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A NEWTON-IMPLICIT ITERATIVE METHOD FOR NONLINEAR INVERSE PROBLEMS†

ZEHONG MENG* AND ZHENYU ZHAO

Abstract. A regularized Newton method for nonlinear ill-posed problems is considered. In each Newton step an implicit iterative method with an appropriate stopping rule is proposed and analyzed. Under certain assumptions on the nonlinear operator, the convergence of the algorithm is proved and the algorithm is stable if the discrepancy principle is used to terminate the outer iteration. Numerical experiment shows the effectiveness of the method.

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1. Introduction

In many applied science and technology fields, such as earthquake prospecting[14], CT technology[11], groundwater hydrology[3], vane design[9], etc. various inverse problems are proposed. Generally they can be formulated as the following nonlinear ill-posed operator equation

$$F(a) = u \tag{1.1}$$

where $F : D(F) \subset \mathcal{X} \to \mathcal{Y}$ is a nonlinear operator between Hilbert space $\mathcal{X}$ and $\mathcal{Y}$. Inverse problems are generally nonlinear and ill-posed in the sense that even when a is uniquely determined by the right-hand side u the mapping $u \mapsto a$ lacks continuity. This is a severe numerical problem when the given right-hand side is noisy data u^δ. Such problems need to be regularized[1]. The perturbed right-hand side u^δ generally satisfies

$$\|u^\delta - u\| \le \delta \tag{1.2}$$

where $\delta \ge 0$ is a known error level.

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When the problem (1.1) is well-posed, Newton-type methods are one effective option for solving (1.1) and they have been applied with success in various applications. However, only few rigorous theoretical analysis of Newton-type methods for ill-posed problems can be found in the literature[2,4,7,10,13].

The present paper develops a Newton-type method for nonlinear ill-posed equation (1.1). The basic idea is the resolution of the linearized equation, which generally is also ill-posed, by an implicit iterative method with invariable control parameters[5,6]. A posteriori stopping rules for the inner and the outer iterations are suggested that make the algorithm a regularizing method.

This paper is organized as follows. In section 2, the algorithm for solving the nonlinear ill-posed problem (1.1) is presented. In section 3, the monotonicity result concerning the iteration error is proved. In section 4, the convergence results of the algorithm for both cases $\delta = 0$ and $\delta > 0$ are derived. Section 5 gives some numerical experiment results.

2. Algorithm

Newton-type methods for solving (1.1) are based on the Taylor expansion of the operator F. Assuming that $a^\dagger$ is a solution of the nonlinear equation (1.1) and a_n is some approximation of $a^\dagger$, then

$$F(a^\dagger) = F(a_n) + F'(a_n)(a^\dagger - a_n) + R(a^\dagger, a_n) \tag{2.1}$$

where $R(a^\dagger, a_n)$ is the Taylor remainder. Adding the noise term u^δ to (2.1) and hence $a^\dagger - a_n$ satisfies

$$F'(a_n)(a^\dagger - a_n) = u^\delta - F(a_n) + u - u^\delta - R(a^\dagger, a_n) \tag{2.2}$$

The right-hand side of (2.2) splits into two parts:

$$\tilde{y}_n = u^\delta - F(a_n),\ \tilde{z}_n = u - u^\delta - R(a^\dagger, a_n)$$

the first part is computable, whereas the second part is not. In other words, the ideal update $x := a^\dagger - a_n$ solves the linear equation

$$T_n x = y_n \tag{2.3}$$

with $T_n = F'(a_n)$ and y_n as the right-hand side in (2.2). The corresponding Newton iterative equation is

$$T_n x = \tilde{y}_n \tag{2.4}$$

which can be regarded as the perturbed equation of the exact equation (2.3). Here the perturbed right hand $\tilde{y}_n$ and the exact right hand y_n satisfy

$$\|\tilde{y}_n - y_n\| \le \delta + \|R(a^\dagger, a_n)\|$$

In general equation (2.4) is still ill-posed. There is well-developed theory on how to regularize linear ill-posed problems with inexact data when the error bound $\|\tilde{y}_n - y_n\|$ is known, cf.[1], however, the difficulty in the present situation is that $\|R(a^\dagger, a_n)\|$ can hardly be estimated accurately, so is $\|\tilde{y}_n - y_n\|$. Thus the most of regularization parameter choice strategies will not be applicable. To get rid of the difficulty, Hanke proposed a parameter choice strategy for Tikhonov and CG methods for equation (2.4)[3,4]. Since implicit iterative method is a very efficient approach for linear ill-posed equations [5,6], we consider in this paper the possibility of application of the implicit iterative method for equation (2.4) combining with Hanke's iterative stopping strategy.

