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COMPLETELY INTEGRABLE COUPLED POTENTIAL KDV EQUATIONS

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Abstract. We make use of the simplified Hirota's bilinear method with computer symbolic computation to study a variety of coupled potential KdV (pKdV) equations. Each coupled equation is completely integrable and gives multiple soliton solutions and multiple singular soliton solutions. The phase shifts for all coupled pKdV equations are identical whereas the coefficients of the obtained solitons are not identical. The four coupled pKdV equations are resonance free.

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1. Introduction

Recently, many nonlinear coupled evolution equations, such as the coupled KdV equation, the coupled mKdV equation, and others appear in scientific applications [1–13]. The coupled evolution equations attracted a considerable amount of research work in the literature, mainly for the determination of soliton solutions or periodic solutions. The goal of other works has been the proof of complete integrability of these coupled equations [14–26].

Many reliable methods are used in the literature to examine the completely integrable equations. The Hirota bilinear method, the Bäcklund transformation method, the inverse scattering method, the Painlevé analysis, Darboux transformation, the Pfaffian technique, the Hereman's simplified form, and others were effectively used in [1–13] to determine multiple soliton solutions for completely integrable equations. Each method has its own significant properties. The Hirota's bilinear method [1–9], the Hietarinta approach [3–4], and the Hereman's et. al. simplified form [2] are rather heuristic and significant. These approaches posses powerful features that make it practical for the determination of multiple soliton solutions [12–26] for a wide class of nonlinear evolution equations. The

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computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations. It was proved in [5] that two solitons become singular after colliding with each other, in the sense that regular solitons with sech^2 profiles are transmitted into singular solitons with cosech^2 profiles through the interaction.

In this work, a variety of coupled potential KdV (pKdV) equations will be investigated for complete integrability and for the determination of multiple soliton solutions. The coupled pKdV equations that we selected are

$$\begin{aligned} u_t + u_{xxx} + \alpha v_x^2 &= 0, \\ v_t + v_{xxx} + \beta u_x^2 &= 0, \end{aligned} \tag{1}$$

$$\begin{aligned} u_t + u_{xxx} + \alpha u_x^2 + \beta u_x v_x &= 0, \\ v_t + v_{xxx} + \alpha v_x^2 - \beta u_x v_x &= 0, \end{aligned} \tag{2}$$

$$\begin{aligned} u_t + u_{xxx} + 3uu_{xx} - 3vu_{xx} + \alpha u_x^2 + 3u_x v_x + 3u^2 u_x - 6uvu_x + 3v^2 u_x &= 0, \\ v_t + v_{xxx} - 3uv_{xx} + 3vv_{xx} + \alpha v_x^2 + 3u_x v_x + 3u^2 v_x - 6uvu_x + 3v^2 v_x &= 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} & u_t - \frac{1}{2}u_{xxx} + \frac{3}{2}v_{xxx} + 6uu_{xx} - 6vu_{xx} + \alpha u_x^2 \\ & \quad - 6u_x v_x - \alpha v_x^2 - 12(u-v)^2 v_x - 3(u-v)^4 = 0, \\ & v_t + \frac{3}{2}u_{xxx} - \frac{1}{2}v_{xxx} - 6uv_{xx} + 6vv_{xx} - \alpha u_x^2 \\ & \quad - 6u_x v_x + \alpha v_x^2 - 12(u-v)^2 u_x - 3(u-v)^4 = 0. \end{aligned} \tag{4}$$

The Hirota's bilinear method, the Hietarinta approach, and the Hereman's simplified form will be used in this work. Our goal is to construct multiple regular soliton solutions and multiple singular soliton solutions for each coupled pKdV equation. Hirota and Ito in [5] examined the phenomena of two solitons near resonant state, two solitons at the resonant state, and two solitons after colliding with each other. The four coupled pKdV equations (1)–(4) will be examined for complete integrability and for resonance effects.

