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SOME GENERALIZATIONS OF THE FEJÉR AND HERMITE-HADAMARD INEQUALITIES IN HöLDER SPACES

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Abstract. In this article, by considering error inequalities, we propose a new way to treat the Fejér and Hermite-Hadamard inequalities involving n knots and m-th derivative on Hölder spaces. Moreover, some new related estimations are also given.

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1. Introduction and Preliminaries

In recent years, a number of authors have studied error inequalities for some known and some new quadrature formulas. Sometimes they have considered generalizations of these formulas, see [6, 7, 8, 10, 11, 12, 13] and their references therein where the mid point and trapezoid quadrature rules are considered.
In this article, we are concerned with an interesting inequality, which is called the Hermite-Hadamard inequality, and stated in [9] as follows

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

for any convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ and $a, b \in I$. This result was extended by L. Fejér [5], in which the author showed that

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \le \int_a^b f(x)p(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b p(x)dx \tag{2}$$

holds for any convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ where $a, b \in I$ and the function $p : [a, b] \to \mathbb{R}$ is non-negative integrable and symmetric about $x = \frac{a+b}{2}$. The important point in [5] is the presence of the function $p : [a, b] \to \mathbb{R}$, which improved (1), especially in the case the value of $p(.)$ is small enough. It is clear

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that if $p(x) \equiv 1$, relation (2) comes back (1). Regarding some extensions of (2), we refer the readers to some recent works [2, 3, 14, 15].

In the first part of this article, we will improve the results introduced in [5, 14, 15]. We consider the situation $f : I \subset \mathbb{R} \to \mathbb{R}$ is such that the first derivative of f, namely f', belongs to the space $C^\alpha[a,b]$ with $0 \le \alpha \le 1$, defined by

$$C^\alpha[a,b] = \{u : I \to \mathbb{R} \,|\, |u(x) - u(y)| \le K\,|x-y|^\alpha\}.$$

Then, a careful analysis of [5, 14, 15] helps us to obtain some better estimations (see Theorems 1 and 2).

Next, we refer to [4], one of the improvements for (1), in which S.S. Dragomir et al. proved that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{K}{(\alpha+2)(\alpha+3)}(b-a)^{\alpha+1}, \tag{3}$$

where $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function, such that $f' \in C^\alpha[a,b]$ and $K > 0$ is a constant.

Inspired by the interesting ideas introduced in [6,7,8], we can strengthen (3) by enlarging the number of knots (two knots in (3)) (see Theorem 3). It is worth noticing that our results seem to be better than (3) in some sense, especially when $b - a << 1$.

In our proofs, we use the following result which is well-known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1 (See [1])**.** *Let $f : [a,b] \to \mathbb{R}$ and let r be a positive integer. If f is such that $f^{(r-1)}$ is absolutely continuous on $[a,b]$, $x_0 \in (a,b)$ then for all $x \in (a,b)$ we have*

$$f(x) = T_{r-1}(f,x_0,x) + R_{r-1}(f,x_0,x)$$

where $T_{r-1}(f,x_0,\cdot)$ is Taylor's polynomial of degree $r-1$, that is,

$$T_{r-1}(f,x_0,x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

and the remainder can be given by

$$R_{r-1}(f,x_0,x) = \int_{x_0}^{x} \frac{(x-t)^{r-1} f^{(r)}(t)}{(r-1)!}dt.$$

Remark 1. *A simple calculation helps us to show that the remainder $R_{r-1}(f,x_0,x)$ in Lemma 1 can be rewritten as*

$$R_{r-1}(f,x_0,x) = \int_0^{x-x_0} \frac{(x-x_0-t)^{r-1} f^{(r)}(x_0+t)}{(r-1)!}dt$$

and thus,

$$f(x+u) = \sum_{k=0}^{r-1} \frac{u^k}{k!} f^{(k)}(x) + \int_0^u \frac{(u-t)^{r-1}}{(r-1)!} f^{(r)}(x+t)dt. \tag{4}$$

2. Main Results

2.1.Some new estimations concerning (2). In this section, we would like to give some extensions of (2) by considering the situation $f^{'} \in C^{\alpha}[a,b]$ and $p:[a,b]\to\mathbb{R}$ is non-negative integrable and symmetric about $x=\frac{a+b}{2}$. The first result of ours can be described as follows.

