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A HIGHER ORDER NUMERICAL SCHEME FOR SINGULARLY PERTURBED BURGER-HUXLEY EQUATION†

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ABSTRACT. In this article, we present a numerical scheme for solving singularly perturbed (i.e. highest -order derivative term multiplied by small parameter) Burgers-Huxley equation with appropriate initial and boundary conditions. Most of the traditional methods fail to capture the effect of layer behavior when small parameter tends to zero. The presence of perturbation parameter and nonlinearity in the problem leads to severe difficulties in the solution approximation. To overcome such difficulties the present numerical scheme is constructed. In construction of the numerical scheme, the first step is the dicretization of the time variable using forward difference formula with constant step length. Then, the resulting non linear singularly perturbed semidiscrete problem is linearized using quasilinearization process. Finally, differential quadrature method is used for space discretization. The error estimate and convergence of the numerical scheme is discussed. A set of numerical experiment is carried out in support of the developed scheme.

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1. Introduction

In this paper, we develop a numerical method for solving the one-dimensional time dependent Burgers-Huxley equation

$$\frac{\partial u}{\partial t} - \epsilon\frac{\partial^2 u}{\partial x^2} + \alpha u\frac{\partial u}{\partial x} = \beta(1-u)(u-\gamma)u, \tag{1}$$

with initial condition

$$u(x,0) = \phi(x), \quad x \in (0,1) \tag{2}$$

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and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T] \tag{3}$$

where $\alpha, \beta \geq 0$ and $\gamma \in (0,1)$ and $0 < \epsilon \leq 1$ are the parameters. The equation describes the interaction between convection, diffusion and reaction. When $\alpha = 0$ and $\epsilon = 1$, equation (1) reduces to Huxley equation [1]. This equation describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals [2-5]

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \beta(1-u)(u-\gamma)u, \tag{4}$$

On the other hand, when $\beta = 0$, equation [1] reduces to the Burgers' equation at high Reynolds number that establishes a balance between time evolutions, nonlinearity and diffusion:

$$\frac{\partial u}{\partial t} - \epsilon\frac{\partial^2 u}{\partial x^2} + \alpha u\frac{\partial u}{\partial x} = 0 \tag{5}$$

The nonlinear partial differential equation [3] is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modeling of turbulent fluid and in continuous stochastic processes. Such type of equation firstly introduced by Bateman [6] in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger [7, 8] introduced this equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. It shows a structure roughly similar to that of Navier-Stokes equations due to the form of the non-linear convection term and the occurrence of the viscosity term. So this equation can be considered as a simplified form of the Navier-Stokes equations [9]. Many other problems can also be modeled by the Burgers' equation [10].

We are interested in understanding in an approximate way the structure of the solution of the perturbed problem for the Burgers-Huxley equation. Singular perturbation problems belong to the class in which a very small positive parameter $\epsilon \in (0,1)$ is multiplied by the highest order derivative term in the differential equation. This small parameter is known as the singular perturbation parameter. It is well established fact that nonlinear diffusion equations (1) - (3) with small parameter play an important role in nonlinear physics. Also it is of great practical interest to study the nonlinear phenomena. In order to solve equation (1), various numerical approaches were adopted by the researchers. Ismail et al. [11] proposed the adomain decomposition method for Burgers-Huxley and Burger-Fisher equation. More recently, Javidi [12] has given a pseudospectral method and Darvishi's preconditioning for generalized Burgers-Huxley equation. In this paper, the main aim is to construct and analyses a numerical scheme for the singularly perturbed Burgers-Huxley equation. In construction of the numerical scheme, the first step is the dicretization of the time variable using forward difference formula with constant step length. Then, the resulting non linear singularly perturbed semidiscrete problem is linearized using quasi-linearization process. Finally, differential quadrature method is used for space discretization.

The error estimate and convergence of the numerical scheme is discussed. The error estimate depends on the perturbation parameter, so the scheme is point-wise convergent. A set of numerical experiment is carried out in support of the developed scheme.

2. The Time Semidiscretization

The first step in the construction of numerical scheme consists of discretizing the time variable with forward difference formula for time derivative of the unknown u with constant time step Δt.This produces a set of stationary singularly perturbed problems of type

$$u_0 = \phi(x) \tag{6}$$

$$\frac{u_{j+1} - u_j}{\Delta t} = [\epsilon u_{xx} - \alpha(uu_x)]_{j+1} + \beta[(1-u_{j+1})(u_{j+1}-\gamma)u_{j+1}], 0 \le j \le M-1 \tag{7}$$

with the boundary conditions

$$u_{j+1}(0) = u_{j+1}(1) = 0, j = 0, 1, ..., M-1 \tag{8}$$

where u_{j+1} is the solution of the above differential equation at the $j+1th$ time step. The above differential equation can be written as follows

$$u_0 = \phi(x) \tag{9}$$

$$\begin{aligned} u_j = &-\epsilon\Delta t(u_{j+1})_{xx} + \alpha\Delta t(u_{j+1})(u_{j+1})_x \\ &- \beta\Delta t(1-u_{j+1})(u_{j+1}-\gamma)u_{j+1} + u_{j+1}, \quad 0 \le j \le M-1. \end{aligned} \tag{10}$$

