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GLOBAL STABILITY OF A NONLINEAR DIFFERENCE EQUATION†

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Abstract. In this paper, we investigate the local asymptotic stability, the invariant intervals, the global attractivity of the equilibrium points, and the asymptotic behavior of the solutions of the difference equation $x_{n+1} = \frac{a+bx_nx_{n-k}}{A+Bx_n+Cx_{n-k}}$, $n = 0, 1, \dots$, where the parameters a, b, A, B, C and the initial conditions $x_{-k}, \dots, x_{-1}, x_0$ are positive real numbers.

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1. Introduction and preliminaries

In this paper, we consider the following rational difference equation

$$x_{n+1} = \frac{a + bx_nx_{n-k}}{A + Bx_n + Cx_{n-k}}, \quad n = 0, 1, \dots , \tag{1.1}$$

where the parameters $a, b, A, B, C > 0$, and the initial conditions $x_{-k}, \dots, x_{-1}$, $x_0 > 0$. $k \in \{1, 2, 3, \dots\}$.

We investigate the local and global asymptotic behaviour and the invariant intervals of the solutions of the difference Eq.(1.1). It is worth mentioning that our main results are motivated by the results in [3,5,6]. For some related works see [1-15].

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$f : I^{k+1} \to I$

be a continuously differentiable function .Then for every pair of initial conditions $x_{-k}, \dots, x_0 \in I$, the difference equation

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$$x_{n+1} = f(x_n, x_{n-k}), \quad ,n = 0, 1, \dots , \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.(see [2])

A point $\overline{x}$ is called an equilibrium point of (1.2) if

$\overline{x} = f(\overline{x}, \overline{x})$.

That is, $x_n = \overline{x}$, for $n \geq -k$ is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of *f*.

Definition 1.1 (Stability). Let $\overline{x}$ be an equilibrium point of Eq.(1.2) and assume that I is some interval of real numbers.

(i) The equilibrium $\overline{x}$ of Eq. (1.2) is called locally stable (or stable) if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x_{-k}, \dots , x_{-1}, x_0 \in I$ and $|x_{-k} - \overline{x}| + \dots + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta$, then

$|x_n - \overline{x}| < \epsilon$ for all$n \geq -k$.

(ii) The equilibrium $\overline{x}$ of Eq. (1.2) is called locally asymptotically stable (or asymptotically stable) if it is stable and if there exist $\gamma > 0$ such that if

$x_{-k}, \dots , x_{-1}, x_0 \in I$ and $|x_{-k} - \overline{x}| + \dots + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma$,

then

$\lim_{n\to\infty} x_n = \overline{x}$

(iii) The equilibrium $\overline{x}$ of Eq. (1.2) is called a global attractor if for every $x_{-k}, \dots , x_{-1}, x_0 \in I$, we have

$\lim_{n\to\infty} x_n = \overline{x}$

(iv) The equilibrium $\overline{x}$ of Eq. (1.2) is called a globally asymptotically stable (or globally stable) if it is stable and is a global attractor.

(v) The equilibrium $\overline{x}$ of Eq. (1.2) is called unstable if it is not stable.

Let

$u = \frac{\partial f}{\partial x}(\overline{x}, \overline{x})$ and $v = \frac{\partial f}{\partial y}(\overline{x}, \overline{x})$,

where $f(x, y)$ is the function in Eq.(1.2) and $\overline{x}$ is an equilibrium of the equation. Then the equation

$$y_{n+1} = uy_n + vy_{n-k}, \quad n = 0, 1, \dots \tag{1.3}$$

is called the linearized equation associated with Eq.(1.2) about the equilibrium point $\overline{x}$.

Lemma 1.1 (see [1, 12]). Assume that $u, v \in R$ and $k \in \{1, 2, \dots\}$. Then

$$|u| + |v| < 1 \tag{1.4}$$

is a sufficient condition for the asymptotic stability of the difference equation (1.3). Suppose in addition that one of the following two cases holds:

(i) k is odd and $v > 0$;

(ii) k is even and $uv > 0$.

Then (1.4) is also a necessary condition for the asymptotic stability of Eq.(1.3).