The implicit iterative method for equation (2.4) is as follows

$$(T_n^*T_n + \alpha_n I)x_k = T_n^*\tilde{y}_n + \alpha_n x_{k-1},\ x_0 = 0,\ k = 1, 2, \cdots \tag{2.5}$$

where α_n is a positive constant which is used to control the rates of the convergence. Assume that they have positive lower and upper bound, i.e.

$$0 < \underline{\alpha} \le \alpha_n \le \overline{\alpha} \tag{2.6}$$

Iteration (2.5) can be rewritten as

$$(T_n^*T_n + \alpha_n I)(x_k - x_{k-1}) = T_n^*(\tilde{y}_n - T_n x_{k-1}), x_0 = 0, k = 1, 2, \cdots \tag{2.7}$$

Let $r_k = \tilde{y}_n - T_n x_k$, and (2.7) becomes $x_k = x_{k-1} + (T_n^*T_n + \alpha_n I)^{-1}T_n^* r_{k-1}$. Repeat use of this formula gives

$$r_k = \alpha_n^k (T_n^*T_n + \alpha_n I)^{-k}\tilde{y}_n \tag{2.8}$$

$$x_k = x_{k-1} + T_n^* w_{k-1} \tag{2.9}$$

$$x_k = T_n^* \sum_{i=0}^{k-1} w_i \tag{2.10}$$

where $w_i = \alpha_n^i \triangle^{-(i+1)}\tilde{y}_n = \triangle^{-1} r_i$ and $\triangle = T_n^*T_n + \alpha_n I$. In the following, we give the Newton-Implicit iterative algorithm for solving equation (1.1).

Algorithm 1.

step 1. Given $\tau > 0$, $\rho > 0$. Choose the initial guess a_0, $n = 0$.

step 2. Compute $\tilde{y}_n = u^\delta - F(a_n)$, $T_n = F'(a_n)$.

step 3. Let $x_0 = 0$, $k = 1$. Given α_n.

step 4. Compute $x_k = x_{k-1} + (T_n^*T_n + \alpha_n I)^{-1}T_n^*(\tilde{y}_n - T_n x_{k-1})$.

step 5. If $\|\tilde{y}_n - T_n x_k\| \le \rho\|\tilde{y}_n\|$, then do step 6, else $k = k + 1$ and do step 4.

step 6. Compute $a_{n+1} = a_n + x_k$.

step 7. If $\|u^\delta - F(a_n)\| \le \tau\delta$, then stop, else $n = n + 1$ and turn to step 2.

In the algorithm, the reason why we let $x_0 = 0$ can refer to [5]. Algorithm 1 requires an initial estimate a_0 of $a^\dagger$, and two tolerance parameters ρ and τ in the stopping rules of the inner and the outer iterations, respectively.

3. Monotonicity

In order to prove the convergence of Algorithm 1, we need to prove the monotonicity of the iteration errors. Some assumptions which are very common for nonlinear ill-posed problems are proposed[3,4].

(i) $F'(\cdot)$ is locally bounded and denote

$$M = \sup\{\|F'(a)\|,\ a \in \mathcal{B}(a^\dagger, r)\} \tag{3.1}$$

where $\mathcal{B}(a^\dagger, r)$ is a ball around $a^\dagger$ with radius $r > 0$.

(ii) For a certain ball $\mathcal{B} \subset D(F)$ around the exact solution $a^\dagger$ of (1.1), and some $C > 0$,

$$\|F(\tilde{a}) - F(a) - F'(a)(\tilde{a} - a)\| \leq C\|\tilde{a} - a\|\|F(\tilde{a}) - F(a)\| \tag{3.2}$$

for all $\tilde{a}, a \in \mathcal{B}$.

For the sake of convenience, let $\tilde{y} = \tilde{y}_n$, $T = T_n$, $\alpha = \alpha_n$. In order to prove the convergence properties of Algorithm 1, first we need to prove the monotonicity of the iteration errors.