with the boundary conditions

$$u_{j+1}(0) = u_{j+1}(1) = 0, j = 0, 1, ..., M-1 \tag{11}$$

3. Quasilinearization Process

In the second step, we linearize the non linear problem (10) by using the quasi-linearization process. Using quasilinearization process, we linearize the non-linear problem and followed by simplification yields

$$u_0^{(n+1)} = \phi(x) \tag{12}$$

$$\begin{aligned} u_j = &-\epsilon\Delta t(u_{j+1}^{(n+1)})_{xx} + \alpha\Delta t(u_{j+1}^{(n+1)}(u_{j+1}^{(n)})_x + u_{j+1}^{(n)}(u_{j+1}^{(n+1)})_x - u_{j+1}^{(n)}(u_{j+1}^{(n)})_x) \\ &+ u_{j+1}^{(n+1)} - \beta\Delta t(u_{j+1}^{(n+1)}(1-u_{j+1}^{(n)})(u_{j+1}^{(n)}-\gamma) + u_{j+1}^{(n)}(1-u_{j+1}^{(n)}) \\ &- u_{j+1}^{(n)}(u^{(n)}-\gamma)) - \beta\Delta t(u_{j+1}^{(n)})^2((1-u_{j+1}^{(n)}) + (u^{(n)}-\gamma)), \end{aligned} \tag{13}$$

with the boundary conditions

$$u_{j+1}^{(n+1)}(0) = u_{j+1}^{(n+1)}(1) = 0, j = 0, 1, ..., M-1 \tag{14}$$

where $u^{(n)}$ is the nominal solution of the problem (10) with initial guess $u^{(0)}$ and n is the iteration index.

Using the notation $u^{(n+1)} = U$ the above equation can be written as

$$U_0 = \phi(x) \tag{15}$$

$$\begin{aligned}u_j =& -\epsilon\Delta t(U_{j+1})_{xx} + \alpha\Delta t(U_{j+1}(u^{(n)}_{j+1})_x + u^{(n)}_{j+1}(U_{j+1})_x - u^{(n)}_{j+1}(u^{(n)}_{j+1})_x) + U_{j+1}\\ &- \beta\Delta t(U_{j+1}(1-u^{(n)}_{j+1})(u^{(n)}_{j+1}-\gamma) + u^{(n)}_{j+1}(1-u^{(n)}_{j+1}) - u^{(n)}_{j+1}(u^{(n)}-\gamma))\\ &- \beta\Delta t(u^{(n)}_{j+1})^2((1-u^{(n)}_{j+1}) + (u^{(n)}-\gamma)),\ \ 0 \le j \le M-1\end{aligned} \tag{16}$$

For the simplicity we use the following notations

$$a^{(n)}(x) = [\frac{1}{\Delta t}+\alpha(u^{(n)}_{j+1})_x-\beta(1-u^{(n)}_{j+1})(u^{(n)}_{j+1}-\gamma)+u^{(n)}_{j+1}(1-u^{(n)}_{j+1})-u^{(n)}_{j+1}(u^{(n)}_{j+1}-\gamma)], \tag{17}$$

$$b^{(n)}(x) = \alpha u^{(n)}_{j+1}, \tag{18}$$

$$F^{(n)} = \frac{u^{(n+1)}_{j+1}}{\Delta t}u^{(n)}_{j+1} + \alpha(u^{(n)}_{j+1})(u^{(n)}_{j+1})_x + \beta\{2((u^{(n)}_{j+1}))^3 - (1+\gamma)(u^{(n)}_{j+1})^2\} \tag{19}$$

We assume that τ and ρ are the lower bounds of the $b^{(n)}(x)$ and $a^{(n)}(x)$ respectively i.e.

$$b^{(n)}(x) \ge \tau > 0, x \in \overline{\Omega} \tag{20}$$

$$a^{(n)}(x) \ge \rho > 0, x \in \overline{\Omega}. \tag{21}$$

Using the above notations equation (16) can be written in the following form

$$U_0 = \phi(x) \tag{22}$$

$$-\epsilon(U_{j+1}(x))_{xx} + b^{(n)}(x)(U_{j+1}(x))_x + a^{(n)}(x)U_{j+1}(x) = F^{(n)}(x) \tag{23}$$