Theorem 1. *If $f^{'} \in C^{\alpha}[a,b]$ and $p:[a,b]\to\mathbb{R}$ is non-negative integrable and symmetric about $x=\frac{a+b}{2}$, then it holds that*

$$\left|\int_a^b f(x)p(x)dx - f\Big(\frac{a+b}{2}\Big)\int_a^b p(x)dx\right| \le \frac{K}{2(\alpha+1)}\int_a^{\frac{a+b}{2}} (a+b-2x)^{\alpha+1}p(x)dx.$$

Proof. Firstly, we observe that

$$\int_a^b f(x)p(x)dx = \int_a^b f(a+b-x)p(a+b-x)dx,$$

which implies since $p(.)$ is symmetric about $\frac{a+b}{2}$ that

$$\int_a^b f(x)p(x)dx = \int_a^b f(a+b-x)p(x)dx$$

and then

$$\begin{aligned}\int_a^b f(x)p(x)dx &= \frac{1}{2}\int_a^b (f(a+b-x)+f(x))p(x)dx \\ &= \int_a^{\frac{a+b}{2}} (f(a+b-x)+f(x))p(x)dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_a^b f(x)p(x)dx - f\Big(\frac{a+b}{2}\Big)\int_a^b p(x)dx \\ &\qquad = \int_a^{\frac{a+b}{2}} \Big[f(a+b-x)+f(x)-2f\Big(\frac{a+b}{2}\Big)\Big]p(x)dx \qquad (5)\end{aligned}$$

On the other hand, we have

$$f(a+b-x)-f\Big(\frac{a+b}{2}\Big) = \int_{\frac{a+b}{2}}^{a+b-x} f^{'}(t)dt,$$

$$f\Big(\frac{a+b}{2}\Big)-f(x) = \int_x^{\frac{a+b}{2}} f^{'}(t)dt = \int_{\frac{a+b}{2}}^{a+b-x} f^{'}(a+b-t)dt$$

and then

$$f(x)+f(a+b-x)-2f\Big(\frac{a+b}{2}\Big) = \int_{\frac{a+b}{2}}^{a+b-x} (f^{'}(t)-f^{'}(a+b-t))dt.$$

Combining this with (5), we conclude that

$$\begin{aligned}\Big|\int_a^b f(x)p(x)dx - f\Big(\frac{a+b}{2}\Big)\int_a^b p(x)dx\Big| &\le \int_a^{\frac{a+b}{2}} \Big|f(x)+f(a+b-x)-2f\Big(\frac{a+b}{2}\Big)\Big|p(x)dx \\ &\le \int_a^{\frac{a+b}{2}} \left(K\int_{\frac{a+b}{2}}^{a+b-x}(2t-a-b)^{\alpha}dt\right)p(x)dx \\ &= \frac{K}{2(\alpha+1)}\int_a^{\frac{a+b}{2}}(a+b-2x)^{\alpha+1}p(x)dx\end{aligned}$$

and the proof of Theorem 1 is now completed. □

Corollary 1. *Let $f^{'} \in C^{\alpha}[a,b]$ and let $p:[a,b]\to\mathbb{R}$ be non-negative integrable and symmetric about $x=\frac{a+b}{2}$. Then we have*

$$\left|\int_a^b f(x)p(x)dx - f\Big(\frac{a+b}{2}\Big)\int_a^b p(x)dx\right| \le \frac{K}{2(\alpha+1)}(b-a)^{\alpha+1}\int_a^{\frac{a+b}{2}} p(x)dx.$$

Corollary 2. *Assume that $p:[a,b]\to\mathbb{R}$ is non-negative integrable and symmetric about $x=\frac{a+b}{2}$ and $f^{'}$ is Lipschitz continuous on [a,b]. i.e.,*

$$|f^{'}(x)-f^{'}(y)| \le K|x-y|.$$

Then, it holds that

$$\left|\int_a^b f(x)p(x)dx - f\Big(\frac{a+b}{2}\Big)\int_a^b p(x)dx\right| \le \frac{K}{4}\int_a^{\frac{a+b}{2}}(b+a-2x)^2p(x)dx.$$