Lemma 1.2 (see [2]). Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, 2, \ldots$$

where $k \in \{1, 2, ...\}$. Let $I = [c, d]$ be some interval of real numbers and assume that
$f : [c, d] \times [c, d]] \to [c, d]$
is a continuous function satisfying the following properties:
(i) $f(x, y)$ is increasing in each argument in $[c, d]$;
(ii) the equation $f(x, x) = x$ has a unique positive solution.
Then Eq.(1.2) has a unique equilibrium $\overline{x} \in [c, d]$ and every solution of Eq.(1.2) converges to $\overline{x}$.

Lemma 1.3 (see [2, 4]). Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, 2, \ldots$$

where $k \in \{1, 2, ...\}$. Let $I = [c, d]$ be some interval of real numbers and assume that
$f : [c, d] \times [c, d] \to [c, d]$
is a continuous function satisfying the following properties:
(i) $f(x, y)$ is non-decreasing in $x \in [c, d]$ for each $y \in [c, d]$, and $f(x, y)$ is non-increasing in $y \in [c, d]$ for each $x \in [c, d]$;
(ii) If $(m, M) \in [c, d] \times [c, d]$ is a solution of the system $f(m, M) = m$ and $f(M, m) = M$, then $m = M$.
Then Eq.(1.2) has a unique equilibrium $\overline{x} \in [c, d]$ and every solution of Eq.(1.2) converges to $\overline{x}$.

Lemma 1.4 (see [4, 5]). Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, 2, \ldots$$

where $k \in \{1, 2, \ldots\}$. Let $I = [c, d]$ be some interval of real numbers and assume that
$f : [c, d] \times [c, d] \to [c, d]$
is a continuous function satisfying the following properties:
(i) $f(x, y)$ is nonincreasing in each of its arguments;
(ii) If $(m, M) \in [c, d] \times [c, d]$ is a solution of the system $f(m, m) = M$ and $f(M, M) = m$, then $m = M$.
Then Eq.(1.2) has a unique equilibrium $\overline{x} \in [c, d]$ and every solution of Eq.(1.2) converges to $\overline{x}$.

2. Local stability and periodic character

In this section, we consider the local stability and periodic character of the positive solutions of Eq.(1.1).

The change of variables $y_n = \sqrt{b}x_n$ followed by the change $x_n = y_n$ reduces Eq.(1.1) to the difference equation

$$x_{n+1} = \frac{a + x_n x_{n-k}}{r + px_n + qx_{n-k}}, \quad n = 0, 1, \dots, \tag{2.1}$$

where $r = \frac{A}{\sqrt{b}}$, $p = \frac{B}{b}$, $q = \frac{C}{b}$, and the initial conditions $x_{-k}, \dots, x_{-1}, x_0$ are positive real numbers. Hereafter, we focus our attention on Eq.(2.1) instead of Eq.(1.1).

The equilibrium points of Eq.(2.1) have the following four cases:

Case 1: $p + q > 1$, Eq.(2.1) has a unique positive equilibrium point

$$\alpha = \frac{-r + \sqrt{r^2 + 4a(p + q - 1)}}{2(p + q - 1)}.$$

Case 2: $p + q = 1$, Eq.(2.1) has a unique positive equilibrium point

$$\beta = \frac{a}{r}.$$

Case 3: $p + q < 1$, $r^2 - 4a(1 - p - q) = 0$, Eq.(2.1) has a unique positive equilibrium point

$$\gamma = \frac{r}{2(1 - p - q)}.$$

Case 4: $p + q < 1$, $r^2 - 4a(1 - p - q) > 0$, Eq.(2.1) has the following two equilibrium points:

$$\eta_1 = \frac{r + \sqrt{r^2 - 4a(1 - p - q)}}{2(1 - p - q)}, \qquad \eta_2 = \frac{r - \sqrt{r^2 - 4a(1 - p - q)}}{2(1 - p - q)}.$$

First, we consider the local asymptotic behavior of the equilibrium point α in *Case 1.*

Let $f : (0, \infty)^2 \to (0, \infty)$ be a function defined by

$$f(x, y) = \frac{a + xy}{r + px + qy}.$$

Therefore it follows that

$$f_x = \frac{yr + qy^2 - ap}{(r + px + qy)^2}, \quad f_y = \frac{xr + px^2 - aq}{(r + px + qy)^2}.$$

For simplicity and convenience in the following discussion, we define:

$$g_1 := g_1(y) = yr + qy^2 - ap, \quad g_2 := g_2(x) = xr + px^2 - aq.$$

$$\theta_1 = \frac{-r + \sqrt{r^2 + 4apq}}{2q}, \qquad \theta_2 = \frac{-r + \sqrt{r^2 + 4apq}}{2p},$$

where θ_1, θ_2 is the unique positive solution of equation $g_1 = 0$ and equation $g_2 = 0$ respectively.