The equation (23) can be written in the following form

$$U_0 = \phi(x) \tag{24}$$

$$L_\epsilon U_{j+1}(x) = F^{(n)}(x), 0 \le j \le M-1, \tag{25}$$

where

$$L_\epsilon U_{j+1}(x) = -\epsilon(U_{j+1}(x))_{xx} + b^{(n)}(x)(U_{j+1}(x))_x + a^{(n)}(x)U_{j+1}(x) \tag{26}$$

with the boundary conditions

$$U_{j+1}(0) = U_{j+1}(1) = 0. \tag{27}$$

3.1. Existence and Uniform bounded. We now establish the existence and uniform boundedness of the sequence $\langle U^{(n)}_{j+i}(x)\rangle$ for b sufficiently small. For the sake of convenience, we consider the following form of equation (23)

$$(U_{j+1})_{xx}(x) = H(U_{j+1}), \tag{28}$$

with boundary conditions

$$U_{j+1}(0) = U_{j+1}(b) = 0. \tag{29}$$

After using the quasilinearization process, we obtain a sequence $\langle U^{(n)}_{j+i}(x)\rangle$ of linear equations determined by the recurrence

$$(U^{(n+1)}_{j+1})_{xx} = H(U^{(n)}_{j+1}) + (U^{(n+1)}_{j+1} - U^{(n)}_{j+1})H_{U_{j+1}}(U^{(n)}_{j+1}), \tag{30}$$

with boundary conditions

$$U^{(n+1)}_{j+1}(0) = U^{(n+1)}_{j+1}(b) = 0, \tag{31}$$

Converting this into an integral equation using Green's function, we get

$$U_{j+1}^{(n+1)} = \int_0^b K(x,y)(H(U_{j+1}^{(n)}) + (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)}))dy \tag{32}$$

where the Green's function

$$K(x,y) = \begin{cases} x(y-b)/b, 0 \le x \le y \le b \\ (x-b)y/b, 0 \le y \le x \le b \end{cases} \tag{33}$$

Observe that

$$\max_{(x,y)} |K(x,y)| = \frac{b}{4},$$

where the maximum is over the region $0 \le x \le y \le b$. let

$$\max_{\|U_{j+1}\| \le 1} (|H(U_{j+1})|, |H^{'}(U_{j+1})) = m_1,$$

assuming that $m_1 < \infty$ and choose $U_{j+1}^{(0)}(x)$ so that $|U_{j+1}^0(x)| \le 1$, for $0 \le x \le b$.
Turning to (32) and taking modulus both side, we have

$$|U_{j+1}^{(n+1)}| \le \int_0^b |K(x,y)||(H(U_{j+1}^{(n)}) + (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)}))|dy, \tag{34}$$

Writing

$$m_2 = \max_{0 \le x \le b} |U_{j+1}^{(1)}|,$$

for $n = 0$, we have

$$m_2 \le \frac{b}{4}\int_0^b (2m_1 + m_1 m_2)dy \le \frac{b^2 m_1}{2} + \frac{b^2 m_1}{4} m_2 \tag{35}$$

Thus, we obtain

$$m_2 \le \frac{b^2 m_1/2}{(1 - b^2 m_1/4)}, \tag{36}$$

provided $\frac{b^2 m_1}{4} < 1$. This upper bound is itself less than 1 if $b^2 \le \frac{4}{3m_1}$, a constraint that can be met by taking the interval$[0,b]$to be sufficiently small, Thus, under these conditions,$m_2 \le 1$. It is clear that this procedure can be continued inductively with the result that we can assert that $|U_{j+1}^{(n)}| \le 1$, for $0 \le x \le b$ provided that $b^2 \le \frac{4}{3m_1}$. Hence, we have demonstrated that the inductive definition of the sequence $\langle U_{j+i}^{(n)}(x) \rangle$ is meaningful.

3.2. Convergence of Quasilinearization Process. For the sake of convenience, we consider the following form of equation (23)

$$(U_{j+1})_{xx}(x) = H(U_{j+1}), \tag{37}$$

with boundary conditions

$$U_{j+1}(0) = U_{j+1}(b) = 0. \tag{38}$$

Consider the sequence obtained by the following relation

$$(U_{j+1}^{(n+1)})_{xx} = H(U_{j+1}^{(n)}) + (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)}), \tag{39}$$

with boundary conditions

$$U_{j+1}^{(n+1)}(0) = U_{j+1}^{(n+1)}(b) = 0, \tag{40}$$

Subtracting the $(n)th$ step from $(n+1)th$ step of equation (39), we have

$$\begin{aligned}\left(U_{j+1}^{(n+1)} - U_{j+1}^{(n)}\right)_{xx} =& H\left(U_{j+1}^{(n)}\right) - H\left(U_{j+1}^{(n-1)}\right) - \left(U_{j+1}^{(n)} - U_{j+1}^{(n-1)}\right) H_{U_{j+1}}\left(U_{j+1}^{(n-1)}\right) \\ &+ \left(U_{j+1}^{(n+1)} - U_{j+1}^{(n)}\right) H_{U_{j+1}}\left(U_{j+1}^{(n)}\right),\end{aligned} \tag{41}$$

The above equation is a differential equation for $(U_{j+1}^{(n+1)} - U_{j+1}^{(n)})$, and converting into an integral equation, we have

$$\begin{aligned}(U_{j+1}^{(n+1)} - U_{j+1}^{(n)}) = \int_0^1 &[K(x,y)H(U_{j+1}^{(n)}) - H(U_{j+1}^{(n-1)}) \\ &- (U_{j+1}^{(n)} - U_{j+1}^{(n-1)})H_{U_{j+1}}(U_{j+1}^{(n-1)}) \\ &+ (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)})]dy,\end{aligned} \tag{42}$$

where the Green's function

$$K(x,y) = \begin{cases} x(y-1), 0 \le x \le y \le 1 \\ (x-1)y, 0 \le y \le x \le 1 \end{cases} \tag{43}$$