Remark 2. *If f is a convex funtion then we have*

$$f(x)+f(a+b-x)-2f\Big(\frac{a+b}{2}\Big)\ge 0, \quad \forall x\in[a,b]$$

and hence, it follows from (5) that

$$\int_a^b f(x)p(x)dx - f\Big(\frac{a+b}{2}\Big)\int_a^b p(x)dx \ge 0.$$

Theorem 2. *If $f^{'} \in C^{\alpha}[a,b]$ and $p:[a,b]\to\mathbb{R}$ is non-negative integrable and symmetric about $x=\frac{a+b}{2}$, then it holds that*

$$\begin{aligned}&\left|\int_a^b f(x)p(x)dx - \frac{f(a)+f(b)}{2}\int_a^b p(x)dx\right| \\ &\qquad \le \frac{K}{2(\alpha+1)}\int_a^{\frac{a+b}{2}}\Big((b-a)^{\alpha+1}-(a+b-2x)^{\alpha+1}\Big)p(x)dx. \quad (6)\end{aligned}$$

Proof. We have known that

$$\int_a^b f(x)p(x)dx - \frac{f(a)+f(b)}{2}\int_a^b p(x)dx$$
$$= \int_a^{\frac{a+b}{2}} \left[f(a+b-x)+f(x)-f(a)-f(b)\right]p(x)dx, \quad (7)$$

where,

$$f(x)-f(a) = \int_a^x f^{'}(t)dt,$$
$$f(b)-f(a+b-x) = \int_{a+b-x}^b f^{'}(t)dt = \int_a^x f^{'}(a+b-t)dt$$

and then

$$f(x)+f(a+b-x)-f(a)-f(b) = \int_a^x (f^{'}(t)-f^{'}(a+b-t))dt.$$

So, for any $x \in [a, \frac{a+b}{2}]$, it follows that

$$|f(x)+f(a+b-x)-f(a)-f(b)| \le \int_a^x |f^{'}(t)-f^{'}(a+b-t)|dt.$$
$$\le K\int_a^x |a+b-2t|^\alpha dt$$
$$= \frac{K}{2(\alpha+1)}\Big((b-a)^{\alpha+1}-(a+b-2x)^{\alpha+1}\Big).$$

Combining this with (7), we obtain the proof of Theorem 2. □

Corollary 3. *If $f^{'} \in C^\alpha[a,b]$ and $p:[a,b]\to\mathbb{R}$ is non-negative integrable and symmetric about $x=\frac{a+b}{2}$, then we have*

$$\left|\int_a^b f(x)p(x)dx - \frac{f(a)+f(b)}{2}\int_a^b p(x)dx\right| \le \frac{K}{2(\alpha+1)}(b-a)^{\alpha+1}\int_a^{\frac{a+b}{2}} p(x)dx.$$

Corollary 4. *If $f^{'}$ is Lipschitz continuous on [a,b]. i.e.,*

$$|f^{'}(x)-f^{'}(y)| \le K|x-y|.$$

Then, we have

$$\left|\int_a^b f(x)p(x)dx - \frac{f(a)+f(b)}{2}\int_a^b p(x)dx\right| \le K\int_a^{\frac{a+b}{2}} (b-x)(x-a)p(x)dx.$$

Remark 3. *We see that if f is a convex funtion, then*

$$f(x)+f(a+b-x)-f(a)-f(b) \le 0, \quad \forall x \in [a,b]$$

Hence, by (7) we get

$$\int_a^b f(x)p(x)dx - \frac{f(a)+f(b)}{2}\int_a^b p(x)dx \le 0.$$

2.2.Error inequalities involving n knots and the m-th derivative on Hölder spaces. In this section, we prove some new inequalities of Hermite-Hadamard and Fejér type, involving n knots and the m-th derivative on Hölder spaces. The proofs rely essentially on Taylor's formula (see Lemma 1). Let $0 \le x_i \le 1$, $i = 1, 2, ..., n$ be solven the following linear system

$$\begin{cases} x_1 + x_2 + \cdots + x_n &= \frac{n}{2}, \\ \cdots\cdots\cdots\cdots & \\ x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1} &= \frac{n}{m}, \\ x_1^m + x_2^m + \cdots + x_n^m &= \frac{n}{m+1}. \end{cases} \tag{8}$$

