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SHARP THRESHOLDS OF BOSE-EINSTEIN CONDENSATES WITH AN ANGULAR MOMENTUM ROTATIONAL TERM†

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Abstract. In this paper, we establish a sharp condition of global existence for the solution of the Gross-Pitaevskii equation with an angular momentum rotational term. This condition is related to the ground state solution of some steady-state nonlinear Schrödinger equation.

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1. Introduction

Since its realization in dilute bosonic atomic gases [1-3], Bose-Einstein condensation of alkali atoms and hydrogen has been produced and studied extensively in the laboratory [4] and has permitted an intriguing glimpse into the macroscopic quantum world. In view of potential applications [5-7], the study of quantized vortices, which are well-known signatures of superfluidity, is one of the key issues. Different research groups have obtained quantized vortices in Bose-Einstein condensates (BEC) experimentally, e.g., the JILA group [8], the ENS group [9, 10], and the MIT group [4]. Currently, there are at least two typical ways to generate quantized vortices from the ground state of BEC: (i) impose a laser beam rotating with an angular velocity on the magnetic trap holding the atoms to create an harmonic anisotropic potential [11-14]; (ii) add to the stationary magnetic trap a narrow, moving Gaussian potential, representing a far-blue detuned laser [15, 16]. The recent experimental and theoretical advances in the exploration of quantized vortices in BEC have spurred great excitement in the atomic physics community and renewed interest in studying superfluidity.

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The properties of BEC in a rotational frame at temperature T much smaller than the critical condensation temperature T_c are well described by the macroscopic wave function $\psi(x,t)$, whose evolution is governed by a self-consistent, mean field nonlinear Schrödinger equation (NLSE) in a rotational frame, also known as the Gross.Pitaevskii equation (GPE) with an angular momentum rotation term [5, 11, 17-19]:

$$ih\psi_t + \frac{h^2}{2m}\triangle\psi = V(x)\psi + NU_0|\psi|^2\psi - \Omega L_z\psi, x \in \mathbb{R}^3, t \geq 0, \qquad (1.1)$$

where $x = (x_1, x_2, x_3)^T$ is the Cartesian coordinate vector,m is the atomic mass, h is the Planck constant, N is the number of atoms in the condensate, Ω is the angular velocity of the rotating laser beam, and $V(x)$ is an external trapping potential. When an harmonic trap potential is considered, $V(x) = \frac{m}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2)$ with ω_1, ω_2 and ω_3 being the trap frequencies in the x_1-, x_2- and x_3- direction, respectively. The local nonlinearity term $NU_0|\psi|^2\psi$ arises from am assumption about the delta-shape interatomic potential. $U_0 = \frac{4\pi h^2}{m}$ describes the interaction between atoms in the condensate with a_s(positive for repulsive interaction and negative for attractive interaction) the $s-$wave scattering length, and $L_z = -ih(x_1\partial_{x_2} - x_2\partial_{x_1})$ is the third component of the angular momentum $L = x \times P$ with the momentum operator $P = -ih\nabla$.

It is convenient to normalize the wave function by requiring that

$$\|\psi\|^2 := \int_{\mathbb{R}^3} |\psi(x,t)|^2 dx = \int_{\mathbb{R}^3} |\psi(x,0)|^2 dx = 1,\ t > 0. \qquad (1.2)$$

Under such a normalization, we introduce the dimensionless variables as follows: $t \to t/\omega_m$ with $\omega_m = \min\{\omega_x, \omega_y, \omega_z\}, \Omega \to \omega_m\Omega, x \to a_0 x$ with $a_0 = \sqrt{\frac{h}{m\omega_m}}$, and $\psi \to \frac{\psi}{a_0^{3/2}}$. We also let

$$\gamma_1 = \frac{\omega_1}{\omega_m}, \gamma_2 = \frac{\omega_2}{\omega_m}, \gamma_3 = \frac{\omega_3}{\omega_m}, \beta = \frac{U_0 N}{a_0^3 h\omega_m} = \frac{4\pi a_s N}{a_0}.$$