The linearized equation associated wih Eq.(2.1) about α is

$$z_{n+1} = \frac{r\alpha + q\alpha^2 - ap}{[r + (p+q)\alpha]^2} z_n + \frac{r\alpha + p\alpha^2 - aq}{[r + (p+q)\alpha]^2} z_{n-k}, \quad n = 0, 1, 2, \dots ,$$

The local asymptotic behavior of α is characterized by the following result.

Theorem 2.1. *Assume $p+q > 1,\ p < q$. Then α is locally asymptotically stable if one of the following three conditions is satisfied:*

(i) $p < q \le 1$,

(ii) $p < 1 < q$ *and* $r^2 - a(q-p) \ge 0$,

(iii) $1 < p < q$ *and* $r^2 \ge a(p+q)$.

Proof. When $p < q,\ \theta_1 < \theta_2$. Let

$$u = f_x(\alpha, \alpha) = \frac{r\alpha + q\alpha^2 - ap}{[r + (p+q)\alpha]^2}, \quad v = f_y(\alpha, \alpha) = \frac{r\alpha + p\alpha^2 - aq}{[r + (p+q)\alpha]^2}.$$

(i) If $p < q \le 1$, then $\alpha \ge \theta_2$ and $g_1 > 0,\ g_2 > 0$ in $[\theta_2, \infty)$. So, $|u| + |v| = u + v < 1$.
(ii) If $p < 1 < q$ and $r^2 - a(q-p) \ge 0$, then $\theta_1 < \alpha < \theta_2$, and $g_1 > 0,\ g_2 < 0$ in (θ_1, θ_2). Thus, $|u| + |v| = u - v < 1$.
(iii) If $1 \le p < q$ and $r^2 \ge a(p+q)$, then $\alpha < \theta_1$, and $g_1 < 0,\ g_2 < 0$ in $(0, \theta_1]$. Therefore, $|u| + |v| = -u - v < 1$. The result follows from Lemma 1.1. □

Theorem 2.2. *Assume $p+q > 1,\ p > q$. Then α is locally asymptotically stable if one of the following three conditions is satisfied:*

(i) $1 \ge p > q$,

(ii) $p > 1 > q$,

(iii) $p > q \ge 1$ *and* $r^2 \ge a(p+q)$.

Proof. The proof is similar to that of Theorem 2.1, and we omit it. □

Theorem 2.3. *Assume $p+q > 1,\ p = q$. Then α is locally asymptotically stable if one of the following three conditions is satisfied:*

(i) $p = q < 1$,

(ii) $p = q > 1$, *and* $r^2 \ge a(p+q)$.

Proof. The proof is similar to that of Theorem 2.1, and we omit it. □

Next, we investigate the local asymptotic behavior of the remaining equilibrium points $\beta,\ \eta_1,\ \eta_2,\ \gamma$ and have the following result.

Theorem 2.4.

(i) *If $p+q = 1$, then β is locally asymptotically stable.*

(ii) *If $p+q < 1,\ r^2 - 4a(1-p-q) = 0$, then γ is locally asymptotically stable.*

(iii) *If $p+q < 1,\ r^2 - 4a(1-p-q) > 0$, then η_1 is unstable, while η_2 is locally asymptotically stable.*

Proof. (i)The linearized equation associated wih Eq.(2.1) about the equilibrium point β is

$$z_{n+1}=\frac{r\beta+q\beta^2-ap}{[r+(p+q)\beta]^2}z_n+\frac{r\beta+p\beta^2-aq}{[r+(p+q)\beta]^2}z_{n-k},\quad n=0,1,2,\dots.$$

Set

$$u=f_x(\beta,\beta)=\frac{r\beta+q\beta^2-ap}{[r+(p+q)\beta]^2},\quad v=f_y(\beta,\beta)=\frac{r\beta+p\beta^2-aq}{[r+(p+q)\beta]^2}.$$

If $p+q=1$, then $\beta>\theta_1,\quad \beta>\theta_2$, and $g_1(\beta)>0,\ g_2(\beta)>0$. Thus, $|u|+|v|=u+v<1$, the result (i) follows from Lemma1.1.