2. The first coupled pKdV equation

We begin our analysis by considering the first coupled pKdV equation

$$\begin{aligned} u_t + u_{xxx} + \alpha v_x^2 &= 0, \\ v_t + v_{xxx} + \beta u_x^2 &= 0, \end{aligned} \tag{5}$$

2.1. Multiple soliton solutions. To determine the regular soliton solutions, we substitute

$$\begin{aligned} u(x,t) &= e^{k_i x - c_i t},\\ v(x,t) &= A e^{k_i x - c_i t}, \end{aligned} \tag{6}$$

where A is a constant, into the linear terms of (5). This in turn gives the dispersion relation by

$$c_i = k_i^3, \tag{7}$$

and as a result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{8}$$

The multi soliton solutions of (5) is assumed to be

$$\begin{aligned} u(x,t) &= R_1 (\ln f)_x = R_1 \frac{f_x}{f},\\ v(x,t) &= R_2 u(x,t) = R_1 R_2 \frac{f_x}{f}, \end{aligned} \tag{9}$$

where R_1, and R_2 are constants that will be determined. The auxiliary function $f(x,t)$ for the single soliton solution is given by

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - k_1^3 t}. \tag{10}$$

Substituting (10) and (9) in (5), and solving for R_1 and R_2 we find

$$R_1 = \frac{6}{(\alpha\beta^2)^{\frac{1}{3}}},\ \ R_2 = \frac{\beta}{(\alpha\beta^2)^{\frac{1}{3}}}. \tag{11}$$

Substituting (10) and (11) in (9) gives the single soliton solutions

$$u(x,t) = \frac{6k_1\, e^{k_1 x - k_1^3 t}}{(\alpha\beta^2)^{\frac{1}{3}}(1 + e^{k_1 x - k_1^3 t})},\ \ v(x,t) = \frac{6\beta k_1\, e^{k_1 x - k_1^3 t}}{(\alpha\beta^2)^{\frac{2}{3}}(1 + e^{k_1 x - k_1^3 t})}. \tag{12}$$

For the two soliton solutions we substitute the auxiliary function $f(x,t)$, given by

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}\\ &= 1 + e^{k_1 x - k_1^3 t} + e^{k_2 x - k_2^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3+k_2^3)t}, \end{aligned} \tag{13}$$

into (5) to obtain the phase shift

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \tag{14}$$

and hence we set

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \le i < j \le 3. \tag{15}$$

The two soliton solutions are obtained by substituting (14) and (13) into (9), where R_1 and R_2 are given above in (11).

It is interesting to point out that equation (5) does not show any resonant phenomenon [5] because the phase shift term a_{12} in (14) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

For the three soliton solutions, we set

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3} \\ &= 1 + e^{k_1x-k_1^3t} + e^{k_2x-k_2^3t} + e^{k_3x-k_3^3t} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t} \\ &\quad + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{(k_1+k_3)x-(k_1^3+k_3^3)t} + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\ &\quad + b_{123}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}. \end{aligned} \tag{16}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{17}$$

The three soliton solutions are obtained by substituting (16) into (9). This shows that the coupled pKdV equation (5) is completely integrable and N-soliton solutions can be determined for $u(x,t)$ and $v(x,t)$, for finite N, where $N \geq 1$.

2.2. Multiple singular soliton solutions. For singular soliton solution, we follow [10–26] and set the auxiliary function $f(x,t)$ by

$$f(x,t) = 1 - e^{\theta_1} = 1 - e^{k_1x-k_1^3t}. \tag{18}$$

Substituting this in (9) and using (11) we obtain the singular soliton solutions

$$\begin{aligned} u(x,t) &= -\frac{6k_1\, e^{k_1x-k_1^3t}}{(\alpha\beta^2)^{\frac{1}{3}}(1-e^{k_1x-k_1^3t})}, \\ v(x,t) &= -\frac{6\beta k_1\, e^{k_1x-k_1^3t}}{(\alpha\beta^2)^{\frac{2}{3}}(1-e^{k_1x-k_1^3t})}. \end{aligned} \tag{19}$$

For the two singular soliton solutions we substitute

$$f(x,t) = 1 - e^{k_1x-k_1^3t} - e^{k_2x-k_2^3t} + a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}, \tag{20}$$

in (9) to find that the phase shift is given by

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{21}$$

and hence

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \leq i < j \leq 3. \tag{22}$$

The two singular solutions are obtained by substituting (20), using the phase shift a_{12} into (9).