Lemma 1. *Let $\gamma \geq 2$, $k_* \in \mathbb{N}$, $x \in \mathcal{X}$ satisfies $\|\tilde{y} - Tx\| \leq \epsilon$ and assumption (3.1) holds. If*

$$\|\tilde{y} - Tx_k\|^2 + \|\tilde{y} - Tx_{k+1}\|^2 > \gamma\epsilon(\alpha + M^2)\|w_k\|, k = 0, 1, \cdots, k_* - 1 \tag{3.3}$$

then $\|x - x_k\|$ is strictly monotonically decreasing for $k = 0, 1, \cdots, k_*$ and

$$|x\|^2 - \|x - x_{k_*}\|^2 > (\gamma - 2)\epsilon \sum_{k=0}^{k_*-1} \|w_k\| \tag{3.4}$$

Proof. By (2.9),

$$\begin{aligned} \|x - x_{k+1}\|^2 &= \|x - x_k - T^* w_k\|^2 \\ &= \|x - x_k\|^2 - \langle 2x - 2x_k - T^* w_k, T^* w_k\rangle \\ &= \|x - x_k\|^2 - \langle Tx - Tx_k, w_k\rangle - \langle Tx - Tx_{k+1}, w_k\rangle \\ &= \|x - x_k\|^2 - \langle \tilde{y} - Tx_k, w_k\rangle - \langle \tilde{y} - Tx_{k+1}, w_k\rangle + 2\langle \tilde{y} - Tx, w_k\rangle \end{aligned}$$

The given assumptions and the estimate $\|\triangle^{-1}\| \geq \dfrac{1}{\alpha + M^2}$ yield

$$\begin{aligned} \|x - x_k\|^2 - \|x - x_{k+1}\|^2 &= \langle \tilde{y} - Tx_k, w_k\rangle + \langle \tilde{y} - Tx_{k+1}, w_k\rangle - 2\langle \tilde{y} - Tx, w_k\rangle \\ &= \langle r_k, w_k\rangle + \langle r_{k+1}, w_k\rangle - 2\langle \tilde{y} - Tx, w_k\rangle \\ &\geq \frac{1}{\alpha + M^2}[\langle r_k, r_k\rangle + \langle r_{k+1}, r_{k+1}\rangle] - 2\langle \tilde{y} - Tx, w_k\rangle \\ &> \gamma\epsilon\|w_k\| - 2\epsilon\|w_k\| = (\gamma - 2)\epsilon\|w_k\| \end{aligned} \tag{3.5}$$

for all $k = 0, 1, \cdots, k_* - 1$. Since $\gamma \geq 2$, the right-hand side of (3.5) is nonnegative which shows that the sequence $\{\|x - x_k\|\}$ is strictly decreasing for k in the given range. Furthermore, since $x_0 = 0$, the second assertion of the lemma follows by taking sum of (3.5) from $k = 0$ to $k_* - 1$. □

Lemma 2. *Let $\gamma \geq 2$, $k_* \in \mathbb{N}$, $x \in \mathcal{X}$ satisfies $\|\tilde{y} - Tx\| \leq \epsilon$ and the assumption (3.1) holds and $\|\tilde{y}\| \neq 0$, then the inequalities*

$$\|\tilde{y} - Tx_k\|^2 > \frac{1}{\alpha}\gamma\epsilon(\alpha + M^2)\|\tilde{y}\|, \ k = 1, 2, \cdots, k_* \tag{3.6}$$

imply (3.3). Furthermore, there are only finitely many k for which (3.6) can hold.

Proof. Since $\|w_k\| = \|\alpha^k \triangle^{-(k+1)}\tilde{y}\| \le \frac{1}{\alpha}\|\tilde{y}\|$, and by (3.6) and the assumptions of the lemma, we have

$$\|\tilde{y} - Tx_k\|^2 + \|\tilde{y} - Tx_{k+1}\|^2 > \frac{2}{\alpha}\gamma\epsilon(\alpha + M^2)\|\tilde{y}\| \ge 2\gamma\epsilon(\alpha + M^2)\|w_k\|$$

so that (3.6) imply (3.3). Since $\|\tilde{y} - Tx_k\| \to 0$ as $k \to \infty$, cf.[5], therefore, (3.6) can only hold for finitely many indices k. □

In the following, we denote $T_n = F'(a_n)$, $\tilde{y}_n = u^\delta - F(a_n)$ and α_n again. It will be assumed that equation (1.1) has a solution $a^\dagger \in \mathcal{B}$. The following theorem gives the monotonicity result.

Theorem 3. *Let $\gamma > 2$, $0 < \rho < 1$, $\|\tilde{y}_n\| \ne 0$. Assume (3.1) and*

$$\|\tilde{y}_n - y_n\| = \|u^\delta - F(a_n) - F'(a_n)(a^\dagger - a_n)\| \le \frac{\alpha_n \rho^2}{\gamma(\alpha_n + M^2)}\|u^\delta - F(a_n)\| \quad (3.7)$$

hold. The inner iteration stops as the inequality

$$\|\tilde{y}_n - T_n x_k\| = \|u^\delta - F(a_n) - F'(a_n)(a_{n+1} - a_n)\| \le \rho\|u^\delta - F(a_n)\| \quad (3.8)$$