Put

$$I(f) = \int_a^b f(x)dx,$$

$$Q(f, n, m, x_1, .., x_n) = \frac{b-a}{n} \sum_{i=0}^{n} f(a + x_i(b-a)). \tag{9}$$

Then, we obtain the following:

Theorem 3. *If $f^{(m)} \in C^\alpha[a, b]$ then it holds that*

$$\Big| I(f) - Q(f, n, m, x_1, .., x_n) \Big| \le \frac{KC_{m,\alpha}(2m+1)}{(m+1)!},$$

where

$$C_{m,\alpha} = (b-a) \int_a^b (b-x)^{m-1}(x-a)^\alpha dx.$$

Proof. Let us first define

$$F(x) = \int_a^x f(x)dx.$$

Then, it should be noticed by the Fundamental Theorem of Calculus that

$$I(f) = F(b) - F(a).$$

Now, applying Lemma 1 (see (4)) to the function $F(x)$ with $x = a$ and $u = b-a$, we get

$$F(b) = F(a) + \sum_{k=1}^{m} \frac{(b-a)^k}{k!} F^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} F^{(m+1)}(a+t)dt$$

which yields that

$$I(f) = \sum_{k=1}^{m} \frac{(b-a)^k}{k!} F^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} F^{(m+1)}(a+t)dt.$$

Equivalently,

$$I(f)=\sum_{k=0}^{m-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a)+\int_0^{b-a}\frac{(b-a-t)^m}{m!}f^{(m)}(a+t)dt \tag{10}$$

For each $1\le i\le n$, applying Lemma 1 again to the function $f(x)$ with $x=a$ and $u=x_i(b-a)$, we get

$$\begin{aligned}
&f(a+x_i(b-a))\\
&\quad=\sum_{k=0}^{m-1}\frac{x_i^k(b-a)^k}{k!}f^{(k)}(a)+\int_0^{x_i(b-a)}\frac{(x_i(b-a)-t)^{m-1}}{(m-1)!}f^{(m)}(a+t)dt\\
&\quad=\sum_{k=0}^{m-1}\frac{x_i^k(b-a)^k}{k!}f^{(k)}(a)+\int_0^{b-a}\frac{x_i^m(b-a-u)^{m-1}}{(m-1)!}f^{(m)}(a+x_iu)du.
\end{aligned}$$

By applying to $i=\overline{1,n}$ and then summing up, we deduce that

$$\begin{aligned}
&\sum_{i=1}^n f(a+x_i(b-a))\\
&=\sum_{i=1}^n\sum_{k=0}^{m-1}\frac{x_i^k(b-a)^k}{k!}f^{(k)}(a)+\sum_{i=1}^n\int_0^{b-a}\frac{x_i^m(b-a-u)^{m-1}}{(m-1)!}f^{(m)}(a+x_iu)du\\
&=\sum_{k=0}^{m-1}\frac{\sum_{i=1}^n x_i^k(b-a)^k}{k!}f^{(k)}(a)+\sum_{i=1}^n\int_0^{b-a}\frac{x_i^m(b-a-u)^{m-1}}{(m-1)!}f^{(m)}(a+x_iu)du\\
&=\sum_{k=0}^{m-1}\frac{n(b-a)^k}{(k+1)!}f^{(k)}(a)+\sum_{i=1}^n\int_0^{b-a}\frac{x_i^m(b-a-u)^{m-1}}{(m-1)!}f^{(m)}(a+x_iu)du.
\end{aligned}$$

Thus,

$$\begin{aligned}
Q(f,n,m,x_1,...,x_n)&=\sum_{k=0}^{m-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a)\\
&\quad+\frac{b-a}{n}\sum_{i=1}^n\int_0^{b-a}\frac{x_i^m(b-a-u)^{m-1}}{(m-1)!}f^{(m)}(a+x_iu)du.
\end{aligned} \tag{11}$$

