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MEAN SQUARE STABILITY IN A MODIFIED LESLIE-GOWER AND HOLLING-TYPE II PREDATOR-PREY MODEL

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Abstract. Of concern in the paper is a Holling-Tanner predator-prey model with modified version of the Leslie-Gower functional response. Dynamical behaviours such as stability, permanence and Hopf bifurcation have been carried out deterministically. Using the normal form theory and center manifold theorem, the explicit formulae determining the stability and direction of Hopf bifurcation have been derived . The deterministic model is extended to a stochastic one by perturbing the growth equation of prey and predator by white and colored noises and finally the mean square stability of the stochastic model systems is investigated analytically. An extensive quantitative analysis has been performed based on numerical computation so as to validate the applicability of the proposed mathematical model.

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1. Introduction

Lotka-Volterra model is the simplest model of predator-prey interaction. The simplicity of Lotka-Volterra model relies on certain assumptions. First it is supposed that the prey population has unlimited food supply and will grow exponentially in the absence of the predator. It is also supposed that the predator species feeds on prey only and on nothing else, and will starve and become extinct in the absence of prey, rather than switch to a different types of food. Other simplifying assumptions are also made upon prey searching, prey consumption and environmental complexity. In spite of that, it plays important roles in the history of mathematical ecology to describe various dynamical characteristics of population interaction. The Lotka-Volterra model is one of the earliest predator-prey models based on sound mathematical and ecological principles. It forms the basis of many models used now a day in the analysis of population dynamics and

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received extensive attentions from mathematicians and ecologists (cf. [15, 21, 22] and references cited therein). Based on experiments, Holling [12] suggested three different kinds of functional responses for different species to model the phenomena of predation, which made the standard Lotka-Volterra system more realistic. Many authors investigated the mathematical properties of these models and explained their implication in biology (cf. [13, 26, 27]). Biologically, it is quite natural to study the existence and asymptotic stability of equilibria, and limit cycles for autonomous predator-prey systems with these functional responses. This prompted us to study the predator-prey system with Holling type-II functional response. For Holling type-II functional response, the predation rate increases as prey density rises, eventually levels off due to the predators handling time. The model also incorporates a modified version of the Leslie-Gower functional response (cf. [2, 3, 14, 18, 19, 32]). The Leslie-Gower predator-prey model formulation is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. Indeed, Leslie [16] introduced a predator-prey model where the carrying capacity of the predators environment is proportional to the number of prey. This interesting formulation for the predator dynamics has been discussed in [17] and [24].

Major works in this direction are based on deterministic models of differential or difference equations. The deterministic approach has some limitations in mathematical modelling of ecological systems. It is quite difficult to predict the future dynamics of the system accurately. This happens due to the fact that deterministic models in ecology do not incorporate the effect of fluctuating environment (cf. [9, 23, 26]) based upon the idea that in case of large populations, stochastic deviations are small enough to be ignored. Stochastic differential equation models (cf. [4, 5, 8, 25, 28, 29, 30, 31]) play a significant role in various branches of applied sciences including biology and population dynamics, as they provide some additional degree of realism compared to their deterministic counterpart. In reality, demographic parameters involved with the modelling approach of ecological systems are not absolute constants, they always fluctuate around some average value due to continuous fluctuation in environment (e.g. variation in intensity of sunlight, temperature, water level etc.). As a result, the population density never attains a fixed value with advancement of time rather exhibit continuous oscillation around some average values. Based upon these factors, we extend our deterministic model to a stochastic one with the assumption that fluctuations in the environment will manifest themselves mainly as fluctuations in the natural growth rate of both the prey and predator species. These fluctuations are taken in terms of both white and colored noises followed by Wiener and Ornstein-Uhlenbeck processes. In this paper, an attempt has also been made to study the mean square stability of the model system in presence of both white and colored noises showing that the colored noise has a stabilizing effect with respect to white noise.

The paper is organized as follows: In Section 2, we present a mathematical model of the Holling-Tanner predator-prey model with modified Leslie-Gower functional response and discuss the boundedness and permanence of the model system. In Section 3, we study the local asymptotic stability, Hopf-bifurcation and Global stability for the model system. Also in this section, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are determined analytically by using the normal form and the center manifold theory. In Section 4, we consider the effect of multiplicative white noise and find an estimation of the noise intensities to investigate the exponential mean square stability of the stochastic model system. Section 5 consists of the effect of multiplicative colored noise to investigate stability in mean square sense. We present a comparative analysis of stability properties in the concluding section.