occurs. Then the inner iteration terminates after $k_n < \infty$ steps, and

$$a_{n+1} = a_n + x_{k_n} = a_n + F'(a_n)^* v_n$$

with some $v_n \in \mathcal{Y}$. Moreover, the following inequalities hold

$$\|a^\dagger - a_n\|^2 - \|a^\dagger - a_{n+1}\|^2 > \frac{(\gamma - 2)\alpha_n \rho^2}{\gamma(\alpha_n + M^2)}\|\tilde{y}_n\|\|v_n\| \quad (3.9)$$

$$\|a^\dagger - a_n\|^2 - \|a^\dagger - a_{n+1}\|^2 > \frac{(\gamma - 2)\alpha_n \rho^2}{\gamma(\alpha_n + M^2)^2}\|\tilde{y}_n\|^2 \quad (3.10)$$

Proof. By (2.2), we know $x = a^\dagger - a_n$ is a solution of equation (2.3). By (3.7),

$$\|\tilde{y}_n - T_n x\| = \|\tilde{y}_n - y_n\| \le \frac{\alpha_n \rho^2}{\gamma(\alpha_n + M^2)}\|\tilde{y}_n\|$$

Hence, let

$$\epsilon = \frac{\alpha_n \rho^2}{\gamma(\alpha_n + M^2)}\|\tilde{y}_n\|$$

Then x satisfies $\|\tilde{y}_n - T_n x\| \le \epsilon$. Substituting $\gamma\epsilon = \frac{\alpha_n \rho^2}{\alpha_n + M^2}\|\tilde{y}_n\|$ into (3.6), it follows from Lemma 2 that the stopping rule (3.8) determines a finite stopping index k_n for the inner iteration and that (3.3) is fulfilled with $k_* = k_n$. In other words (3.4) is fulfilled with $k_* = k_n$.

Consider the updates of a_n in Algorithm 1. It follows that

$$a_{n+1} = a_n + x_{k_n} = a_n + T_n^* \sum_{i=0}^{k_n - 1} w_i = a_n + T_n^* v_n \text{ with } v_n = \sum_{i=0}^{k_n - 1} w_i \quad (3.11)$$

Since $x = a^\dagger - a_n$ and $x - x_{k_n} = a^\dagger - a_{n+1}$, Lemma 1 asserts that $\|a^\dagger - a_{n+1}\| < \|a^\dagger - a_n\|$ and that

$$\|a^\dagger - a_n\|^2 - \|a^\dagger - a_{n+1}\|^2 > (\gamma - 2)\epsilon\|v_n\| = \frac{(\gamma-2)\alpha_n\rho^2}{\gamma(\alpha_n + M^2)}\|\tilde{y}_n\|\|v_n\|$$

which asserts (3.9). Since $v_n = \sum_{i=0}^{k_n-1} \alpha_n^i \triangle^{-(i+1)}\tilde{y}_n = g_{\alpha_n,k_n}(T_n^*T_n)\tilde{y}_n$, where

$$g_{\alpha_n,k_n}(\lambda) = \sum_{i=0}^{k_n-1} \alpha_n^i(\lambda + \alpha_n)^{-(i+1)} = \frac{1}{\lambda}[1 - \alpha_n^{k_n}(\lambda+\alpha_n)^{-k_n}]$$

and

$$\|g^{-1}_{\alpha_n,k_n}(T_n^*T_n)\| \le \sup_{0\le\lambda\le M^2} \frac{1}{g_{\alpha_n,k_n}(\lambda)} \le \sup_{0\le\lambda\le M^2}\frac{1}{g_{\alpha_n,1}(\lambda)} = \alpha_n + M^2$$

yields

$$\|v_n\| \ge \frac{1}{\alpha_n + M^2}\|\tilde{y}_n\| \tag{3.12}$$

(3.9) and (3.12) yield (3.10). The conclusion of the theorem hold. □

It is easy to see that the same inequalities (3.9), (3.10) would hold if the inner iteration before the stopping rule (3.8) is met. This is important for practical purposes because usually the number of inner iteration is constrained by some maximum number.

4. Convergence Analysis

Firstly consider the convergence of Algorithm 1 for exact equation (1.1).