For the three singular soliton solutions, we set

$$\begin{aligned} f(x,t) = 1 - e^{k_1x-k_1^3t} - e^{k_2x-k_2^3t} - e^{k_3x-k_3^3t} + a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3)t} \\ + a_{13}e^{(k_1+k_3)x-(k_1^3+k_3^3)t} + a_{23}e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\ - a_{12}a_{23}a_{13}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}. \end{aligned} \tag{23}$$

The three singular soliton solutions are obtained by substituting the last equation into (9). This shows that the coupled pKdV equation (5) gives N singular soliton solutions for $u(x,t)$ and $v(x,t)$, for finite N, where $N \geq 1$.

3. The second coupled pKdV equation

We next consider the second coupled pKdV equation

$$\begin{aligned} u_t + u_{xxx} + \alpha u_x^2 + \beta u_x v_x = 0, \\ v_t + v_{xxx} + \alpha v_x^2 - \beta u_x v_x = 0, \end{aligned} \tag{24}$$

where $\alpha \neq \beta$.

3.1. Multiple soliton solutions. To determine the regular soliton solutions, we substitute

$$\begin{aligned} u(x,t) = e^{k_ix-c_it}, \\ v(x,t) = Ae^{k_ix-c_it}, \end{aligned} \tag{25}$$

where A is a constant, into the linear terms of (24). This in turn gives the dispersion relation by

$$c_i = k_i^3, \tag{26}$$

and hence

$$\theta_i = k_ix - k_i^3t. \tag{27}$$

The multi soliton solutions of (24) is assumed to be

$$\begin{aligned} u(x,t) = R_1\left(\ln f\right)_x = R_1\frac{f_x}{f}, \\ v(x,t) = R_2u(x,t) = R_1R_2\frac{f_x}{f}, \end{aligned} \tag{28}$$

where R_1, and R_2 are constants that will be determined. The auxiliary function $f(x,t)$ for the single soliton solution is given by

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1x-k_1^3t}. \tag{29}$$

Substituting (29) in (24), and solving for R_1 and R_2 we find

$$R_1 = \frac{6(\alpha-\beta)}{(\alpha^2+\beta^2)},\ \ R_2 = \frac{\alpha+\beta}{(\alpha-\beta)}, \alpha \neq \beta. \tag{30}$$

Substituting (29) and (30) in (28) gives the single soliton solutions

$$\begin{aligned} u(x,t) &= \frac{6(\alpha-\beta)k_1\, e^{k_1x-k_1^3t}}{(\alpha^2+\beta^2)(1+e^{k_1x-k_1^3t})}, \\ v(x,t) &= \frac{6(\alpha+\beta)k_1\, e^{k_1x-k_1^3t}}{(\alpha^2+\beta^2)(1+e^{k_1x-k_1^3t})}. \end{aligned} \tag{31}$$

For the two soliton solutions we substitute the auxiliary function $f(x,t)$, given by

$$\begin{aligned} f(x,t) &= 1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2} \\ &= 1+e^{k_1x-k_1^3t}+e^{k_2x-k_2^3t}+a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}, \end{aligned} \tag{32}$$

into (24) to obtain the phase shift

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \tag{33}$$

and hence we set

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \le i < j \le 3. \tag{34}$$

The two soliton solutions are obtained by substituting (33) and (32) into (28), where R_1 and R_2 are given above in (30).

It is interesting to point out that equation (24) does not show any resonant phenomenon [5] because the phase shift term a_{12} in (33) cannot be 0 or ∞ for $|k_1| \neq |k_2|$. This result is similar to the result obtained before the first coupled pKdV.