2. The Model : Boundedness and Permanence

We consider the following predator-prey model as proposed by Aziz-Alaoui et al. [3].

$$\frac{dN}{d\tau} = rN\Big(1-\frac{N}{K}\Big) - \frac{bNP}{N+K_1}, \tag{2.1a}$$

$$\frac{dP}{d\tau} = P\Big(a-\frac{cP}{N+K_2}\Big), \tag{2.1b}$$

where $N(\tau)$ and $P(\tau)$ are the number of prey and predator species at time τ. The model system is subject to the initial conditions $N(0) > 0$ and $P(0) > 0$. The parameters r, K, b, K_1, K_2, a, c involved with the model system (2.1) are all positive and the sign ($+$ or $-$) in front of each term indicates an increase or loss in the growth rate. In [3] Aziz-Alaoui et al., have discussed stability analysis of the model system (2.1) leaving the analysis of boundedness and permanence. In this present paper, we prove the same results and also extend the model system (2.1) in a random environment.

For the dynamical system (2.1), the basic assumptions and the significance of parameters are as follows:

(A_1) In the absence of predation, the prey population grows logistically with carrying capacity $K(\in \mathbf{R}_+)$ and intrinsic growth rate $r(\in \mathbf{R}_+)$ as follows:

$$\frac{dN}{dt} = r\Big(1-\frac{N}{K}\Big).$$

(A_2) The predator species consumes the prey according to the functional response $\dfrac{bN}{N+K_1}$ and 'a' denotes the growth rate of predator P. K_1 is the measure of the extent to which environment provides protection to prey N.

(A_3) K_2 measures the extent to which the environment provides protection to the predator.

(A_4) b is the maximal predator per capita consumption rate, i.e., the maximum number of preys that can be captured by a predator in each unit time.

(A_5) c is a measure of the food quality that the prey provides for conversion into predator births.

We non-dimensionalize our model system (2.1) with the following scaling

$$t = r\tau, \quad x = \frac{N}{K}, \quad y = \frac{cP}{aK},$$

and this results into

$$\frac{dx}{dt} = x\Big(1 - x\Big) - \frac{\alpha xy}{x+\delta} = F_1(x,y) \tag{2.2a}$$

$$\frac{dy}{dt} = \beta y\Big(1 - \frac{y}{x+\gamma}\Big) = F_2(x,y) \tag{2.2b}$$

where $x(0) > 0$, $y(0) > 0$ and

$$\alpha = \frac{ab}{cr}, \quad \beta = \frac{a}{r}, \quad \gamma = \frac{K_2}{K}, \quad \delta = \frac{K_1}{K}.$$

Considering the biological significance, we investigate the dynamical system (2.2) in the region R_+^2 where $R_+^2 = \Big\{(x,y) \in R^2 : x \geq 0, y \geq 0\Big\}$.

Theorem 1. *R_+^2 is an invariant set.*

Proof. From the first equation of (2.2) it follows that $x = 0$ is an invariant subset *i.e* $x \equiv 0$ if and only if $x = 0$ for some time t. This imply that $x(t) > 0 \; \forall \; t$ if $x(0) > 0$. The same argument follows for the second equation of the system (2.2) i.e., any trajectory starting in R_+^2, cannot cross the coordinate planes. Hence the theorem. $\square$

Theorem 2. *The prey population is always bounded above.*

Proof. The first equation of (2.2) gives

$$\frac{dx}{dt} \leq x\Big(1 - x\Big).$$

Therefore $\limsup_{t\to\infty} x(t) \leq 1$. Hence the theorem. $\square$

Theorem 3. *All the solutions of (2.2) that commences in R_+^2 are uniformly bounded.*

Proof. Let us define a function $W(t) : R_+ \to R_+$ by $W = x + y$.
The time derivative gives

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = x\big(1-x\big) - \frac{\alpha xy}{x+\delta} + \beta y \left(1 - \frac{y}{x+\gamma}\right).$$

For any $\rho > 0$, we get

$$\frac{dW}{dt} + \rho W \leq x\big(1 + \rho - x\big) + y\big(\beta + \rho - \frac{\beta y}{1+\gamma}\big) \leq \frac{(1+\rho)^2}{4} + \frac{(1+\gamma)(\beta+\rho)^2}{4\beta}. \tag{2.3}$$

Thus we can define a constant $\eta > 0$, such that

$$\eta = \frac{(1+\rho)^2}{4} + \frac{(1+\gamma)(\beta+\rho)^2}{4\beta} > 0.$$

This shows

$$\frac{dW}{d\tau} + \rho W \leq \eta.$$

By applying Gronwall's inequality [7], we get

$$0 < W(x, y) \leq \frac{\eta}{\rho}(1 - e^{-\rho t}) + e^{-\eta t} W(x(0), y(0)).$$

Thus for $t \to \infty$, we have $0 < W(x, y) < \frac{\eta}{\rho}$. Hence all the solutions $(x(t), y(t))$ of (2.2) that commence in R_+^2 are restricted in the region

$$H = \left\{ (x, y) \in R_+^2 : W = \frac{\eta}{\rho} + \vartheta, \ \forall \ \vartheta > 0 \right\}$$

for all $t \geq T$, where T depends on the initial values $(x(0), y(0))$. □

Definition 1. *The model system* (2.2) *is said to be permanent if there exist* ξ_1, ξ_2, $0 < \xi_1 < \xi_2$, *such that for all solutions of* (2.2) *with the initial conditions* $x(0) > 0$ *and* $y(0) > 0$,

$$\min\{\liminf_{t\to\infty} x(t), \quad \liminf_{t\to\infty} y(t)\} \geq \xi_1,$$

and

$$\max\{\limsup_{t\to\infty} x(t), \quad \limsup_{t\to\infty} y(t)\} \leq \xi_2.$$

Theorem 4. *The model system* (2.2) *is permanent if* $\alpha(1 + \gamma) < \delta$.