Theorem 4. *Assume $u^\delta = u = F(a^\dagger)$ for some $a^\dagger \in D(F)$, $0 < \rho < 1$, and assume $F'(\cdot)$ is locally bounded with (3.1) and that F satisfies (3.2) for some $C > 0$ in a ball $\mathcal{B} \in \mathcal{D}(\mathcal{F})$ around $a^\dagger$. If $a_0 \in \mathcal{B}$ and $\|a^\dagger - a_0\| < \dfrac{\underline{\alpha}\rho^2}{2C(\underline{\alpha} + M^2)}$, where $\underline{\alpha}$ satisfies (2.6), then the iterates $\{a_n\}$ of Algorithm 1 converge to a solution of (1.1) as $n \to \infty$.*

Proof. Define $\gamma = \dfrac{\underline{\alpha}\rho^2}{C(\underline{\alpha}+M^2)\|a^\dagger - a_0\|}$ which is greater than 2 by assumption. Therefore (3.2) with $\tilde{a} = a^\dagger,\ a = a_0$ implies (3.7) and hence

$$\|a^\dagger - a_{n+1}\| < \|a^\dagger - a_n\| \tag{4.1}$$

for $n = 0$ by virtue of Theorem 3.3. We assume this inequality is true for $n \le l$, and will prove the inequality (4.1) remains true for $n = l + 1$. Again define $\gamma = \dfrac{\underline{\alpha}\rho^2}{C(\underline{\alpha}+M^2)\|a^\dagger - a_{l+1}\|}$ which is greater than 2 by the assumption of induction, and hence (3.2) with $\tilde{a} = a^\dagger,\ a = a_{l+1}$ implies (3.7) and (4.1) holds for $n = l + 1$. Thus we prove that the sequence $\{\|a^\dagger - a_n\|\}$ is monotonously decreasing during the entire iteration by induction.

It will be shown that the iteration errors $e_n = a^\dagger - a_n,\ n \in \mathbb{N}$, form a Cauchy sequence. Given $m,\ n \in \mathbb{N}$ with $m > n$, let $l \in \{n, n+1, \cdots, m\}$ be chosen in such a way that

$$\|u - F(a_l)\| \le \|u - F(a_i)\|,\ i = n, n+1, \cdots, m \tag{4.2}$$

Consider now

$$\|e_l - e_n\|^2 = \|e_n\|^2 - \|e_l\|^2 + 2\langle e_l - e_n, e_l\rangle \tag{4.3}$$

From (3.11), it follows that

$$|\langle e_l - e_n, e_l\rangle| = |\langle \sum_{i=n}^{l-1} F'(a_i)^* v_i, e_l\rangle| \le \sum_{i=n}^{l-1} \|v_i\|\|F'(a_i)e_l\| \tag{4.4}$$

The last factor $\|F'(a_i)e_l\|$ can be estimated by using (3.2) as follows

$$\begin{aligned}\|F'(a_i)e_l\| &= \|F'(a_i)e_i - F'(a_i)(a_l - a_i)\| \\ &\le \|u - F(a_i) - F'(a_i)e_i\| + \|F(a_l) - F(a_i) - F'(a_i)(a_l - a_i)\| + \|u - F(a_l)\| \\ &\le C\|a^\dagger - a_i\|\|u - F(a_i)\| + C\|a_l - a_i\|\|F(a_l) - F(a_i)\| + \|u - F(a_l)\|\end{aligned}$$

By the monotonicity of $\|a^\dagger - a_n\|$, $\|a^\dagger - a_0\| < \dfrac{\underline{\alpha}\rho^2}{2C(\underline{\alpha} + M^2)}$ and (4.2), we have

$$\begin{aligned}\|F'(a_i)e_l\| &\le \frac{\underline{\alpha}\rho^2}{2(\underline{\alpha} + M^2)}\|u - F(a_i)\| + \frac{\underline{\alpha}\rho^2}{\underline{\alpha} + M^2}\|F(a_l) - F(a_i)\| + \|u - F(a_l)\| \\ &\le \frac{3\underline{\alpha}\rho^2}{2(\underline{\alpha} + M^2)}\|u - F(a_i)\| + (\frac{\underline{\alpha}\rho^2}{\underline{\alpha} + M^2} + 1)\|u - F(a_l)\| \\ &\le (\frac{5\underline{\alpha}\rho^2}{2(\underline{\alpha} + M^2)} + 1)\|u - F(a_i)\|\end{aligned}$$

