely many distinct elements.

By Lemma 3.1, we know $E(z) = E_\infty(z)$ when $E_\infty(z) < 0$, so we show that there are infinitely many critical points of $E(z)$. Theorem 1.1 is proved. □

4. Proof of Theorem 1.2.

In this section, we will study problem (1.1) with $1 < q < p < r < p^*$, and will prove Theorem 1.2 by the following general version of the Mountain Pass Lemma(see[3]).

Lemma 4.1. *Let E be a functional on a Banach space H, $E \in C^1(H, R)$. Let us assume that there exists $\rho, R > 0$ such that*

(i) $E(z) > \rho$, $\forall z \in H$ with $\|z\|_p = R$.

(ii) $E(0) = 0$, and $E(w_0) < \rho$ for some $w_0 \in H$, with $\|w_0\|_p > R$.

Let us define $\Gamma = \{\gamma \in C([0,1], H) |\ \gamma(0) = 0, \gamma(1) = w_0\}$, and

$$\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)). \tag{4.1}$$

Then there exists a sequence $\{z_n\} \subset H$, such that $E(z_n) \to \mu$, and $E'(z_n) \to 0$ in H' (dual of H) as $n \to \infty$.

Now similar to Lemma 2.5 in section 2, we have the following result.

Lemma 4.2. *Suppose $1 < q < p \le r < p^*$ hold, then any $(PS)_c$ sequence $\{z_n\} \subset H$ of $E(z)$ contains a convergent subsequence when*

$$c < \frac{2}{N} S_{\alpha,\beta}^{\frac{N}{p}}. \tag{4.2}$$

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. From (4.1) and (4.2), we only need to show

$$\mu < \frac{2}{N} S_{\alpha,\beta}^{\frac{N}{p}}, \tag{4.3}$$

then Lemma 4.1 and Lemma 4.2 give the existence of the critical point of E.

To obtain (4.3), Let us choose $z_0 = (u_0, u_0) \in H$, with

$$|z_0|_{p^*} = 1,\ \lim_{t\to\infty} E(tz_0) = -\infty,$$

then there exists a $t_{\theta\lambda} > 0$ such that $\sup_{t\ge0} E(tz_0) = E(t_{\theta\lambda} z_0)$ holds, and then $t_{\theta\lambda}$ satisfies

$$0 = t_{\theta\lambda}^{p-1}\|z_0\|_p^p + t_{\theta\lambda}^{q-1}\|z_0\|_q^q - (\lambda+\theta) t_{\theta\lambda}^{r-1} \int_\Omega V(x)|u_0|^r dx - t_{\theta\lambda}^{p^*-1}$$

then we get

$$(\lambda+\theta)\int_\Omega V(x)|u_0|^r dx = t_{\theta\lambda}^{p-r}\|z_0\|_p^p + t_{\theta\lambda}^{q-r}\|z_0\|_q^q - t_{\theta\lambda}^{p^*-r}$$

from $1 < q < p \leq r < p^*$, we get $t_{\theta\lambda} \to 0$ as $(\lambda+\theta) \to \infty$. Then there exists $\Lambda_* > 0$ such that for any $(\lambda+\theta) > \Lambda_*$, we have

$$\sup_{t\geq 0} E(tz_0) < \frac{2}{N} S_{\alpha,\beta}^{\frac{N}{p}}.$$

Now we take $w_0 = t_0 z_0$ with t_0 large enough to verify $E(w_0) < 0$, we get

$$\alpha \leq \max_{t\in[0,1]} E(\gamma_0(t))$$

where $\gamma_0(t) = tw_0$. Therefore,

$$\mu \leq \sup_{t\geq 0} E(tw_0) < \frac{2}{N} S_{\alpha,\beta}^{\frac{N}{p}}.$$

then we have proved (4.3), that's complete the proof. □

Now let's assume $1 < q < \frac{N(p-1)}{N-1} < p \leq \max\{p, p^* - \frac{q}{p-1}\} < r < p^*$, and define, for $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{\psi(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}, \ v_\varepsilon(x) = \frac{u_\varepsilon(x)}{|u_\varepsilon(x)|_{p^*}}$$

where $\psi(x) \in C_0^\infty(B(0,2R))$ is such that $0 \leq \psi(x) \leq 1$, and $\psi(x) \equiv 1$ on $B(0,R)$.

We obtain the following estimates(see[13]).

$$\int_\Omega |u_\varepsilon|^t dx = \begin{cases} K_1 \varepsilon^{\frac{N(p-1)-t(N-p)}{p}} + O(1), & t > \frac{N(p-1)}{N-p} \\ K_1|ln\varepsilon| + O(1), & t = \frac{N(p-1)}{N-p} \\ O(1), & t < \frac{N(p-1)}{N-p} \end{cases} \tag{4.4}$$

$$\int_\Omega |\nabla u_\varepsilon|^t dx = \begin{cases} K_2 \varepsilon^{\frac{t+N(p-1)-tN}{p}} + O(1), & t > \frac{N(p-1)}{N-1} \\ K_2|ln\varepsilon| + O(1), & t = \frac{N(p-1)}{N-1} \\ O(1), & t < \frac{N(p-1)}{N-1} \end{cases} \tag{4.5}$$

In particular, we have

$$\int_\Omega |\nabla u_\varepsilon|^p dx = K_2 \varepsilon^{\frac{p-N}{p}} + O(1) \tag{4.6}$$

and

$$(\int_\Omega |u_\varepsilon|^{p^*} dx)^{\frac{p}{p^*}} = K_3 \varepsilon^{\frac{p-N}{p}} + O(1) \tag{4.7}$$

$$\int_\Omega |u_\varepsilon|^p dx = \begin{cases} K_1 \varepsilon^{\frac{p^2-N}{p}} + O(1), & p^2 < N \\ K_1|ln\varepsilon| + O(1), & p^2 = N \\ O(1), & p^2 > N \end{cases} \tag{4.8}$$

where K_1, K_2, K_3 are positive constants independent of ε, and $S = \frac{K_2}{K_3}$ is the best Sobolev constant given in section 2.