Proof. We have $x(t) \leq 1$ for all values of t. Now from the second equation of (2.2), we get

$$\frac{dy}{dt} = \beta y \left(1 - \frac{y}{x + \gamma}\right) = \beta y \left(\frac{x + \gamma - y}{x + \gamma}\right) \leq \beta y \left(\frac{1 + \gamma - y}{x + \gamma}\right). \tag{2.4}$$

Therefore,

$$\limsup_{t\to\infty} y(t) \leq 1 + \gamma.$$

From the prey equation of the system (2.2), we have

$$\frac{dx}{dt} = x\left[1 - x - \frac{\alpha y}{x + \delta}\right] \geq x\left\{(1 - x) - \frac{\alpha(1 + \gamma)}{\delta(1 + \frac{x}{\delta})}\right\} \geq x\left\{(1 - x) - \frac{\alpha(1 + \gamma)}{\delta}\right\}. \tag{2.5}$$

Therefore, if we choose $\underline{x} = 1 - \dfrac{\alpha(1 + \gamma)}{\delta}$, then

$$\liminf_{t\to\infty} x(t) \geq \underline{x} \quad \text{if} \quad \underline{x} > 0 \quad \text{i.e if} \quad \frac{\alpha(1 + \gamma)}{\delta} < 1.$$

Hence for large t, we have $x(t) > \underline{x}$.
Also for large t,

$$\frac{dy}{dt} = \beta y \left(1 - \frac{y}{x + \gamma}\right) \geq \beta y \left(1 - \frac{y}{\underline{x} + \gamma}\right). \tag{2.6}$$

Let $\underline{y}$ be the root of the equation

$$1 - \frac{y}{\underline{x} + \gamma} = 0,$$

and this equation gives

$$y = \left(\underline{x} + \gamma\right) = \underline{y} > 0.$$

Choosing a positive number ϵ such that $\epsilon < \min\left\{\underline{x}, \underline{y}\right\}$, we arrive at the following conclusion

$$\liminf_{t\to\infty} x(t) > \epsilon, \text{ and } \liminf_{t\to\infty} y(t) > \epsilon.$$

Hence the theorem. □

3. Local Asymptotic Stability

The equilibrium points for the model system (2.2) are given by (i) $E_0(0,0)$ (trivial equilibrium), (ii)$E_1(1,0)$ (axial equilibrium) (iii) $E_2(0,\gamma)$ (axial equilibrium) and (iv) $E^*(x^*,y^*)$ (positive equilibrium), where $y^* = x^* + \gamma$ and x^* is the positive root of the quadratic equation

$$x^{*2} + (\alpha + \delta - 1)x^* + \alpha\gamma - \delta = 0. \tag{3.1}$$

The quadratic equation (3.1) has a positive root if $\delta > \alpha\gamma$ and is given by

$$x^* = \frac{1 - \alpha - \delta + \sqrt{(1-\alpha-\delta)^2 + 4(\delta - \alpha\gamma)}}{2}. \tag{3.2}$$

In order to find the stability of the above mentioned equilibria we have to determine the Jacobian matrix $J(x,y)$ or simply J for the dynamical system (2.2) at the point (x,y) within the first quadrant of x-y plane and is given by

$$J = \begin{bmatrix} 1 - 2\,x - \frac{\alpha\, y\delta}{(x+\delta)^2} & -\frac{\alpha\, x}{x+\delta} \\ \frac{\beta\, y^2}{(x+\gamma)^2} & \beta - 2\,\frac{\beta\, y}{x+\gamma} \end{bmatrix}. \tag{3.3}$$

At $E_0(0,0)$ the eigenvalues of the corresponding Jacobian matrix are 1 and β both of which are positive. Therefore, E_0 is unstable.

At $E_1(1,0)$ the eigenvalues of the corresponding Jacobian matrix are $-1 < 0$ and $\beta > 0$ and consequently E_1 is a saddle point.

At $E_2(0,\gamma)$ the eigenvalues of the corresponding Jacobian matrix are $1 - \frac{\alpha\gamma}{\delta}$ and $-\beta$. Therefore, E_2 is either a stable-node or a saddle according as $\delta < \alpha\gamma$ or $\delta > \alpha\gamma$.

Let J^* be the Jacobian matrix at E^*. Now we study the stability of the positive equilibrium $E^*(x^*,y^*)$.

Theorem 5. *The positive equilibrium E^* of the model system (2.2) is stable if* $\beta > -x^* + \dfrac{\alpha\, x^* y^*}{(x^*+\delta)^2}$ *with* $\delta > \gamma$.