(ii)If $p+q<1,\ r^2-4a(1-p-q)=0$, then $\gamma>\theta_1,\quad \gamma>\theta_2$. Similarly to the proof of (i), the result follows from Lemma1.1.

(iii)The linearized equation associated wih Eq.(2.1) about the equilibrium point η_1 is

$$z_{n+1}=\frac{r\eta_1+q{\eta_1}^2-ap}{[r+(p+q)\eta_1]^2}z_n+\frac{r\eta_1+p{\eta_1}^2-aq}{[r+(p+q)\eta_1]^2}z_{n-k},\quad n=0,1,2,\dots.$$

Set

$$u=f_x(\eta_1,\eta_1)=\frac{r\eta_1+q{\eta_1}^2-ap}{[r+(p+q)\eta_1]^2},\quad v=f_y(\eta_1,\eta_1)=\frac{r\eta_1+p{\eta_1}^2-aq}{[r+(p+q)\eta_1]^2}.$$

If $p+q<1,\ r^2-4a(1-p-q)>0$, then $\eta_1>\theta_1,\quad \eta_1>\theta_2$, and $g_1(\eta_1)>0,\ g_2(\eta_1)>0$. But $|u|+|v|=u+v>1$, and $v>0,\quad uv>0$. So, η_1 is unstable by Lemma1.1(i-ii).

As for as η_2, similarly to the proof of (i), the result follows from Lemma1.1. □

In the following, we consider the periodic character of the positive solutions of Eq.(2.1).

Theorem 2.5. *Eq.(2.1) has no positive solutions with prime period two.*

Proof. Assume, for the sake of contradiction, that

$\dots,\Phi,\Psi,\Phi,\Psi,\dots$

is a prime period-two positive solution of Eq.(2.1).There are two cases to be considered.

Case 1: k is odd.

In this case, $x_{n+1}=x_{n-k}$, Φ and Ψ satisfy the system

$a+\Psi\Phi=\Phi r+p\Phi\Psi+q\Phi^2$ and $a+\Phi\Psi=\Psi r+p\Phi\Psi+q\Psi^2$.

Subtracting these two equations, we obtain

$(\Phi-\Psi)[r+q(\Phi+\Psi)]=0$.

Since $r+q(\Phi+\Psi)>0$, we have $\Phi=\Psi$. This is a contradiction.

Case 2: k is even.

In this case, $x_n=x_{n-k}$, Φ and Ψ satisfy the system

$a+\Psi^2=\Phi r+(p+q)\Phi\Psi$ and $a+\Phi^2=\Psi r+(p+q)\Phi\Psi$.

Subtracting these two equations, we obtain

$(\Psi-\Phi)(\Psi+\Phi+r)=0$.

Since $\Psi+\Phi+r > 0$, we have $\Phi = \Psi$. This is a contradiction. □

3. Global attractivity of the positive equilibrium points

In this section, we investigate the global attractivity of the four equilibrium points $\alpha,\ \beta,\ \gamma,\ \eta_2$.

First, we consider the global attractivity of α and have the following results.

Theorem 3.1. *Assume that* $p+q > 1,\ p < q$. *Then the positive equilibrium* α *of Eq.(2.1) is a global attractor.*

Proof. When $p < q,\ \theta_1 < \theta_2$. Set

$$f(x,y) = \frac{a+xy}{r+px+qy}.$$

Observe that the function $f(x,y)$ increases in each argument in $[\theta_2,\ \infty)$, increases in x for all $y \in (\theta_1,\ \theta_2)$, decreases in y for all $x \in (\theta_1,\ \theta_2)$, decreases in each argument in $(0,\ \theta_1]$.

We divide the proof into the following three case:

Case 1: $p < q \leq 1$.

Since $g_1 > 0,\ g_2 > 0$ for all $x, y \in [\theta_2,\ \infty)$, $f(x,y)$ increases in each argument in $(\theta_2,\ \infty)$. $\alpha \geq \theta_2$ if $p < q < 1$.

The equation $f(x,x) = x$ has a unique solution α in $[\theta_2,\ \infty)$. Thus by Lemma1.2, Eq.(2.1) has a unique equilibrium point α and every solution of Eq.(2.1) converges to α in $[\theta_2,\ \infty)$.