Observe that

$$\max_{(x,y)} |K(x,y)| = \frac{1}{4},$$

The mean-value theorem gives us

$$\begin{aligned}H(U_{j+1}^{(n)}) =& H(U_{j+1}^{(n-1)}) + (U_{j+1}^{(n)} - U_{j+1}^{(n-1)})H_{U_{j+1}}(U_{j+1}^{(n-1)}) \\ &+ \frac{(U_{j+1}^{(n)} - U_{j+1}^{(n-1)})^2}{2} H_{U_{j+1}U_{j+1}}(\xi)\end{aligned} \tag{44}$$

where ξ lies between U_{j+1}^{n-1} and U_{j+1}^{n}. Using equation (44) into (42), we have

$$\begin{aligned}(U_{j+1}^{(n+1)} - U_{j+1}^{(n)}) = \int_0^1 &K(x,y)(\frac{(U_{j+1}^{(n)} - U_{j+1}^{(n-1)})^2}{2} H_{U_{j+1}U_{j+1}}(\xi) \\ &+ (U_{j+1}^{(n+1)} - U_{j+1}^{(n)})H_{U_{j+1}}(U_{j+1}^{(n)}))dy,\end{aligned} \tag{45}$$

Define

$$k = \max_{|U_{j+1}| \le 1} |H_{U_{j+1}U_{j+1}}(U_{j+1})|$$

and

$$m = \max_{|U_{j+1}| \le 1} |H_{U_{j+1}}(U_{j+1})|$$

Now equation (45) becomes

$$|(U_{j+1}^{(n+1)} - U_{j+1}^{(n)})| \leq \frac{1}{4}\int_0^1 \frac{k}{2}(\frac{(U_{j+1}^{(n)} - U_{j+1}^{(n-1)})^2}{2} + |(U_{j+1}^{(n+1)} - U_{j+1}^{(n)})|)dy, \quad (46)$$

Taking the maximum over x on both sides of the above inequality and after simplification, we have

$$max|(U_{j+1}^{(n+1)} - U_{j+1}^{(n)})| \leq \frac{k}{8(1-m/4)} max(U_{j+1}^{(n+1)} - U_{j+1}^{(n)})^2 \quad (47)$$

The inequality (47) shows that the convergence of quasilinearization process is quadratic.

4. The spatial Discretization

We consider the polynomial differential quadrature method (PDQM) for spatial discretization. PDQM is an approximation to derivatives of a function at any grid points using weighted sum of all the functional values at certain points in the whole computational domain. Since the weighting coefficients are dependent only the spatial grid spacing, we assume uniformly distributed N grid points $x_1 < x_2 <, ..., < x_N$ on the real axis. The differential quadrature discretization of the first and the second derivatives at a pointx_i is given by the following equations

$$u_x(x_i, t) = \sum_{j=1}^{N} a_{ij} u_x(x_j, t), \quad u_{xx}(x_i, t) = \sum_{j=1}^{N} b_{ij} u_x(x_j, t) \quad (48)$$

where a_{ij} and b_{ij} represent the weighting coefficients [17], $i = 1, 2, ..., N$. The following base functions are used to obtain weighting coefficients

$$g_k(x) = \frac{M(x)}{(x - x_k)M^{(1)}(x_k)}, k = 1, 2, ..., N \quad (49)$$

where

$$M(x) = (x - x_1)(x - x_2)...(x - x_N)$$
$$M^{(1)}(x_i) = \prod_{k=1, k\neq i}^{N} (x_i - x_k) \quad (50)$$

using the set of base functions given in equation (49), the weighting coefficients of the first order derivative are found as [17]

$$a_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, j \neq i \quad (51)$$

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, i = 1, 2, ..., N \quad (52)$$

and for weighting coefficients of the second order derivative, the formula is [17]

$$b_{ij} = 2a_{ij}(a_{ii} - \frac{1}{x_i - x_j}), j \neq i \tag{53}$$

$$b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}, i = 1, 2, ..., N \tag{54}$$

Disrectizing equation (23) by using differential quadrature method mention above from (51) to (54) on the uniform mesh, we get

$$U_{0,i} = \phi(x_i), x_i \in \Omega^N \tag{55}$$

$$-\epsilon \sum_{k=1}^{N} b_{ik} U_{j+1,k} + b_i^{(n)} \sum_{k=1}^{N} a_{ik} U_{j+1,k} + a_i^{(n)} U_{j+1,k} = F^{(n)}(x_i), \tag{56}$$

$$U_{j+1,0} = U_{j+1,N} = 0. \tag{57}$$

This system is solved by Gauss-elimination method.