Proof. The characteristic equation of the Jacobian matrix J^* is

$$\lambda^2 + Q\lambda + R = 0,$$

where

$$Q = -\text{trace}(J^*) = x^* - \frac{\alpha\, x^* y^*}{(x^* + \delta)^2} + \beta, \quad R = \det(J^*) = \beta x^* \left\{ 1 + \frac{\alpha(\delta - \gamma)}{(x^* + \delta)^2} \right\}.$$

According to Routh-Hurwitz criterion the necessary and sufficient conditions for local asymptotical stability are trace of $J^* < 0$ and $det(J^*) > 0$. Hence the theorem. $\square$

Theorem 6. *Suppose E^* exists with $\delta > \gamma$, then the model system (2.2) undergoes a Hopf bifurcation around E^* whenever $\beta = \beta^* = -x^* + \dfrac{\alpha\, x^* y^*}{(x^* + \delta)^2}$.*

Proof. We see that

$$\begin{aligned}
&(i) \quad \text{Trace of } J^* = 0, \text{ if } \beta = \beta^* = -x^* + \frac{\alpha\, x^* y^*}{(x^* + \delta)^2}. \\
&(ii) \quad \det(J^*)|_{\beta=\beta^*} > 0, \text{ if } \delta > \gamma. \\
&(iii) \quad \text{At } \beta = \beta^* \text{ eigenvalues are purely imaginary.} \\
&(iv) \quad \frac{d}{d\beta}(\text{trace of } J^*)|_{\beta=\beta^*} = -1 \neq 0.
\end{aligned}$$

Therefore, all the conditions of Hopf-bifurcation are satisfied and hence the theorem. $\square$

Theorem 7. *If $\alpha < \min\left(\beta, \dfrac{2\delta}{\gamma}\right)$, local asymptotical stability of E^* ensures its global stability.*

Proof. Let us consider a function $h(x, y)$ of the form $h(x, y) = \dfrac{1}{xy}$. Then $h(x, y) > 0$ for $x, y > 0$. Now

$$\Delta(x, y) = \frac{\partial}{\partial x}(F_1 h) + \frac{\partial}{\partial y}(F_2 h) = -\frac{1}{y} - \frac{(\beta - \alpha)x^2 + x(2\delta - \alpha\gamma) + \delta^2}{x(x + \gamma)(x + \delta)^2}.$$

Thus it follows that $\Delta(x, y) < 0$ if $\alpha < \beta$ and $2\delta > \alpha\gamma$. Therefore, by Bendixon-Dulac criterion there will be no limit cycle when $\alpha < \min\left(\beta, \dfrac{2\delta}{\gamma}\right)$. $\square$

3.1. Direction and stability of the Hopf bifurcation. In Theorem 6, we have obtained the conditions which guarantee that the system undergoes a Hopf bifurcation at the interior equilibrium $E^*(x^*, y^*)$ when β takes some critical values β^*. Now, we shall study the direction and stability of Hopf bifurcation by applying the techniques from normal form and center manifold theory introduced by Hassard et al. [10].

Based on the analysis in Appendix, we can compute the following values:

$$C_1(0) = \frac{i}{2\omega_0}(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \tag{3.4a}$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(0)\}}, \tag{3.4b}$$

$$\beta_2 = 2Re\{C_1(0)\}, \tag{3.4c}$$

$$\tau_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im(\lambda'(0))}{\omega_0}, \tag{3.4d}$$

where g_{ij} are given in (8.7) (Appendix), μ_2 determines the direction of the Hopf bifurcation, β_2 determines the stability of the bifurcating periodic solution, τ_2 determines the period of the bifurcating periodic solution and for the explanations of all the notations involved here we refer to Hassard et al. [10].

Therefore we are in a position to summarize the properties of the Hopf bifurcation at the critical value of $\beta = \beta^*$ in the following theorem:

Theorem 8. *(i) If $\mu_2 > 0$ (< 0); the Hopf bifurcation is supercritical (subcritical),*
(ii) If $\beta_2 < 0$ (> 0); the bifurcated periodic solutions are stable (unstable),
(iii) If $\tau_2 > 0$ (< 0); period of the bifurcating periodic solution increases (decreases).

4. The Model with White Noise: Mean Square Stability

In this section we study the effect of random fluctuation on the model system after introducing stochastic perturbation terms in the growth equation of prey and predator species. Here we assume that the stochastic perturbations of the variables around their value at E^* are of Gaussian white noise type, which are proportional to the distances of x, y from the values x^*, y^* (cf. [6]). So the stochastic version corresponding to the deterministic model system (2.2) takes the following form:

$$dx = F_1(x, y)dt + \sigma_1(x - x^*)d{\xi_t}^{(1)}, \tag{4.1}$$

$$dy = F_2(x, y)dt + \sigma_2(y - y^*)d{\xi_t}^{(2)}, \tag{4.2}$$

where σ_1, σ_2 are real constants known as the intensity of environmental fluctuations and ${\xi_t}^{(1)}$, ${\xi_t}^{(2)}$ are independent standard Wiener process (standard Brownian motion) (cf. [6]).