For the three soliton solutions, we set

$$\begin{aligned} f(x,t) &= 1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+b_{123}e^{\theta_1+\theta_2+\theta_3} \\ &= 1+e^{k_1x-k_1^3t}+e^{k_2x-k_2^3t}+e^{k_3x-k_3^3t}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t} \\ &\quad + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{(k_1+k_3)x-(k_1^3+k_3^3)t}+\frac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\ &\quad + b_{123}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}. \end{aligned} \tag{35}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{36}$$

The three soliton solutions are obtained by substituting (35) into (28). This shows that the coupled pKdV equation (24) is completely integrable and N-soliton solutions can be determined for $u(x,t)$ and $v(x,t)$, for finite N, where $N \ge 1$.

3.2. Multiple singular soliton solutions. For singular soliton solution, we follow [13–26] and set the auxiliary function $f(x,t)$ by

$$f(x,t) = 1 - e^{\theta_1} = 1 - e^{k_1 x - k_1^3 t}. \tag{37}$$

Substituting this in (28) and using (30) we obtain the singular soliton solutions

$$\begin{aligned} u(x,t) &= -\frac{6(\alpha-\beta)k_1\, e^{k_1 x - k_1^3 t}}{(\alpha^2+\beta^2)(1-e^{k_1 x - k_1^3 t})}, \\ v(x,t) &= -\frac{6(\alpha+\beta)k_1\, e^{k_1 x - k_1^3 t}}{(\alpha^2+\beta^2)(1-e^{k_1 x - k_1^3 t})}. \end{aligned} \tag{38}$$

For the two singular soliton solutions we substitute

$$f(x,t) = 1 - e^{k_1 x - k_1^3 t} - e^{k_2 x - k_2^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3+k_2^3)t}, \tag{39}$$

in (28) to find that the phase shift is given by

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{40}$$

and hence

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \le i < j \le 3. \tag{41}$$

The two singular soliton solutions are obtained by substituting (39), using the phase shift a_{12}, into (28).

For the three singular soliton solutions, we set

$$\begin{aligned} f(x,t) = 1 &- e^{k_1 x - k_1^3 t} - e^{k_2 x - k_2^3 t} - e^{k_3 x - k_3^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3+k_2^3)t} \\ &+ a_{13} e^{(k_1+k_3)x - (k_1^3+k_3^3)t} + a_{23} e^{(k_2+k_3)x - (k_2^3+k_3^3)t} \\ &- a_{12}a_{23}a_{13} e^{(k_1+k_2+k_3)x - (k_1^3+k_2^3+k_3^3)t}. \end{aligned} \tag{42}$$

The three singular soliton solutions are obtained by substituting the last equation into (28). This shows that the coupled pKdV equation (24) gives N singular soliton solutions for $u(x,t)$ and $v(x,t)$.

4. The third coupled pKdV equation

We next consider the third coupled pKdV equation [1]

$$\begin{aligned} u_t + u_{xxx} + 3uu_{xx} - 3vu_{xx} + \alpha u_x^2 + \beta u_x v_x + 3u^2 u_x - 6uvu_x + 3v^2 u_x = 0, \\ v_t + v_{xxx} - 3uv_{xx} + 3vv_{xx} + \alpha v_x^2 + \beta u_x v_x + 3u^2 v_x - 6uvu_x + 3v^2 v_x = 0. \end{aligned} \tag{43}$$

4.1. Multiple soliton solutions. Proceeding as before we get the dispersion relation

$$c_i = k_i^3, \tag{44}$$

and also

$$\theta_i = k_i x - k_i^3 t. \tag{45}$$

The multi soliton solutions of (43) is assumed to be

$$\begin{aligned} u(x,t) &= R_1 (\ln f)_x = R_1 \frac{f_x}{f}, \\ v(x,t) &= R_2 u(x,t) = R_1 R_2 \frac{f_x}{f}. \end{aligned} \tag{46}$$