Lemma 1. *Maximum principle*
Assume that any function $\Psi(x,t) \in C^2(\overline{\Omega})$ satisfies $\psi(0,t) \geq 0$ and $\psi(1,t) \geq 0$. Then $L_\epsilon \psi(x,t) \geq 0$ for all $x \in \Omega$ implies that $\psi(x,t) \geq 0$ for all $x \in \overline{\Omega}$.

Proof. Let (z^*, t^*) be such that $\psi z^*, t^*) = min_{\overline{\Omega}}$ and suppose that $(\psi z^*, t^*) < 0$. It is clear that $(z^*, t^*) \in 0, 1$. Therefore $\psi_x(z^*, t^*) = 0, \psi_{xx}(z^*, t^*) \geq 0$, and
$L_\epsilon \psi(z^*, t^*) = -\epsilon \psi_{xx}(z^*, t^*) + b^{(n)}(z^*, t^*)\psi_x(z^*, t^*) + a^{(n)}(z^*, t^*)\psi z^*, t^*)$
Since $a^{(n)}(x) \geq 0$ for all $x \in \overline{\Omega}$ therefore $L_\epsilon \psi(z^*, t^*) < 0$,
which contradicts the assumption, therefore it follows that $\psi z^*, t^*) \geq 0$ and thus $\psi(x,t) \geq 0$ for all $x \in \overline{\Omega}$. □

Lemma 2. *Let $U_{j+1}(x)$ be the solution of equation (10), there exits a constant C such that*

$$\|U_{j+1}(x)\| \leq C \quad \forall x \in \overline{\Omega} \tag{58}$$

$$\|U_{j+1}^{(k)}(x)\| \leq C(1 + \epsilon^{-k} e^{-\tau(1-x)/\epsilon}), \quad \forall x \in \overline{\Omega} \tag{59}$$

where $U_{j+1}^{(k)}$ denote the kth derivative of $U_{j+1}(x)$

Proof. See [19]. □

Error Estimate and Convergence Analysis

Lemma 3. *Under the assumption that all the coordinates are in the interval h(i.e. uniform mesh with length h) and the Nth order derivative of the function $U_{j+1}(x)$ is bounded (as in **lemma 2**), then*

$$|M^{(m)}(x)| \leq N(N-1)...(N-m+1)h^{N-m} \tag{60}$$

$$|\overline{M}(\overline{x})| \leq h^{N-m} \tag{61}$$

Proof. For detail see [pp. 42] of [17]. □

Lemma 4. *If the error for the mth order derivative approximation is defined as*

$$E_D^{(m)}(U_{j+1}) = \frac{\partial^m U_{j+1}}{\partial x^m} - \frac{\partial^m (P_N U_{j+1})}{\partial x^m} \tag{62}$$

where $P_N U_{j+1}$ is the approximation of U_{j+1} a polynomial of degree $(N-1)$, then

$$E_D^{(m)}(U_{j+1}(\overline{x})) = \frac{U_{j+1}^{(N)}(\overline{\xi})\overrightarrow{M}(\overline{x})}{(N-m)!} \tag{63}$$

and

$$\|E_D^{(m)}(U_{j+1})\| \leq \frac{C(1+\epsilon^{-k}e^{-\tau(1-x)/\epsilon})h^{N-m}}{(N-m)!}, m = 1, 2. \tag{64}$$

Proof. Let $\phi(x) = P_N U_{j+1}$ and define a function $F(z)$ as

$$F(z) = U_{j+1}(z) - \phi(z) - cM(z) \tag{65}$$

since $g(z) = U_{j+1}(z) - \phi(z)$ has N roots in the interval, according to Rolle's Theorem, its mth order derivative $g^{(m)}(z)$ has at least $(N-m)$ roots in the interval, namely $\overline{x}_1, \overline{x}_2, ..., \overline{x}_{N-m}$
Thus, the function

$$F^{(m)}(z) = U_{j+1}^{(m)}(z) - \phi^{(m)}(z) - \overline{c}\overline{M}(z) \tag{66}$$

where $\overline{M}(z) = (z-\overline{x}_1)(z-\overline{x}_2)...(z-\overline{x}_{N-m})$ would vanish at $\overline{x}_1, \overline{x}_2, ..., \overline{x}_{N-m}$. Now, if set $M^{(m)}(\overline{x}) = 0$, where $\overline{x}$ is different from $\overline{x}_1, \overline{x}_2, ..., \overline{x}_{N-m}$, then $F^{(m)}(z)$ has $(N-m+1)$ roots, and

$$E_D^{(m)}(U_{j+1}(\overline{x})) = U_{j+1}^{(m)}(\overline{x}) - \phi^{(m)}(\overline{x}) = \overline{c}\overline{M}(\overline{x}) \tag{67}$$

Using Roll's Theorem repeatedly $(N-m)$ times, the $(N-m)th$ order derivative of $F^{(m)}(z)$ is found to have at least one root $\overline{\xi}$, i. e.