If we choosing w_0 in the proof of Theorm 1.2 carefully, we can prove the following stronger result.

Theorem 4.3. *If* $1 < q < \frac{N(p-1)}{N-1} < p \le \max\{p, p^* - \frac{q}{p-1}\} < r < p^*$ *and* V_0 *hold, then for any* $\theta > 0$, $\lambda > 0$, *problem (1.1) has a nontrivial solution.*

Proof. Following [16], we can take $z_\varepsilon = (v_\varepsilon, v_\varepsilon)$, and

$$g(t) = E(tz_\varepsilon) = \frac{2t^p}{p}\|v_\varepsilon\|_p^p + \frac{2t^q}{q}\|v_\varepsilon\|_p^p - \frac{(\lambda+\theta)t^r}{r}\int_\Omega V(x)|v_\varepsilon|^r dx - \frac{2t^{p^*}}{p^*}.$$

then there exists a $t_\varepsilon > 0$ such that $\sup_{t\ge 0} E(tz_\varepsilon) = E(t_\varepsilon z_\varepsilon)$ hold, and then t_ε satisfies

$$g'(t_\varepsilon) = 2t^{p-1}\|v_\varepsilon\|_p^p + 2t^{q-1}||v_\varepsilon|_p^p - (\lambda+\theta)t^{r-1}\int_\Omega V(x)|v_\varepsilon|^r dx - 2t^{p^*-1} = 0. \quad (4.9)$$

then we have

$$\int_\Omega |\nabla v_\varepsilon|^p dx + t_\varepsilon^{q-p}\int_\Omega |\nabla v_\varepsilon|^q dx > t_\varepsilon^{p^*-p}$$

From (4.4)-(4.8) we can know

$$\int_\Omega |\nabla v_\varepsilon|^p dx = S + O(\varepsilon^{\frac{N-p}{p}}),\ \int_\Omega |\nabla v_\varepsilon|^q dx = O(\varepsilon^{\frac{q(N-p)}{p^2}})$$

set ε small enough, then we can have

$$t_\varepsilon^{p^*-p} \le 2S.$$

here we use the fact that $t_\varepsilon \to t_0 = (\int_\Omega |\nabla v_\varepsilon|^p dx)^{\frac{1}{p^*-p}} > 0$ as $\varepsilon \to 0$, where t_0 will be given later.

Then from (4.9) we obtain

$$2\|v_\varepsilon\|_p^p < (\lambda+\theta)t_\varepsilon^{r-p}\|V(x)\|_\infty |v_\varepsilon|_r^r + 2t_\varepsilon^{p^*-p} \quad (4.10)$$

From (4.4)-(4.8), and (4.10), choose ε small enough, we have

$$t_\varepsilon^{p^*-p} \ge \frac{S}{2}$$

Now we consider

$$h(t) = \frac{t^p}{p}\int_\Omega |\nabla v_\varepsilon|^p dx - \frac{t^{p^*}}{p^*}$$

the function attains its maximum at $t_0 = (\int_\Omega |\nabla v_\varepsilon|^p dx)^{\frac{1}{p^*-p}}$, and again combine with (4.4)-(4.8), we have

$$\begin{aligned} g(t_\varepsilon) &\le 2h(t_\varepsilon) + \frac{t_\varepsilon^q}{q}\int_\Omega |\nabla v_\varepsilon|^q dx - \frac{(\lambda+\theta)t_\varepsilon^r}{r}\int_\Omega V(x)|v_\varepsilon|^r dx \\ &\le 2h((\int_\Omega |\nabla v_\varepsilon|^p dx)^{\frac{1}{p^*-p}}) + \frac{(2S)^q}{q}\int_\Omega |\nabla v_\varepsilon|^q dx - \frac{(\lambda+\theta)(\frac{S}{2})^r}{r}\sigma\int_\Omega |v_\varepsilon|^r dx \\ &\le \frac{2}{N}S^{\frac{N}{p}} + C_6\varepsilon^{\frac{N-p}{p}} + C_7 O(\varepsilon^{\frac{q(N-p)}{p^2}}) - C_8 O(\varepsilon)^{\frac{p-1}{p}(N-r\frac{N-p}{p})} \end{aligned}$$

where C_6, C_7, C_8 are positive constants independent with ε. Since $1 < q < \frac{N(p-1)}{N-1} < p \leq \max\{p, p^* - \frac{q}{p-1}\} < r < p^*$, we obtain that

$$\frac{N-p}{p} > \frac{q(N-p)}{p^2} > \frac{p-1}{p}(N - r\frac{N-p}{p}),$$

then we choose ε small enough, by Lemma 2.1, we get $g(t_\varepsilon) = \sup_{t\geq 0} E(tv_\varepsilon) < \frac{2}{N} S^{\frac{N}{p}} \leq \frac{2}{N} S_{\alpha,\beta}^{\frac{N}{p}}$, by Lemma 4.1 and Lemma 4.2, we complete the proof. □

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