The auxiliary function $f(x,t)$ for the single soliton solution is given by

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - k_1^3 t}. \tag{47}$$

Substituting auxiliary function (29) in (43), and solving for R_1 and R_2 we find

$$R_1 = \frac{6}{\alpha+\beta}, \quad R_2 = 1. \tag{48}$$

This means that for the single soliton solutions $u(x,t) = v(x,t)$, therefore we set

$$u(x,t) = v(x,t) = \frac{6k_1\, e^{k_1 x - k_1^3 t}}{(\alpha+\beta)(1+e^{k_1 x - k_1^3 t})}. \tag{49}$$

For the two soliton solutions we substitute the auxiliary function $f(x,t)$, given by

$$f(x,t) = 1 + e^{k_1 x - k_1^3 t} + e^{k_2 x - k_2^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3+k_2^3)t}, \tag{50}$$

into (43), we obtain the phase shift exactly as obtained before in the first two coupled pKdV equations. The two soliton solutions $u(x,t)$ and $v(x,t)$ are also identical and obtained by substituting the phase shift and (50) into (46), where R_1 and R_2 are given above in (48).

It is interesting to point out that equation (43) is resonance free because the phase shift term a_{12} in (40) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

For the three soliton solutions $u(x,t)$ and $v(x,t)$ are also identical and can be obtained by substituting

$$\begin{aligned} f(x,t) = 1 &+ e^{k_1 x - k_1^3 t} + e^{k_2 x - k_2^3 t} + e^{k_3 x - k_3^3 t} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x - (k_1^3+k_2^3)t} \\ &+ \frac{(k_1-k_3)^2}{(k_1+k_3)^2} e^{(k_1+k_3)x - (k_1^3+k_3^3)t} + \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{(k_2+k_3)x - (k_2^3+k_3^3)t} \\ &+ b_{123} e^{(k_1+k_2+k_3)x - (k_1^3+k_2^3+k_3^3)t}, \end{aligned} \tag{51}$$

where we found

$$b_{123} = a_{12} a_{23} a_{13}. \tag{52}$$

This shows that the coupled pKdV equation (43) is completely integrable and N-soliton solutions can be determined for $u(x,t)$ and $v(x,t)$, for finite N, where $N \geq 1$.

4.2. Multiple singular soliton solutions. For singular soliton solution, we set the auxiliary function $f(x,t)$ by

$$f(x,t) = 1 - e^{k_1 x - k_1^3 t}. \tag{53}$$

Substituting this in (46) and using (48) we obtain the singular soliton solutions

$$u(x,t) = v(x,t) = -\frac{6k_1\, e^{k_1 x - k_1^3 t}}{(\alpha+\beta)(1 - e^{k_1 x - k_1^3 t})}. \tag{54}$$

For the two singular soliton solutions and the three singular soliton solutions are identical and obtained by substituting

$$f(x,t) = 1 - e^{k_1 x - k_1^3 t} - e^{k_2 x - k_2^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3+k_2^3)t} \tag{55}$$

and

$$\begin{aligned} f(x,t) = 1 &- e^{k_1 x - k_1^3 t} - e^{k_2 x - k_2^3 t} - e^{k_3 x - k_3^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3+k_2^3)t} \\ &+ a_{13} e^{(k_1+k_3)x - (k_1^3+k_3^3)t} + a_{23} e^{(k_2+k_3)x - (k_2^3+k_3^3)t} \\ &- a_{12}a_{23}a_{13} e^{(k_1+k_2+k_3)x - (k_1^3+k_2^3+k_3^3)t} \end{aligned} \tag{56}$$

into (43) respectively.