Equations (4.1) and (4.2) can be represented as an Ito stochastic differential system of the type

$$dX_t = f(t, X_t)dt + g(t, X_t)d\xi_t, \quad X_t(t = 0) = X_0 \tag{4.3}$$

whose solution X_t, for all positive time t is an Ito process and the components of (4.3) are given by

$$X_t = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \xi_t = \begin{bmatrix} \xi_t^{(1)} \\ {\xi_t}^{(2)} \end{bmatrix}, \; f(t, X_t) = \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix}, \tag{4.4}$$

and

$$g(t, X_t) = \begin{bmatrix} \sigma_1 (x - x^*) & 0 \\ 0 & \sigma_2 (y - y^*) \end{bmatrix}. \tag{4.5}$$

The function $f(t, X_t)$ is a slowly varying continuous component called as 'drift coefficient' and $g(t, X_t)$ is the rapidly varying continuous random component called as 'diffusion coefficient' and ξ_t is a 2-dimensional Wiener process whose increments $\Delta\xi_t{}^j = \xi_j(t + \Delta t) - \xi_j(t),\ j = 1,\ 2$ are independent Gaussian random variables $x(0, \Delta t)$.

Since the diffusion matrix $g(t, X_t)$ depends on the solution vector X_t, the stochastic system (4.1) - (4.2) is said to have multiplicative noise. Moreover, due to diagonal form of the diffusion matrix $g(t, X_t)$, the system (4.1) - (4.2) is said to have diagonal noise.

It is quite tricky to obtain the stability conditions for stochastic differential equations (4.1) - (4.2) in mean square sense by means of an appropriate Lyapunov functions method working on the complete nonlinear equations(4.1) $-$ (4.2). For the sake of simplicity, we consider the linearized version of stochastic differential equations (4.1) - (4.2) by introducing new variables $u_1 = x - x^*$, and $u_2 = y - y^*$.

The linearized version of (4.1) - (4.2) around $E^*(x^*, y^*)$ is given by

$$du(t) = f(u(t))dt + g(u(t))d\xi(t), \tag{4.6}$$

where

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},\ f(u(t)) = \begin{bmatrix} -a_{11}u_1 - a_{12}u_2 \\ a_{21}u_1 - a_{22}u_2 \end{bmatrix} \text{ and } g(u(t)) = \begin{bmatrix} \sigma_1 u_1 & 0 \\ 0 & \sigma_2 u_2 \end{bmatrix}. \tag{4.7}$$

In (4.7), the constants a_{ij} are given by

$$a_{11} \;=\; x^* - \frac{\alpha x^* y^*}{(x^* + \delta)^2}, \quad a_{12} = \frac{\alpha x^*}{x^* + \delta}, \quad a_{21} = \beta,\ a_{22} = \beta. \tag{4.8}$$

Clearly, in (4.6) the positive equilibrium E^* corresponds to the trivial solution $(u_1, u_2) = (0, 0)$.

We consider a set $\Omega = \left\{(t \geq t_0) \times R^2, t_0 \in R^+\right\}$. If $V \in C_2(\Omega)$ is a twice continuously differentiable function with respect to u and a continuous function with respect to t, then we state the following theorem due to Afanasev [1] regarding the mean square stability of the stochastic model system governed by (4.6).

Theorem 9. *Suppose there exists a function $V(u, t) \in C_2(\Omega)$ satisfying the inequalities*

$$K_1|u|^p \leq V(t, u) \leq K_2|u|^p, \tag{4.9}$$

$$LV(t, u) \leq K_3|u|^p, \tag{4.10}$$

where $K_i > 0,\ i = 1, 2, 3$ and $p > 0$ are some suitable constants. Then the trivial solution of (4.6) is exponentially p-stable for all $t \geq 0$. If $p = 2$, then the trivial solution of (4.6) is exponentially mean square stable. Moreover, the trivial solution of (4.6) is globally asymptotically stable in probability. Note that $|u|$ represents the modulus of u, L is the differential operator associated with the equation (4.6) and defined by

$$LV(t, u) = \frac{\partial V(t, u)}{\partial t} + f^T(u(t))\frac{\partial V(t, u)}{\partial u} + \frac{1}{2}Tr[g^T(u(t))\frac{\partial^2 V(t, u)}{\partial u^2}g(u(t))], \tag{4.11}$$

where T means usual matrix transposition and the first and second order partial derivatives of V with respect to u are defined as follows:

$$\frac{\partial V(t,u)}{\partial u} = \begin{bmatrix} \dfrac{\partial V}{\partial u_1} \\ \dfrac{\partial V}{\partial u_2} \end{bmatrix}, \ \frac{\partial^2 V(t,u)}{\partial u^2} = \begin{bmatrix} \dfrac{\partial^2 V}{\partial {u_1}^2} & \dfrac{\partial^2 V}{\partial u_1 \partial u_2} \\ \dfrac{\partial^2 V}{\partial u_2 \partial u_1} & \dfrac{\partial^2 V}{\partial {u_2}^2} \end{bmatrix}. \tag{4.12}$$