$$U_{j+1}^{(N)}(\overline{\xi}) = -\overline{c}(N-m)! = 0 \tag{68}$$

Using Lemma 3, we have

$$E_D^{(m)}(U_{j+1}(\overline{x})) = \frac{U_{j+1}^{(N)}(\overline{\xi})\overrightarrow{M}(\overline{x})}{(N-m)!} \tag{69}$$

Taking norm and using Lemma 2 and Lemma 3, we have

$$\|E_D^{(m)}(U_{j+1})\| \leq \frac{C(1+\epsilon^{-k}e^{-\tau(1-x)/\epsilon})h^{N-m}}{(N-m)!}, m = 1, 2 \tag{70}$$

□

Theorem 1. *Let $U_{j+1}(x)$ be the solution of equation (23) and $U_{h,j+1}$ be the solution of the discrete problem (57) at the $(j+1)th$ level, then*

$$\|U_{j+1}(x) - U_{h,j+1}\|_h \leq \frac{C(1+\epsilon^{-k}e^{-\tau(1-x)/\epsilon})h^{N-2}}{\rho(N-2)}(\epsilon + \frac{c_1 h}{N-1}) \tag{71}$$

where $\|\|_h$ denote the discrete maximum norm.

Proof. Let $U_{j+1}(x)$ be the solution of equation (23) and $U_{h,j+1}$ be the solution of the discrete problem (57) at the $(j+1)th$ level. Subtracting equation (23) from the equation (57) and using the Lemma 4, we have

$$(U_{j+1}(x) - U_{h,j+1}) = \frac{1}{a^{(n)}(x_i)}(\epsilon E_D^{(2)}(U_{j+1}(x_i)) - b^{(n)}(x_i)E_D^{(1)}(U_{j+1}(x_i))) \quad (72)$$

Taking modulus both side, we get

$$|(U_{j+1}(x) - U_{h,j+1})| \leq \frac{\epsilon}{|a^{(n)}(x_i)|}|(E_D^{(2)}(U_{j+1}(x_i))| + \frac{|b^{(n)}(x_i)|}{|a^{(n)}(x_i)|}|E_D^{(1)}(U_{j+1}(x_i)))| \quad (73)$$

Using the Lemma 4 and the boundedness of $b^{(n)}$ and $a^{(n)}$, and taking the discrete maximum norm, we have

$$\|U_{j+1}(x) - U_{h,j+1}\|_h \leq \frac{C(1+\epsilon^{-k}e^{-\tau(1-x)/\epsilon})h^{N-2}}{\rho(N-2)}(\epsilon + \frac{c_1 h}{N-1}). \quad (74)$$

□

5. Numerical Results and Discussions

In this section, to illustrate the efficiency and accuracy of the numerical method, a set of numerical experiments is carried out.

Example 1. *We consider Burgers-Huxley equation with* $\alpha = \beta = 1, \gamma = 0.5$ *and initial and boundary conditions in the following form*

$$u(x,0) = \sin(\pi x), 0 \leq x \leq 1 \quad (75)$$

$$u(0,t) = u(0,t) = 0, 0 \leq x \leq T. \quad (76)$$

Example 2. *We consider Burgers-Huxley equation with* $\alpha = 1, \beta = 0$ *and initial and boundary conditions in the following form*

$$u(x,0) = x(1-x^2), 0 \leq x \leq 1 \quad (77)$$

$$u(0,t) = u(0,t) = 0, 0 \leq x \leq T. \quad (78)$$

Example 3. *We consider Burgers-Huxley equation with* $\alpha = 3, \beta = 9.8, \gamma = 0.7$ *and initial and boundary conditions in the following form*

$$u(x,0) = (1 - \cos(x)), 0 \leq x \leq 1 \quad (79)$$

$$u(0,t) = u(0,t) = 0, 0 \leq x \leq T. \quad (80)$$

Throughout the numerical experiment the time step length $\Delta t = 0.001$ is used. Table 1-3 displays the computed solutions of Examples 1-3 for different values of N(number of grid points) and at different time $t = 0.1, 0.9$. The tables show that the computed solutions converge as the number of grid point increase. The Figures 1-11 show the layer behavior of the problem at different values of time t and ϵ . It can be seen from the figures that this scheme faithfully mimic the dynamics of the corresponding nonlinear time dependent partial differential equation.

TABLE 1. Computed numerical solution of example 1 at different number of grid points for different value of ϵ.

2 t	ϵ	x	N=7	N=9	N=17	N=21	N23
0.1	2^{-3}	0.25	0.52587	0.52588	0.52588	0.52588	0.52588
		0.50	0.86138	0.86147	0.86147	0.86147	0.86147
		0.75	0.73938	0.74023	0.74030	0.74030	0.74030
	2^{-7}	0.25	0.57049	0.56850	0.56855	0.56854	0.56854
		0.50	0.95056	0.95109	0.95111	0.95111	0.95111
		0.75	0.86501	0.86705	0.86710	0.86710	0.86710
0.9	2^{-3}	0.25	0.14655	0.15035	0.15062	0.15062	0.15062
		0.50	0.26229	0.26340	0.26496	0.26498	0.26498
		0.75	0.24125	0.23874	0.24194	0.24198	0.24198
	2^{-7}	0.25	0.20604	0.18612	0.18260	0.18172	0.18173
		0.50	0.39594	0.39458	0.39243	0.39244	0.39244
		0.75	0.62581	0.60975	0.61369	0.61361	0.61361