5. The fourth coupled pKdV equation

We close this work by studying the fourth coupled pKdV equation [1]

$$\begin{aligned} &u_t - \frac{1}{2}u_{xxx} + \frac{3}{2}v_{xxx} + 6uu_{xx} - 6vu_{xx} + \alpha u_x^2 \\ &\quad - 6u_x v_x - \alpha v_x^2 - 12(u-v)^2 v_x - 3(u-v)^4 = 0, \\ &v_t + \frac{3}{2}u_{xxx} - \frac{1}{2}v_{xxx} - 6uv_{xx} + 6vv_{xx} - \alpha u_x^2 \\ &\quad - 6u_x v_x + \alpha v_x^2 - 12(u-v)^2 u_x - 3(u-v)^4 = 0. \end{aligned} \tag{57}$$

5.1. Multiple soliton solutions. Proceeding as before we get the dispersion relation

$$c_i = k_i^3, \tag{58}$$

and also

$$\theta_i = k_i x - k_i^3 t. \tag{59}$$

The multi soliton solutions of (57) is assumed to be

$$\begin{aligned} u(x,t) &= R_1 \left(\ln f\right)_x = R_1 \frac{f_x}{f}, \\ v(x,t) &= R_2 u(x,t) = R_1 R_2 \frac{f_x}{f}. \end{aligned} \tag{60}$$

Using the auxiliary function

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - k_1^3 t} \tag{61}$$

and proceeding as before we find

$$R_1 = -1, \;\; R_2 = 1. \tag{62}$$

This means that for the single soliton solutions $u(x,t) = v(x,t)$, therefore we set

$$u(x,t) = v(x,t) = -\frac{6k_1\, e^{k_1 x - k_1^3 t}}{(1 + e^{k_1 x - k_1^3 t})}. \tag{63}$$

Notice that R_1, R_2, and the solitons solutions (63) do not depend on α.

For the two soliton solutions we substitute the auxiliary function $f(x,t)$, given by

$$f(x,t) = 1 + e^{k_1 x - k_1^3 t} + e^{k_2 x - k_2^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3 + k_2^3)t} \tag{64}$$

into (60).

For the three soliton solutions $u(x,t)$ and $v(x,t)$ are also identical and can be obtained by proceeding as before. This shows that the coupled pKdV equation (57) is completely integrable and N-soliton solutions can be determined for $u(x,t)$ and $v(x,t)$, for finite N, where $N \geq 1$.

5.2. Multiple singular soliton solutions. For singular soliton solution, we set the auxiliary function $f(x,t)$ by

$$f(x,t) = 1 - e^{k_1 x - k_1^3 t}. \tag{65}$$

Proceeding as before, we obtain the singular soliton solutions

$$u(x,t) = v(x,t) = \frac{6k_1\, e^{k_1 x - k_1^3 t}}{1 - e^{k_1 x - k_1^3 t}}. \tag{66}$$

For the two singular soliton solutions and the three singular soliton solutions are identical and obtained by substituting

$$f(x,t) = 1 - e^{k_1 x - k_1^3 t} - e^{k_2 x - k_2^3 t} + a_{12} e^{(k_1+k_2)x - (k_1^3 + k_2^3)t} \tag{67}$$

and

$$\begin{aligned} f(x,t) = 1 &- e^{k_1 x - k_1^3 t} - e^{k_2 x - k_2^3 t} - e^{k_3 x - k_3^3 t} \\ &+ a_{12} e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + a_{13} e^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\ &+ a_{23} e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\ &- a_{12}a_{23}a_{13} e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t} \end{aligned} \tag{68}$$

into (57) respectively.

6. Discussion

In this work four coupled potential KdV equations were studied. The following conclusions can be concluded:
1. The four coupled pKdV equations give the same phase shift a_{ij}.
2. The four coupled pKdV equations are resonance free because the phase shift term a_{12} cannot be 0 or ∞ for $|k_1| \neq |k_2|$.
3. The four coupled pKdV equations are completely integrable and multiple soliton solutions were formally derived for each one.
4. The only difference we observed is in the coefficients of the obtained soliton solutions for the four coupled pKdV equations. The parameter α in the fourth coupled pKdV equation has no effect on the obtained solutions.

The Hirota's bilinear method was employed to achieve the goal we set for this work. The analysis confirms the fact that certain equations which have N-soliton solutions, have simultaneously, N-singular soliton solutions.

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