Thus (4.4) and (3.9) imply that

$$\begin{aligned}|\langle e_l - e_n, e_l\rangle| &\le (\frac{5\underline{\alpha}\rho^2}{2(\underline{\alpha} + M^2)} + 1)\sum_{i=n}^{l-1} \|v_i\|\|u - F(a_i)\| \\ &< \frac{\gamma(\underline{\alpha} + M^2)}{(\gamma - 2)\underline{\alpha}\rho^2}(\frac{5\underline{\alpha}\rho^2}{2(\underline{\alpha} + M^2)} + 1)(\|a^\dagger - a_n\|^2 - \|a^\dagger - a_l\|^2)\end{aligned}$$

which together with (4.3) yields

$$\|e_l - e_n\|^2 < c(\|a^\dagger - a_n\|^2 - \|a^\dagger - a_l\|^2)$$

where $c = \dfrac{\gamma(\underline{\alpha} + M^2)}{(\gamma - 2)\underline{\alpha}\rho^2}(\dfrac{5\underline{\alpha}\rho^2}{2(\underline{\alpha} + M^2)} + 1) + 1$ does not depend on $l,\ n,\ m$. In the same way one obtains

$$\|e_m - e_l\|^2 < c(\|a^\dagger - a_l\|^2 - \|a^\dagger - a_m\|^2)$$

so that

$$\|a_m - a_n\|^2 = \|e_m - e_n\|^2 \le 2\|e_m - e_l\|^2 + 2\|e_l - e_n\|^2 < 2c(\|a^\dagger - a_n\|^2 - \|a^\dagger - a_m\|^2)$$

The right-hand side tends to zero for $n,\ m \to \infty$ because of the monotonicity of the iteration error, and hence $\{a_n\}$ is a Cauchy sequence.

Denote the limit of a_n by a. From (3.10), by summation that $\sum_{n=0}^{\infty} \|u - F(a_n)\|^2$ converges, and therefore $F(a_n) \to u$ as $n \to \infty$. Thus, it has been shown that a is a solution of (1.1). □

In practice, only the perturbed right-hand side u^δ with (1.2) will be known. Hence we need to discuss the convergence of Algorithm 1 with inexact right-hand side. To emphasize this point the corresponding iterates will be denoted by a_n^δ further on. In case of perturbed data it is another key point to stop the outer iteration appropriately early to prevent divergence. Algorithm 1 terminates the outer loop as soon as the residual norm $\|u^\delta - F(a_n^\delta)\|$ is of the order of the noise level δ: more precisely, if τ is a fixed positive number, then the stopping index $n(\delta)$ is the smallest iteration index $n \in \mathbb{N}$ for which

$$\|u^\delta - F(a_n^\delta)\| \le \tau\delta \tag{4.5}$$

The following result shows that this stopping rule is well defined and provides a stable approximation of a solution of $F(a) = u$.

Theorem 5. *Let $0 < \rho < 1$, and $\frac{\underline{\alpha}\rho^2\tau}{\underline{\alpha} + M^2} > 2$, where $\underline{\alpha}$ satisfies (2.6). Assume that $F'(\cdot)$ is locally bounded in $D(F)$ with (3.1) and that F satisfies (3.2) for some $C > 0$ in a ball $\mathcal{B} \subset \mathcal{D}(\mathcal{F})$ around $a^\dagger$. If $\|u - u^\delta\| \le \delta$ and if $a_0^\delta \in \mathcal{B}$ is sufficiently close to a solution $a^\dagger$ of $F(a) = u$, then the discrepancy principle (4.5) terminates Algorithm 1 with (3.8) terminating the inner iteration after $n(\delta) < \infty$ iterations. Moreover, the corresponding approximation $a_{n(\delta)}^\delta$ converges to a solution of $F(a) = u$ as $\delta \to 0$.*

Proof. At first it will be shown that

$$\|a^\dagger - a_n^\delta\| < \|a^\dagger - a_{n-1}^\delta\|,\ n = 1, 2, \cdots, n(\delta) \tag{4.6}$$

Assume an open ball around $a^\dagger$ of radius $\frac{\underline{\alpha}\rho^2\tau - 2(\underline{\alpha} + M^2)}{2C(1+\tau)(\underline{\alpha} + M^2)}$ including a_0^δ belongs to $\mathcal{B}$. In this case it follows from (3.2) that

$$\begin{aligned}\|u^\delta - F(a_0^\delta) - F'(a_0^\delta)(a^\dagger - a_0^\delta)\| &\le \delta + C\|a^\dagger - a_0^\delta\|\|u - F(a_0^\delta)\| \\ &\le (1 + C\|a^\dagger - a_0^\delta\|)\delta + C\|a^\dagger - a_0^\delta\|\|u^\delta - F(a_0^\delta)\|\end{aligned}$$

If $n(\delta) > 0$, then $\delta < \frac{\|u^\delta - F(a_0^\delta)\|}{\tau}$, hence from above

$$\|u^\delta - F(a_0^\delta) - F'(a_0^\delta)(a^\dagger - a_0^\delta)\| \le \frac{1 + (1+\tau)C\|a^\dagger - a_0^\delta\|}{\tau}\|u^\delta - F(a_0^\delta)\|$$

This shows that (3.7) holds for $n = 0$ with $\gamma = \dfrac{\underline{\alpha}\rho^2\tau}{(\underline{\alpha} + M^2)[1 + (1+\tau)C\|a^\dagger - a_0^\delta\|]}$ which is greater than 2 by assumption, so is (4.6) for $n = 1$. By induction (3.7) holds for $n < n(\delta)$. Consequently Theorem 3 deduces the monotonicity assertion (4.6).