TABLE 2. Computed numerical solution of example 2 at different number of grid points for different value of ϵ.

t	ϵ	x	N=7	N=9	N=17	N=21	N23
0.1	2^{-3}	0.25	0.20155	0.20156	0.20162	0.20162	0.20162
		0.50	0.33095	0.33085	0.33085	0.33085	0.33085
		0.75	0.29131	0.29084	0.29072	0.29072	0.29072
	2^{-7}	0.25	0.21555	0.21554	0.21554	0.21554	0.21554
		0.50	0.36193	0.36194	0.36194	0.36194	0.36194
		0.75	0.34562	0.34571	0.34573	0.34573	0.34573
0.9	2^{-3}	0.25	0.07784	0.07825	0.07820	0.07820	0.07820
		0.50	0.12143	0.12198	0.12202	0.12202	0.12202
		0.75	0.09543	0.09619	0.09629	0.09629	0.09629
	2^{-7}	0.25	0.12608	0.12805	0.12856	0.12867	0.12868
		0.50	0.25011	0.24804	0.24829	0.24829	0.24829
		0.75	0.34650	0.34085	0.34146	0.34146	0.34146

6. Conclusion

In this paper, a numerical scheme is constructed for solving the singularly perturbed time dependent Burgers-Huxley equation. To solve such a type of problem, one encounters the problem of nonlinearity and perturbation parameter. The nonlinearity is tackled by quasilinearzation process and quadratic convergence of the process is shown. Finally, the error estimate and convergence analysis of the scheme is discussed. A set of numerical experiment iscarried out in support of the predicted theory and found that the numerical solutions match with predicted theory.

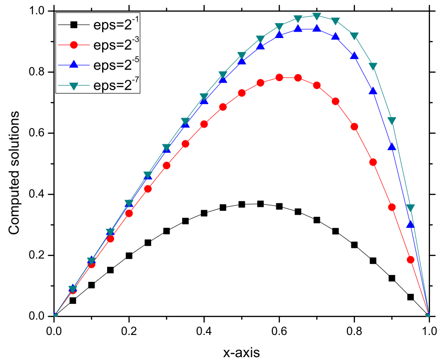

FIGURE 1. Computed solutions of example 1 for different values of ϵ at $T = 0.2$.

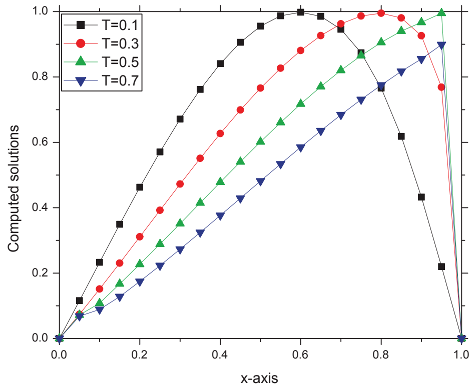

FIGURE 2. Computed solutions of example 1 for different values of time at $\epsilon = 2^{-5}$.

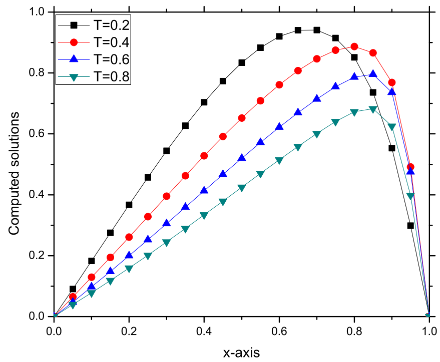

FIGURE 3. Computed solutions of example 1 for different values of time at $\epsilon = 2^{-9}$.

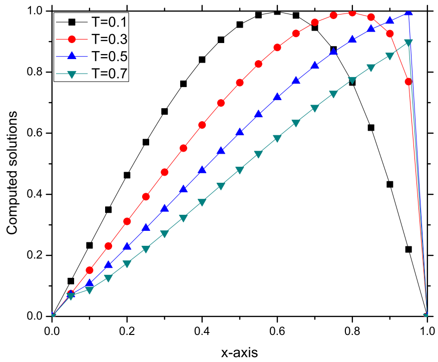

FIGURE 4. Computed solutions of example 2 for different values of ϵ at $T = 0.1$.

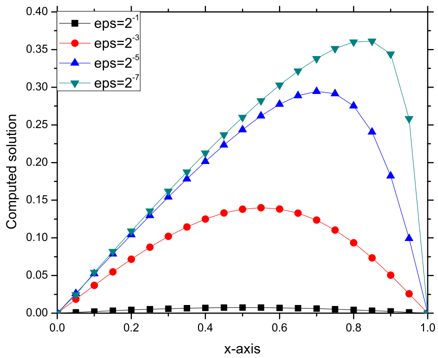

FIGURE 5. Computed solutions of example 2 for different values of ϵ at $T = 0.8$.