The dimensionless angular momentum rotational term then becomes

$$L_z = -i(x_1\partial_{x_2} - x_2\partial_{x_1}) = -i\partial_\theta$$

with (r,θ) being the polar coordinates in two dimensions (2D) and (r,θ,z) the cylindrical coordinates in three dimensions (3D). In the disk-shaped condensation, i.e., $\omega_2 = \omega_1$ and $\omega_3 \gg \omega_1 (\Leftrightarrow \gamma_1 = 1, \gamma_2 \approx 1$, and $\omega_3 \gg 1$ with choosing $\omega_m = \omega_1)$, the three-dimensional GPE can be reduced to a two-dimensional GPE [14]. Thus, here we consider the dimensionless GPE with a rotational term in the d-dimensions (d = 2, 3) [14]:

$$i\psi_t = -\frac{1}{2}\triangle\psi + V_d(x)\psi + \beta_d|\psi|^2\psi - \Omega L_z\psi, x \in \mathbb{R}^d, t > 0, \qquad (1.4)$$

$$\psi x, 0 = \psi_0(x), x \in \mathbb{R}^d,\ \textit{with}\ \ \|\psi_0\|^2 := \int_{\mathbb{R}^d} |\psi_0|^2 dx = 1, \qquad (1.5)$$

where

$$\beta_d = \begin{cases} \beta\sqrt{\frac{\gamma_3}{2\pi}}, \\ \beta, \end{cases} \qquad V_d(x) = \begin{cases} \frac{\gamma_1^2 x_1^2+\gamma_2^2 x_2^2}{2}, \ d=2, \\ \frac{\gamma_1^2 x_1^2+\gamma_2^2 x_2^2+\gamma_3^2 x_3^2}{2}, \ d=3, \end{cases} \tag{1.6}$$

with $\gamma_1 > 0, \gamma_2 > 0$, and $\gamma_3 > 0$ being constants.

In [20], authors establish the global well-posedness of the following Cauchy problem for the Gross-Pitaevskii equation with an angular momentum rotational term in two dimensions

$$i\psi_t = -\frac{1}{2}\triangle\psi + \frac{\omega^2}{2}|x|^2\psi + \beta|\psi|^{2\sigma}\psi - \Omega L_z\psi, x \in \mathbb{R}^2, t > 0, \tag{1.7}$$

$$\psi(x,0) = \psi_0(x), x \in \mathbb{R}^2, \tag{1.8}$$

where $\omega > 0, \beta > 0$ and $\sigma \in [1/\omega, \infty)$ are constants.

On the contrary, in this paper, we consider the following Cauchy problem of the Gross-Pitaevskii equation with an angular momentum rotational term in two dimensions

$$i\psi_t = -\frac{1}{2}\triangle\psi + \frac{1}{2}|x|^2\psi + |\psi|^{2\sigma}\psi - L_z\psi, x \in \mathbb{R}^2, t > 0, \tag{1.9}$$

$$\psi(x,0) = \psi_0(x), x \in \mathbb{R}^2, \tag{1.10}$$

Our aim is to observe the sharp condition of global existence for solution of (1.9)-(1.10). We are concerned with the relations between the global existence of the Cauchy problem of (1.9)-(1.10) and the ground state, which is the positive solution of the steady-state nonlinear Schrödinger equation given by

$$\triangle u - u + u^3 = 0. \tag{1.11}$$

The existence of ground state solution for nonlinear Schröinger equations (1.11), has been studied by many authors(for example, Berger[21], Coffman [22], and Strauss[23]).

We organize this paper as follows: In Section 2, we give some preliminaries and lemmas. In Section 3, we prove the sharp condition of global existence for the solution of (1.9)-(1.10).

2. Notation and preliminaries

We define a space H by

$$H := H^1(\mathbb{R}^2) \cap \{\psi : |x|\psi \in L^2(\mathbb{R}^2)\},$$

with the inner product

$$\langle\psi,\phi\rangle := \int_{\mathbb{R}^2} (\nabla\psi\cdot\nabla\overline{\phi} + \psi\overline{\phi} + |x|^2\psi\overline{\phi})dx,$$

for all $\psi, \phi \in H$. The norm of H is denoted by $\|\cdot\|_H$.

It is well known that (see [24]), two important invariants of the CGPE (1.9)-(1.10) are the normalization of the wave function

$$N_\psi(t) = \|\psi\|^2 := \int_{\mathbb{R}^2} |\psi(x,t)|^2 dx = \int_{\mathbb{R}^2} |\psi(x,0)|^2 dx = N_\Psi(0) = 1,\ t \geq 0 \quad (2.1)$$

and the energy per particle

$$H_\psi(t) = \int_{\mathbb{R}^2} (\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}|x|^2|\psi|^2 - \frac{1}{2}|\psi|^4 - Re(\overline{\psi}L_z\psi))dx \equiv H_\psi(0),\ t \geq 0, \quad (2.2)$$

where $Re(f)$ denotes the real part of the function f. Moreover, an additional important invariant is the total angular momentum expectation [24]

$$\langle L_z\rangle(t) = \langle L_z\rangle(0),\ t \geq 0 \quad (2.3)$$

where

$$\langle L_z\rangle(t) = \int_{\mathbb{R}^2} \overline{\psi} L_z \psi dx = i\int_{\mathbb{R}^2} \overline{\psi}(x_2\partial_{x_1} - x_1\partial_{x_2})\psi dx. \quad (2.4)$$

Furthermore, we have the following local existence theorem for the Cauchy problem (1.9)-(1.10).