Now we can prove the following theorem,

Theorem 10. *Assume that $\omega_1(2a_{11} - \sigma_1^2) > (a_{21}\omega_2 - a_{12}\omega_1)$ with $a_{11} > 0$ and $\omega_2(2a_{22} - \sigma_2^2) > (a_{21}\omega_2 - a_{12}\omega_1)$, then the zero solution of the system (4.6) is asymptotically mean square stable.*

Proof. Let us consider the Lyapunov function

$$V(u) = \frac{1}{2}[\omega_1 {u_1}^2 + \omega_2 {u_2}^2], \tag{4.13}$$

where ω_1 and ω_2 are real positive constant to be chosen latter. Then (4.12) gives

$$\frac{\partial V(t,u)}{\partial u} = \begin{bmatrix} \omega_1 u_1 \\ \omega_2 u_2 \end{bmatrix}, \quad \frac{\partial^2 V(t,u)}{\partial u^2} = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}, \tag{4.14}$$

$$g^T(u(t)) \frac{\partial^2 V(t,u)}{\partial u^2} g(u(t)) = \begin{bmatrix} \omega_1 {\sigma_1}^2 {u_1}^2 & 0 \\ 0 & \omega_2 {\sigma_2}^2 {u_2}^2 \end{bmatrix}, \tag{4.15}$$

and

$$\frac{1}{2} Tr[g^T(u(t)) \frac{\partial^2 V(t,u)}{\partial u^2} g(u(t))] = \frac{1}{2}[\omega_1 {\sigma_1}^2 {u_1}^2 + \omega_2 {\sigma_2}^2 {u_2}^2]. \tag{4.16}$$

Therefore,

$$\begin{aligned} LV(u(t)) &= \omega_1(-a_{11}u_1 - a_{12}u_2)u_1 + \omega_2(a_{21}u_1 - a_{22}u_2)u_2 + \frac{1}{2}[\omega_1 {\sigma_1}^2 {u_1}^2 + \omega_2 {\sigma_2}^2 {u_2}^2] \\ &= -{u_1}^2 \left\{ a_{11} - \frac{{\sigma_1}^2}{2} \right\} \omega_1 - {u_2}^2 \left\{ a_{22} - \frac{{\sigma_2}^2}{2} \right\} \omega_2 - u_1 u_2 \{ a_{12}\omega_1 - a_{21}\omega_2 \}. \end{aligned}$$

If we chose $\dfrac{a_{21}}{a_{12}} > \dfrac{\omega_1}{\omega_2} > 0$, Using the inequality $u_1 u_2 < \frac{u_1^2 + u_2^2}{2}$ we can rewrite the above expression as

$$LV(u(t)) = -\frac{1}{2}\Big[\left(\left(2\,a_{11} - {\sigma_1}^2 \right) w_1 + w_1 a_{12} - w_2 a_{21} \right) u_1^2 + \left(\left(2\,a_{22} - {\sigma_2}^2 \right) \omega_2 + \omega_1 a_{12} - w_2 a_{21} \right) u_2^2 \Big],$$

which can be written as

$$LV(u(t)) = -\frac{1}{2}[u^T Q u], \tag{4.17}$$

where

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}. \tag{4.18}$$

with

$$q_{11} = (2\,a_{11} - {\sigma_1}^2)\omega_1 + \omega_1 a_{12} - \omega_2 a_{21}, \;\; q_{22} = (2\,a_{22} - {\sigma_2}^2)\omega_2 + \omega_1 a_{12} - \omega_2 a_{21}.$$

The eigenvalues, say λ_1, and λ_2 of the matrix Q will be positive if the following conditions hold:

$$\omega_1(2a_{11} - \sigma_1^2) > (a_{21}\omega_2 - a_{12}\omega_1), \;\; \omega_2(2a_{22} - \sigma_2^2) > (a_{21}\omega_2 - a_{12}\omega_1). \tag{4.19}$$

Now we define $\lambda_m = min(\lambda_1, \lambda_2)$, then from (4.14) we get the subsequent result

$$LV(u(t)) \leq -\lambda_m |u|^2. \tag{4.20}$$

Hence the theorem. □

5. The Model with Colored Noise: Mean Square Stability

In the previous section we have perturbed the system (2.2) by independent white noises due to randomly fluctuating environment but in real ecosystems the external random perturbations, because of interaction with the environment, are correlated within a finite correlation time. When the time scale of random fluctuations is larger than the characteristic time scale of the ecosystem the external noise cannot be considered white noise. A strongly correlated noise, for example, emerges as the result of a coarse graining over a hidden set of slow variables. With this consideration we will study the effect of colored noise perturbation on the system (2.2). Proceeding as earlier the linearized version of the model system (2.2) in presence of colored noise takes the form