Case 2: $p < 1 < q$.

Since $g_1 > 0,\ g_2 < 0$ for all $x, y \in (\theta_1,\ \theta_2)$. $f(x,y)$ increases in x for all $y \in (\theta_1,\ \theta_2)$ and decreases in y for all $x \in (\theta_1,\ \theta_2)$. $y_2 < \alpha < \theta_2$ if $p < 1 < q$. The only solution of the system

$$m = \frac{a+mM}{r+pm+qM}, \quad M = \frac{a+Mm}{r+pM+qm}$$

is $m = M$. Then by Lemma1.3, every solution of Eq.(2.1) converges to α in $(\theta_1,\ \theta_2)$.

Case 3: $1 \leq p < q$.

Since $g_1 < 0,\ g_2 < 0$ for all $x, y \in (0,\ \theta_1]$, $f(x,y)$ decreases in each argument in $(0,\ \theta_1]$. $\alpha \leq \theta_1$ if $1 < p < q$. The only solution of the system

$$M = \frac{a+m^2}{r+(p+q)m}, \quad m = \frac{a+M^2}{r+(p+q)M}$$

is $m = M$. Then following from Lemma1.4, every solution of Eq.(2.1) converges to α in $(0,\ \theta_1]$. □

Theorem 3.2. *Assume that* $p+q > 1,\ p > q$.

Then the positive equilibrium α *of Eq.(2.1) is a global attractor.*

Proof. When $p > q$, $\theta_1 > \theta_2$. Similar to the proof of Theorem 3.1, and it is easy to verify that the following statement is true:
(i) If $1 \geq p > q$, then every solution of Eq.(2.1) converges to α in $[\theta_1, \infty)$.
(ii) If $p > 1 > q$, then every solution of Eq.(2.1) converges to α in (θ_2, θ_1).
(iii) If $p > q \geq 1$, then every solution of Eq.(2.1) converges to α in $(0, \theta_2]$. □

Theorem 3.3. *Assume that $p+q > 1$, $p = q$. Then the positive equilibrium α of Eq.(2.1) is a global attractor.*

Proof. Similar to the proof of Theorem 3.1, the following statement is easily proved to be true:

(i) If $p = q \leq 1$, then Eq.(2.1) has a unique equilibrium point α and every solution of Eq.(2.1) converges to α in $[\theta_2, \infty)$.

(ii) If $p = q > 1$, then every solution of Eq.(2.1) converges to α in $(0, \theta_2)$. □

By Theorem 3.1, 3.2, 3.3 and Lemma2.1, we have the following corollaries.

Corollary 3.1. *Assume $p + q > 1$, $p < q$. Then α is globally asymptotically stable if one of the following three conditions is satisfied:*
(i) $p < q \leq 1$,
(ii) $p < 1 < q$ *and* $r^2 - a(q - p) \geq 0$,
(iii) $1 < p < q$ *and* $r^2 \geq a(p + q)$.

Corollary 3.2. *Assume $p + q > 1$, $p > q$. Then α is globally asymptotically stable if one of the following three conditions is satisfied:*
(i) $1 \geq p > q$,
(ii) $p > 1 > q$,
(iii) $p > q \geq 1$ *and* $r^2 \geq a(p + q)$.

Corollary 3.3. *Assume $p + q > 1$, $p = q$. Then α is globally asymptotically stable if one of the following three conditions is satisfied:*
(i) $p = q \leq 1$,
(ii) $p = q > 1$, *and* $r^2 \geq a(p + q)$.

Next, we examine the global attractivity of β, γ, η_2.

Theorem 3.4.
(i) *If $p + q = 1$, then β is a global attractor.*
(ii) *If $p + q < 1$, $r^2 - 4a(1 - p - q) = 0$, then γ is a global attractor.*
(iii) *If $p + q < 1$, $r^2 - 4a(1 - p - q) > 0$, then η_2 is a global attractor.*

Proof. (i) Set

$$f(x, y) = \frac{a + xy}{r + px + qy}.$$

We divide the proof into the following two case:
Case 1: $p > q$.