Proposition 1. *Let $\psi_0 \in H$. Then there exists a solution $\psi \in C([0,T), H)$ of the Cauchy problem (1.9)-(1.10) for some $T \in (0,\infty]$, and $T = +\infty$ or $T < +\infty$ and $\lim_{t\to T} \|\psi\|_H^2 = \infty$.*

By a direct computation, we have

Proposition 2. *Let $\psi_0 \in H$, and ψ be a solution of the Cauchy problem (1.9)-(1.10) in $C([0,T), H)$. Put $J(t) = \int_{\mathbb{R}^2} |x|^2|\psi|^2 dx$. Then one has*

$$J'(t) = 2\Im\int_{\mathbb{R}^2} \nabla\psi \cdot x\overline{\psi} dx, \quad (2.5)$$

$$J''(t) = 2\int_{\mathbb{R}^2} (|\nabla\psi|^2 - |x|^2|\psi_j|^2 - |\psi|^4)dx = 4H_\psi(0) - 4J(t) + 4Re\langle L_z\rangle(0). \quad (2.6)$$

3. The sharp condition for global existence

From Weinstein [25] and Kwong [26], we have the following lemma.

Lemma 3.1. Equation (1.11) has a ground state solution u_0, then $\frac{1}{2}\int_{\mathbb{R}^2} u_0^2 dx$ is the minimum of the functional

$$I(\psi) = \frac{\int_{\mathbb{R}^2} |\nabla\psi|^2 dx \int_{\mathbb{R}^2} |\psi|^2 dx}{\int_{\mathbb{R}^2} |\psi|^4 dx}, \quad (3.1)$$

where $\psi \in H$.

Remark 3.2. From Lemma 3.1, we can get

$$\int_{\mathbb{R}^2} |\psi|^4 dx \leq 2\left(\int_{\mathbb{R}^2} u_0^2 dx\right)^{-1} \int_{\mathbb{R}^2} |\nabla\psi|^2 dx \int_{\mathbb{R}^2} |\psi|^2 dx, \quad (3.2)$$

which is just the Gagliardo-Nirenberg inequality.

By the same method as in [25], we get the following inequality which will be used in the proof of the main result.

Lemma3.3. Let $\psi \in H$, then we have

$$\int_{\mathbb{R}^2} |\psi|^2 dx \leq \left(\int_{\mathbb{R}^2} |\nabla \psi|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |x|^2 |\psi|^2 dx \right)^{\frac{1}{2}}. \tag{3.3}$$

Then we give a lemma about the blow up of solutions of the Cauchy problem (1.9)-(1.10).

Lemma 3.4. Let $\psi_0 \neq 0$ for all $x \in \mathbb{R}^2$ satisfies that

$$J(0) = \int_{\mathbb{R}^2} |x|^2 |\psi_0|^2 dx \geq 2(H_\psi(0) + Re\langle L_z \rangle(0)),$$

$$J'(0) = 2\Im \int_{\mathbb{R}^2} \nabla \psi_0 \cdot x\overline{\psi_0} dx \leq 0.$$

Then the solution for the Cauchy problem (1.9)-(1.10) blows up in finite time.

Proof. We prove lemma by two different case.

Case 1. $H_\psi(0) + Re\langle L_z \rangle(0) \geq 0$. From Proposition 2.2, we have

$$J(t) = \beta \sin(2t + \theta) + H_\psi(0) + Re\langle L_z \rangle(0),$$

where $\beta \geq 0$ and $\theta \in [0, 2\pi)$ are constants determined by $J(0)$, $J'(0)$ and $H_\psi(0) + Re\langle L_z \rangle(0)$. Moreover,

$$\beta^2 = [J(0) - H_\psi(0) - Re\langle L_z \rangle(0)]^2 + [\frac{J'(0)}{2}]^2. \tag{3.4}$$

For $t \in [0, \frac{\pi}{2}]$, we have

$$0 \leq J(t) = \beta \sin(2t + \theta) + H_\psi(0) + Re\langle L_z \rangle(0). \tag{3.5}$$

Thus, if $J(0) \geq 2(H_\psi(0) + Re\langle L_z \rangle(0))$ and $J'(0) = 2\beta \cos \theta$, we have $\theta \in [\frac{\pi}{2}, \pi)$. By (3.4) and (3.5) we claim that there exists $T \in [\frac{\pi}{4}, \frac{\pi}{2})$, such that $\lim_{t \to T^-} J(t) = 0$. By Lemma 3.3 and (2.1), we get that

$$\lim_{t \to T^-} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx = +\infty.$$

This shows that $\psi(x, t)$ blows up in finite time.