$$\frac{du_1}{dt} = -a_{11}u_1 - a_{12}u_2 + \eta_1(t)u_1, \tag{5.1a}$$

$$\frac{du_2}{dt} = a_{21}u_1 - a_{22}u_2 + \eta_2(t)u_2, \tag{5.1b}$$

where the perturbed terms $\eta_1(t)$ and $\eta_2(t)$ are independent colored noises modeled by Ornstein - Uhlenbeck processes (which are more realistic noises than white noises) and satisfies the following Langevin equation

$$\frac{d\eta_i(t)}{dt} = -\alpha_i \eta_i(t) + \sigma_i \xi_i(t), \; t > 0, \; \eta_i(0) = \eta_{i0} \; i = 1, 2, \tag{5.2}$$

where $\alpha_i > 0$, $\sigma_i > 0$ are constants. The mathematical expectations and correlation functions of the processes $\eta_i(t)$ are given by

$$E\{\eta_i(t)\} = 0, \; E\{\eta_i(s), \eta_i(t)\} = \frac{{\sigma_i}^2}{2\alpha_i} e^{-\alpha_i|t-s|}, \; i = 1, \; 2, \tag{5.3}$$

where ${\sigma_i}^2$ is the intensity of white noise process $\xi_i(t)$. $\xi_i(t)$, $i = 1, 2$ are independent standard Gaussian white noises having the following expectations and correlation functions:

$$E\{\xi_i(t)\} = 0, \; E\{\xi_i(t), \xi_i(s)\} = \delta(s - t), \; i = 1, \; 2, \tag{5.4}$$

where $\delta(t)$ denotes the Dirac delta function. It is to be noted that for $\alpha_i \to \infty$, the Ornstein - Uhlenbeck noise approaches to white-noise limit $\xi_i(t)$. The stochastic dynamic system (5.2) considered separately is not a Markov process but the four component

process $(x(t), y(t), \eta_1(t), \eta_2(t))$ taken together is Markovian. Now we are in a position to derive the sufficient conditions for exponential mean square stability.
Rewriting (5.1) as

$$\frac{du(t)}{dt} = A\,u(t) + \eta(t)\,u(t) \tag{5.5}$$

where $A = \begin{bmatrix} -a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix}$, $\eta(t) = \begin{bmatrix} \eta_1\,(t) & 0 \\ 0 & \eta_2\,(t) \end{bmatrix}$ and u(t) is defined in (4.7).

Let us assume $a_{11} + a_{22} > 0$, so that in absence of multiplicative colored noise, the zero solution of (5.1) is always locally asymptotically stable. We also assume that $a_{11}^2 + a_{22}^2 \neq 2\Big(a_{11}a_{22} + 2a_{12}a_{21}\Big)$, so that all the eigenvalues of the coefficient matrix A are distinct and have negative real parts. Let us denote the eigenvalues of the matrix A as λ_1 and λ_2 and using similarity transformation the matrix A can be transformed into a $(2\times\ 2)$ diagonal matrix with the eigenvalues of A as the entries on the main diagonal. Under these conditions it can be proved that the first moment exponential stability, even moment exponential stability, p-th mean exponential stability and almost sure exponential stability are equivalent to the same properties for the two first order systems

$$\frac{du_i}{dt} = \lambda_i u_i + u_i \eta_i(t),\ i = 1,\, 2. \tag{5.6}$$

The solution of the decoupled SDEs (5.6) is given by

$$u_i(t) = u_i(0)\exp\left\{\lambda_i t + \int_0^t \eta_i(s)ds\right\},\ i = 1,\, 2. \tag{5.7}$$

Therefore

$$|u_i(t)|^p \quad = \quad |u_i(0)|^p \exp\left\{p(Re\lambda_i)t + p\int_0^t \eta_i(s)ds\right\},\ i = 1,\, 2. \tag{5.8}$$

Then the p-th moment of $|u_i(t)|,\ i = 1,\, 2$ is given by

$$E\{|u_i(t)|^p\} = |u_i(0)|^p \exp\{p(Re\lambda_i)t\}\, E\left[\exp\left\{p\int_0^t \eta_i(s)ds\right\}\right], i = 1,\, 2. \tag{5.9}$$

For a Gaussian stochastic process $\theta(t)$, we have

$$E\{\exp(\theta(t))\} = \exp\left\{E(\theta(t)) + \frac{1}{2}E(\theta^2(t))\right\},\ i = 1,\, 2. \tag{5.10}$$

Using (5.10) in (5.9), we get

$$E\{|u_i(t)|^p\} = |u_i(0)|^p \exp\{p(Re\lambda_i)t\} \exp\left\{pE\left(\int_0^t \eta_i(s)ds\right) + \frac{p^2}{2}E(\mu_i^2(t))\right\}, \tag{5.11}$$

where $\mu_i(t) = \displaystyle\int_0^t \eta_i(s)ds$, $i = 1,\, 2$.