Now taking sum of (3.10) from $n = 0$ to $n(\delta) - 1$, one obtains

$$n(\delta)\tau^2\delta^2 \leq \sum_{n=0}^{n(\delta)-1} \|u^\delta - F(a_n^\delta)\|^2 < \frac{\gamma(\underline{\alpha} + M^2)^2}{(\gamma - 2)\underline{\alpha}\rho^2}\|a^\dagger - a_0^\delta\|^2 < \infty$$

This shows that $n(\delta)$ is a finite number.

Next consider $a_{n(\delta)}^\delta$ as $\delta \to 0$ and discuss two special cases firstly. First, if $n(\delta) = n$ for all $\delta > 0$, by continuity, then $a_n^\delta \to a_n$ as $\delta \to 0$, where a_n is the nth iterate with exact right-hand side u. Furthermore, since $\|u^\delta - F(a_n^\delta)\| \leq \tau\delta$ by definition of $n = n(\delta)$ there must hold $F(a_n) = u$ in the limit $\delta \to 0$. Consequently, $a_{n(\delta)}^\delta$ converges to the solution a_n of $F(a) = u$ in the case that $n(\delta) = n$ for all $\delta > 0$.

Second, assume that $n(\delta) \to \infty$ as $\delta \to 0$, and denote by a the limit of $\{a_n\}$ which exists by Theorem 4. Given $\epsilon > 0$, let $m(\epsilon) \in \mathbb{N}$ be such that $\|a - a_m\| < \dfrac{\epsilon}{2}$ for $m \geq m(\epsilon)$, and let $\delta(\epsilon)$ be so small that $n(\delta) > m(\epsilon)$ for $\delta < \delta(\epsilon)$, then it follows from (4.6) that

$$\|a - a_{n(\delta)}^\delta\| < \|a - a_m^\delta\| \leq \|a - a_m\| + \|a_m - a_m^\delta\| \leq \frac{\epsilon}{2} + \|a_m - a_m^\delta\|$$

for all $\delta < \delta(\epsilon)$ and some $m = m(\epsilon)$. Again by continuity it follows that $\|a_m - a_m^\delta\| < \dfrac{\epsilon}{2}$ as δ is sufficiently small and hence $\|a - a_{n(\delta)}^\delta\| < \epsilon$ for δ sufficiently small. This proves $a_{n(\delta)}^\delta \to a$ as $\delta \to 0$ in the case where $n(\delta) \to \infty$.

In the end, we consider the general case. If the convergence result doesn't hold, then there must exist $\sigma > 0$ and subsequence $\{\delta_i\}$, $\delta_i \to 0$, such as

$$\|a_{n(\delta_i)}^{\delta_i} - a\| \geq \sigma \tag{4.7}$$

Since the real sequence $\{n(\delta_i)\}$ must exist convergence subsequence(finite or infinite), we still denote the convergence subsequence by $\{n(\delta_i)\}$. If $n(\delta_i) \to \infty$, as $i \to \infty$, we have from the second special case that

$$\|a_{n(\delta_i)}^{\delta_i} - a\| \to 0 \text{ as } i \to \infty$$

which contradicts (4.7). If $n(\delta_i) \to n$, we can similarly deduce a contradiction by the first special case. This proves $a_{n(\delta)}^\delta \to a$ as $\delta \to 0$ in the general case. □

5. Numerical examples

Consider a numerical example for a parameter estimation problem which is a simplified model of the groundwater hydrology[3], i.e.

$$\begin{cases} -(a(x)u_x(x))_x = -e^x, & x \in (0,1) \\ u(0) = 1, \;\; u(1) = e^1 \end{cases} \tag{5.1}$$

The assumptions (3.1) and (3.2) hold[3]. Estimate the coefficient a in (5.1) for the given perturbed value of u.

If the exact data $u = e^x$, then the solution of the inverse problem $a^\dagger$ is unique and $a^\dagger = 1$. Instead of u we used in our computations the perturbed data u^δ, where

$$u^\delta = u + \delta\sqrt{2}\cos(10\pi x) \tag{5.2}$$

with $\|u^\delta - u\|_{L_2} \le \delta$.