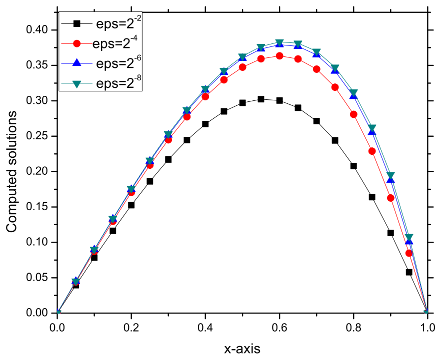

FIGURE 6. Computed solutions of example 2 for different values of time at $\epsilon = 2^{-1}$.

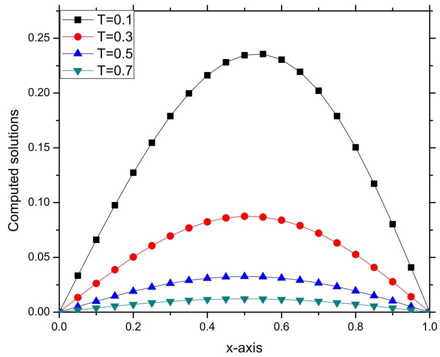

FIGURE 7. Computed solutions of example 2 for different values of time at $\epsilon = 2^{-7}$.

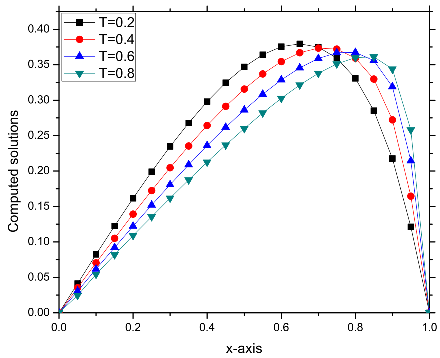

FIGURE 8. Computed solutions of example 3 for different values of ϵ at $T = 0.1$.

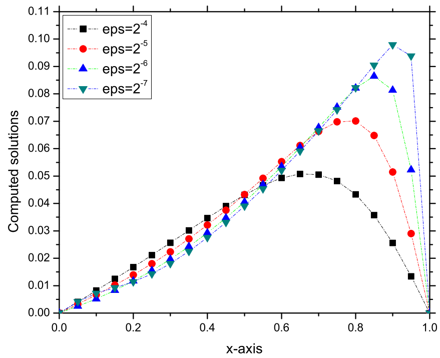

FIGURE 9. Computed solutions of example 3 for different values of ϵ at $T = 0.9$.

TABLE 3. Computed numerical solution of example 2 at different number of grid points for different value of ϵ.

t	ϵ	x	N=7	N=9	N=17	N=21	N23
0.1	2^{-3}	0.25	0.03878	0.03769	0.03680	0.03683	0.03682
		0.50	0.11059	0.11062	0.11063	0.11063	0.11063
		0.75	0.20124	0.20435	0.20309	0.20318	0.20319
	2^{-7}	0.25	0.02301	0.02737	0.02782	0.02783	0.02783
		0.50	0.10415	0.10231	0.10271	0.10271	0.10271
		0.75	0.22613	0.21488	0.21701	0.21702	0.21702
0.9	2^{-3}	0.25	0.01857	0.01843	0.01833	0.01833	0.01833
		0.50	0.03006	0.02955	0.02942	0.02942	0.02942
		0.75	0.02434	0.02385	0.02375	0.02375	0.02375
	2^{-7}	0.25	0.01486	0.01480	0.01468	0.01433	0.01419
		0.50	0.04005	0.03997	0.03904	0.03904	0.03904
		0.75	0.07562	0.07408	0.07422	0.07422	0.07422

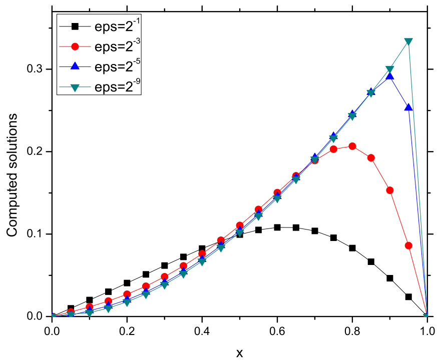

FIGURE 10. Computed solutions of example 3 for different values of time at $\epsilon = 2^{-4}$.

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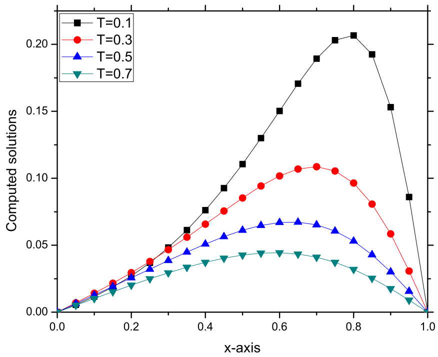

FIGURE 11. Computed solutions of example 3 for different values of time at $\epsilon = 2^{-7}$.

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