Case 2. $H_\psi(0) + Re\langle L_z \rangle(0) < 0$. By an analytical identity

$$J(t) = J(0) + J'(0)t + \int_0^t J''(s)(t - s) ds.$$

By using (2.6), we have

$$J''(t) \leq 4H_\psi(0) + 4Re\langle L_z \rangle(0).$$

Hence

$$J(t) \leq J(0) + J'(0)t + 2(H_\psi(0) + Re\langle L_z \rangle(0))t^2.$$

Since $H_\psi(0) + Re\langle L_z\rangle(0) < 0$, $J(0) \geq 0$ and $J'(0) \leq 0$, it implies that there exists a $0 < T < +\infty$ such that $\lim_{t\to T^-} J(t) = 0$. It follows as Case 1 that $\psi(x,t)$ blows up in finite time. This completes the proof of Lemma 3.4. $\square$

With the lemmas above, we can give a sharp condition of global existence of (1.9)-(1.10).

Theorem 3.5. Let u_0 be radially symmetric solution of equations (1.11). If $\psi_0 \in H$, and $\int_{\mathbb{R}^2} |\psi_0|^2 dx < \int_{\mathbb{R}^2}(|u_0|^2 dx$, then the corresponding solution $\psi(x,t)$ of (1.9)-(1.10) exists globally in time. At the same time, for arbitrary positive ν and complex c satisfying $|c| \geq 1$, if we take initial date $\psi_0 \in H$ and $\psi_0(x,0) = c\nu u_0(\nu x)$, then $\int_{\mathbb{R}^2} |\psi_0|^2 dx \geq \int_{\mathbb{R}^2} |u_0|^2 dx$, and the corresponding solution $\psi(x,t)$ of (1.9)-(1.10) blows up in finite time.

Proof. Let $\psi(x,t) \in C([0,T), H)$ be a solution of the Cauchy problem (1.9)-(1.10), where $[0,T)$ is the maximal existence time. From (2.2),(2.3), and Remark 3.2, we get

$$\int_{\mathbb{R}^2} \frac{1}{2}\left(1 - \frac{\int_{\mathbb{R}^2} |\psi_0|^2 dx}{\int_{\mathbb{R}^2}(|u_0|^2 dx}\right)|\nabla\psi|^2 dx + \int_{\mathbb{R}^2} \frac{1}{2}|x|^2|\psi|^2 dx \leq H_\psi(0) + Re\langle L_z\rangle(0). \tag{3.6}$$

From $\int_{\mathbb{R}^2}(|\psi_0|^2 dx < \int_{\mathbb{R}^2} |u_0|^2 dx$ and (2.2),(2.3), we get $\int_{\mathbb{R}^2} |\nabla\psi|^2 dx$ and $\int_{\mathbb{R}^2} |x|^2|\psi|^2 dx$ are bounded for $t \in [0,T)$. By Lemma 3.3, it yields that $\psi(x,t)$ globally exists in $t \in [0,\infty)$.

Now we take initial date such that $\psi(x,0) = c\nu u_0(\nu x)$, with arbitrary positive ν and complex c satisfying $|c| \geq 1$. Then

$$\int_{\mathbb{R}^2} |\psi_0|^2 dx = |c|^2 \int_{\mathbb{R}^2} |u_0|^2 dx \geq \int_{\mathbb{R}^2} |u_0|^2 dx,$$

and

$$\int_{\mathbb{R}^2} |\nabla\psi_0|^2 dx = |c|^2\nu^2 \int_{\mathbb{R}^2} |\nabla u_0|^2 dx. \tag{3.7}$$

From (2.2), (2.3), (3.7) and the condition that c satisfies $|c| \geq 1$, we have

$$H_\psi(0) + Re\langle L_z\rangle(0) \leq \frac{1}{2}(1 - |c|^2)|c|^2\nu^2 \int_{\mathbb{R}^2} |\nabla u_0|^2 dx + \frac{1}{2}J(0) \leq \frac{1}{2}J(0), \tag{3.8}$$

and

$$J'(0) = -2\Im \int_{\mathbb{R}^2} x\psi)_0 \cdot \nabla\overline{\psi}_0 dx = -2|c|^2\Im \int_{\mathbb{R}^2} xu_0 \cdot \nabla\overline{u}_0 dx = 0. \tag{3.9}$$

Thus Lemma 3.4 and (3.8), (3.9) yields that $\psi(x,t)$ of (1.9)-(1.10) blows up in finite time.

This completes the proof Theorem 3.5. $\square$

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