The differential equation (5.1) was solved with a Galerkin method on the finite dimensional subspace of piecewise linear splines on a uniform grid with subinterval length 1/128. The iterative solution a_n^δ were obtained by solving (2.4) on the finite dimensional subspace of piecewise linear splines on a uniform grid with subinterval length 1/128 with implicit iterative method (2.5).

In order to observe the convergence of Algorithm 1 and the effectiveness of the results, two types of the initial value a_0^δ are used in the computation, i.e. a_0^δ=const and $a_0^\delta = a^\dagger + \sigma \sin 10\pi x (\sigma > 0)$, including those with large initial errors $\|a_0^\delta - a^\dagger\|_{L_2}$. Let $n(\delta)$ denote the outer iteration numbers and $e_0 = \|a_0^\delta - a^\dagger\|_{L_2}$, $e_n = \|a_n^\delta - a^\dagger\|_{L_2}$. The parameter $\tau = 1.1$ while δ in (5.2) is chosen as 10^{-5} and 10^{-8} in Table 1 and Table 2, respectively.

To accelerate the convergence, we used variable parameters $\alpha_k = \frac{1}{2}\alpha_{k-1}$, $\alpha_0 = 0.5$ to replace α_n in the inner iterative procedure (2.5). It can be proved that all the theoretical results in the paper can be expanded to the variable control parameters(cf.[6]).

From the numerical results, we can see the method is effective. Table 1 and Table 2 show that the iteration numbers increase as δ decreases. For the same δ, the iteration numbers increase as the iteration initial errors increase. In the convergence analysis, the initial value of the iteration a_0^δ is required to be close sufficiently to $a^\dagger$. However, in practical computations, even if a_0^δ is far from $a^\dagger$, the algorithm still shows the convergence property.

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Table 1. ($\delta = 10^{-5}$)

a_0^δ	ρ	$n(\delta)$	e_0	e_n	e_n/e_0
0.5	0.8	35	0.496E+0	0.202E+0	0.408E0
3	0.8	29	1.984E+0	1.124E+0	0.566E0
7	0.8	48	5.953E+0	0.192E+0	0.323E-1
8	0.95	132	6.945E+0	1.029E+0	0.148E0
12	0.95	129	1.091E+1	1.035E+0	0.948E-1
15	0.98	332	1.389E+1	1.194E+0	0.860E-1
20	0.98	327	1.885E+1	1.190E+0	0.631E-1
35	0.98	343	3.373E+1	1.188E+0	0.352E-1
1+0.3sin(10πx)	0.8	26	0.212E+0	0.330E-1	0.156E0
1+0.5sin(10πx)	0.8	28	0.353E+0	0.766E-1	0.217E0
1+0.7sin(10πx)	0.8	30	0.495E+0	0.126E+0	0.254E0
1+0.9sin(10πx)	0.8	33	0.636E+0	0.180E+0	0.283E0

Table 2. ($\delta = 10^{-8}$)

a_0^δ	ρ	$n(\delta)$	e_0	e_n	e_n/e_0
0.5	0.8	59	0.496E+0	4.501E-4	9.074E-4
3	0.8	52	1.984E+0	5.520E-4	2.782E-4
7	0.8	73	5.953E+0	4.899E-4	8.229E-5
8	0.9	115	6.945E+0	5.278E-4	7.599E-5
12	0.95	229	1.091E+1	5.171E-4	4.738E-5
15	0.98	549	1.389E+1	5.428E-4	3.908E-5
20	0.98	546	1.885E+1	5.393E-4	2.861E-5
35	0.98	563	3.373E+1	5.502E-4	1.631E-5
44	0.98	556	4.266E+1	5.351E-4	1.254E-5
1+0.3sin(10πx)	0.8	51	0.212E+0	5.408E-4	0.255E-2
1+0.5sin(10πx)	0.8	54	0.353E+0	4.895E-4	0.138E-2
1+0.7sin(10πx)	0.8	54	0.495E+0	4.895E-4	9.892E-4
1+0.9sin(10πx)	0.8	57	0.636E+0	5.323E-4	8.367E-4

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Zehong Meng received her Ph.D at Shanghai University under the direction of computational mathematics. Since 2007 she has been at Zhejiang University Of Finance and Economics. Her research interests focus on the theory analysis and computation of nonlinear ill-posed problems and related modules.

1. School of Mathematics and Statistics,Zhejiang University Of Finance and Economics, Hangzhou, China.
e-mail:zehongmeng78@163.com
2. College of Science, Guangdong Ocean University, Zhanjiang, China.
e-mail:wozitianshanglai@163.com