In this case, $\theta_2 < \theta_1,\ \beta > \theta_1,\ f(x,y)$ is increasing in each argument in (θ_1,∞). What's more, the equation $f(y,y)=y$ has a unique solution β in (θ_1,∞). Thus by Lemma1.2, Eq.(2.1) has a unique equilibrium point β and every solution of Eq.(2.1) converges to α in (θ_1,∞).

Case 2: $p \le q$.

In this case, $\theta_2 \ge \theta_1,\ \beta > \theta_2,\ f(x,y)$ is increasing in each argument in (θ_2,∞). In addition, the equation $f(y,y)=y$ has a unique solution β in (θ_2,∞). Thus by Lemma1.2, Eq.(2.1) has a unique equilibrium point β and every solution of Eq.(2.1) converges to α in (θ_2,∞).

(ii) The proof is similar to that of (i), and we omit it.

(iii) If $p+q<1,\ r^2-4a(1-p-q)>0$, then $\eta_1 > \eta_2 > \max\{\theta_2,\theta_1\}$ and $g_1(\eta_1)>0,\ g_2(\eta_1)>0,\ g_1(\max\{\theta_2,\theta_1\})>0,\ g_2(\max\{\theta_2,\theta_1\})>0$. So, $f(x,y)$ is increasing in each argument in $(\max\{\theta_2,\theta_1\},\eta_1)$. What's more, the equation $f(y,y)=y$ has a unique solution η_2 in $(\max\{\theta_2,\theta_1\},\eta_1)$. Thus by Lemma1.2, Eq.(2.1) has a unique equilibrium point η_2 and every solution of Eq.(2.1) converges to η_2 in $(\max\{\theta_2,\theta_1\},\eta_1)$. □

By Theorem 3.4 and Lemma2.1, we obtain the following corollary.

Corollary 3.4.
(i) *If* $p+q=1$, *then* β *is globally asymptotically stable.*
(ii) *If* $p+q<1,\ r^2-4a(1-p-q)=0$, *then* γ *is globally asymptotically stable.*
(iii) *If* $p+q<1,\ r^2-4a(1-p-q)>0$, *then* η_2 *is globally asymptotically stable.*

4. Invariant intervals

From the discussion of the global attractivity of the positive equilibrium points in section 3, it is easy to have the following results about invariant intervals.

Theorem 4.1. *Assume that* $p+q>1,\ r^2 \ge a(p+q),\ p<q$.
(i) *If* $p<q\le 1$, *then every positive solution of Eq.(2.1) lies eventually in* $[\theta_2,\infty)$.
(ii) *If* $p<1<q$, *then every positive solution of Eq.(2.1) lies eventually in* (θ_1,θ_2).
(iii) *If* $1\le p<q$, *then every positive solution of Eq.(2.1) lies eventually in* $(0,\theta_1]$.

Theorem 4.2. *Assume that* $p+q>1,\ r^2 \ge a(p+q),\ p>q$.
(i) *If* $1\ge p>q$, *then every positive solution of Eq.(2.1) lies eventually in* $[\theta_1,\infty)$.
(ii) *If* $p>1>q$, *then every positive solution of Eq.(2.1) lies eventually in* (θ_2,θ_1).
(iii) *If* $p>q\ge 1$ *then every positive solution of Eq.(2.1) lies eventually in* $(0,\theta_2]$.

Theorem 4.3. *Assume that* $p+q>1,\ r^2 \geq a(p+q),\ p=q$.
(i) *If* $p=q\leq 1$, *then every positive solution of Eq.(2.1) lies eventually in* $[\theta_2,\ \infty)$.
(ii) *If* $p=q>1$, *then every positive solution of Eq.(2.1) lies eventually in* $(0,\ \theta_2)$.

Theorem 4.4. *Assume that* $p+q=1$, *then the following statement is true:*
(i) *If* $p>q$, *then every positive solution of Eq.(2.1) lies eventually in* $(\theta_1,\ \infty)$.
(ii) *If* $p\leq q$, *then every positive solution of Eq.(2.1) lies eventually in* $(\theta_2,\ \infty)$.

Theorem 4.5. *Assume that* $p+q<1,\ r^2-4a(1-p-q)=0$, *then the following statement is true:*
(i) *If* $p>q$, *then every positive solution of Eq.(2.1) lies eventually in* $(\theta_1,\ \infty)$.
(ii) *If* $p\leq q$, *then every positive solution of Eq.(2.1) lies eventually in* $(\theta_2,\ \infty)$.

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