By (10) and (11), we obtain that

$$
\begin{aligned}
&\Big|I(f) - Q(f,n,m,x_1,...,x_n)\Big| \\
&\quad = \Big|\int_0^{b-a} \frac{(b-a-t)^m}{m!} f^{(m)}(a+t)dt \\
&\qquad\qquad - \frac{b-a}{n}\sum_{i=1}^{n}\int_0^{b-a} \frac{x_i^m(b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a+x_i u)du\Big| \\
&\quad = \Big|\int_a^{b} \frac{(b-x)^m}{m!} f^{(m)}(x)dx \\
&\qquad\qquad - \frac{b-a}{n}\sum_{i=1}^{n}\int_a^{b} \frac{x_i^m(b-x)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a+x_i x)dx\Big| \\
&\quad = \Big|\int_a^{b} \frac{(b-x)^m}{m!} [f^{(m)}(x) - f^{(m)}(a)]dx \\
&\qquad - \frac{b-a}{n}\sum_{i=1}^{n}\int_a^{b} \frac{x_i^m(b-x)^{m-1}}{(m-1)!} [f^{(m)}((1-x_i)a+x_i x) - f^{(m)}(a)]dx\Big|,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\Big|I(f) - Q(f,n,m,x_1,...,x_n)\Big| &\le \Big|\int_a^{b} \frac{(b-x)^m}{m!} [f^{(m)}(x) - f^{(m)}(a)]dx\Big| \\
+ \frac{b-a}{n}\sum_{i=1}^{n}\Big|\int_a^{b} &\frac{x_i^m(b-x)^{m-1}}{(m-1)!} [f^{(m)}((1-x_i)a+x_i x) - f^{(m)}(a)]dx\Big|. \quad (12)
\end{aligned}
$$

Thus, by $f^{(m)} \in C^\alpha[a,b]$, we have

$$
\begin{aligned}
&|I(f) - Q(f,n,m,x_1,..,x_n)| \\
&\quad \le \left|\int_a^{b} \frac{(b-x)^m}{m!} K(x-a)^\alpha dx\right| \\
&\qquad + \frac{b-a}{n}\left|\sum_{i=1}^{n}\int_a^{b} \frac{x_i^m(b-x)^{m-1}}{(m-1)!} K((1-x_i)a+x_i x) - a)^\alpha dx\right| \\
&\quad = K\Big(\int_a^{b} \frac{(b-x)^m}{m!} (x-a)^\alpha dx \\
&\qquad + \frac{b-a}{n}\sum_{i=1}^{n}\int_a^{b} \frac{x_i^{m+\alpha}(b-x)^{m-1}}{(m-1)!} (x-a)^\alpha dx\Big)
\end{aligned}
$$

and then

$$
\begin{aligned}
|I(f)-Q(f,n,m,x_1,..,x_n)| &\leq K\Big(\frac{b-a}{m}\int_a^b \frac{(b-x)^{m-1}}{(m-1)!}(x-a)^{\alpha}dx \\
&\quad + \frac{b-a}{n}\sum_{i=1}^{n} x_i^m \int_a^b \frac{(b-x)^{m-1}}{(m-1)!}(x-a)^{\alpha}dx\Big) \\
&\leq K\Big(\frac{b-a}{m}\int_a^b \frac{(b-x)^{m-1}}{(m-1)!}(x-a)^{\alpha}dx \\
&\quad + \frac{b-a}{n}\frac{n}{m+1}\int_a^b \frac{(b-x)^{m-1}}{(m-1)!}(x-a)^{\alpha}dx\Big) \\
&\leq K\frac{2m+1}{(m+1)!}(b-a)\int_a^b (b-x)^{m-1}(x-a)^{\alpha}dx \\
&= \frac{KC_{m,\alpha}(2m+1)}{(m+1)!},
\end{aligned}
$$

where

$$C_{m,\alpha} = (b-a)\int_a^b (b-x)^{m-1}(x-a)^{\alpha}dx$$

and the proof of Theorem 3 is now completed. □

Corollary 5. *If $f^{(m)}$ is Lipschitz continuous on $[a,b]$, i.e.,*

$$|f^{(m)}(x)-f^{(m)}(y)| \leq K|x-y|,$$

then we have

$$\Big|I(f)-Q(f,n,m,x_1,..,x_n)\Big| \leq \frac{K(2m+1)}{(m+1)!m(m+1)}(b-a)^{m+2}.$$

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