Now using the properties of Ornstein-Uhlenbeck process we have

$$E(\mu_i(t)) = 0, \tag{5.12a}$$

$$E(\mu_i^2(t)) = \frac{\sigma_i^2}{\alpha_i^2}t + \frac{\sigma_i^2}{\alpha_i^3}(\exp(-\alpha_i t) - 1), \ i = 1, 2. \tag{5.12b}$$

From (5.11) and (5.12), we get

$$E\{|u_i(t)|^p\} = |u_i(0)|^p \exp\left\{p\left(Re(\lambda_i) + \frac{p\sigma_i^2}{2\alpha_i^2}\right)t\right\}\exp\left\{\frac{p^2\sigma_i^2}{2\alpha_i^3}(\exp(-\alpha_i t) - 1)\right\}, i = 1, 2. \tag{5.13}$$

We summarize the above fact in the following theorem,

Theorem 11. *Let us assume that* $a_{11} + a_{22} > 0$ *and* $a_{11}^2 + a_{22}^2 \neq 2\Big(a_{11}a_{22} + 2a_{12}a_{21}\Big)$. *Then the model system* (5.1) *can be transformed into a set of two decoupled SDEs* (5.6) *and the null solution of* (5.1) *is exponentially mean square stable if and only if*

$$Re(\lambda_i) + \frac{\sigma_i^2}{\alpha_i^2} < 0, \ \ i = 1, 2. \tag{5.14}$$

6. Numerical Simulations

In this section we have shown the numerical simulations of our model system in deterministic environment as well as in stochastic environment using Matlab software, in order to substantiate the analytical results. Here we take the following set of parameter values: $\alpha = 0.4$; $\beta = 0.9$; $\gamma = 0.3$; $\delta = 0.55$. Using the above set, we have estimated the positive equilibrium point as $E^* = (0.6812202375, 0.9812202375)$, $tr(J^*) = -1.404842999 < 0$. The parameters satisfies the existence and stability condition of E^*. Therefore by **Theorem 5** and **7**, E^* is locally as well as globally asymptotically stable. Visibly, for a large number of distinct positive starting values, each of the curve converging to E^* spirally, [cf. Figure 1].

For parameters $\alpha = 1$; $\beta = \beta^* = 0.721$; $\gamma = 0.001$; $\delta = 0.01$, $E^*(0.09, 0.091)$ loses its stability and become unstable. The corresponding phase portraits are shown in **Figure 2**. Also we get $\omega_0 = \sqrt{R(\beta^*)} = 0.3511281817$.

By means of the software Maple, we evaluate from (8.7) (in Appendix),

$$g_{11} = -0.4500000000 - 0.9778524105i, \quad g_{02} = -2.360384612 + 1.918204812i$$
$$g_{20} = 1.460384612 + 2.308347236i, \qquad g_{21} = -11.04634404 + 15.11611588i$$

Then from (8.8) and (3.4), we compute

$$Re\lambda'(0) = \rho'(0) = -0.50, \ \ Im\lambda'(0) = \omega'(0) = 0.2492129164,$$
$$C_1(0) = -2.010499160 + 2.145529445i, \ \ \mu_2 = -4.020998320,$$
$$\beta_2 = -4.020998320 \ \text{ and } \ \tau_2 = -3.256488048.$$

Thus we conclude that since $\mu_2 < 0$; the Hopf bifurcation of system (2.2) occurring at $\beta^* = 0.721$ is subcritical and the bifurcating periodic solution exist when β crosses β^* to the left. Also $\beta_2 < 0$ implies that this hopf bifurcating periodic solutions from $E^*(0.09, 0.091)$ at $\beta^* = 0.721$ are stable. Since $\tau_2 < 0$, the period of the periodic solutions increases as β decrease.

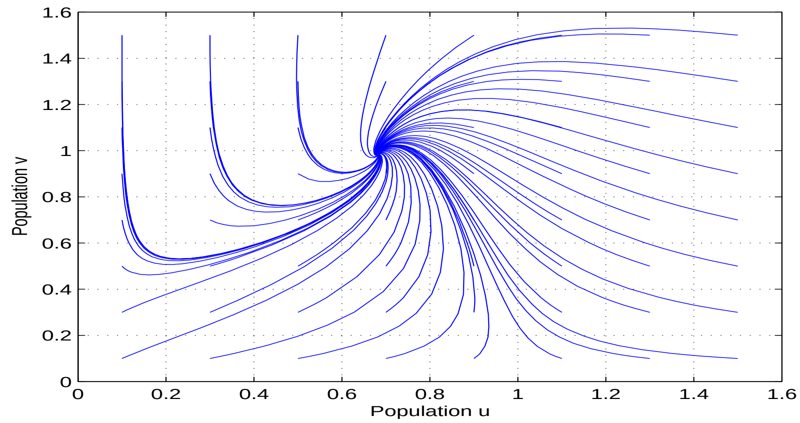

FIGURE 1. The phase portrait showing E^* is a global attractor

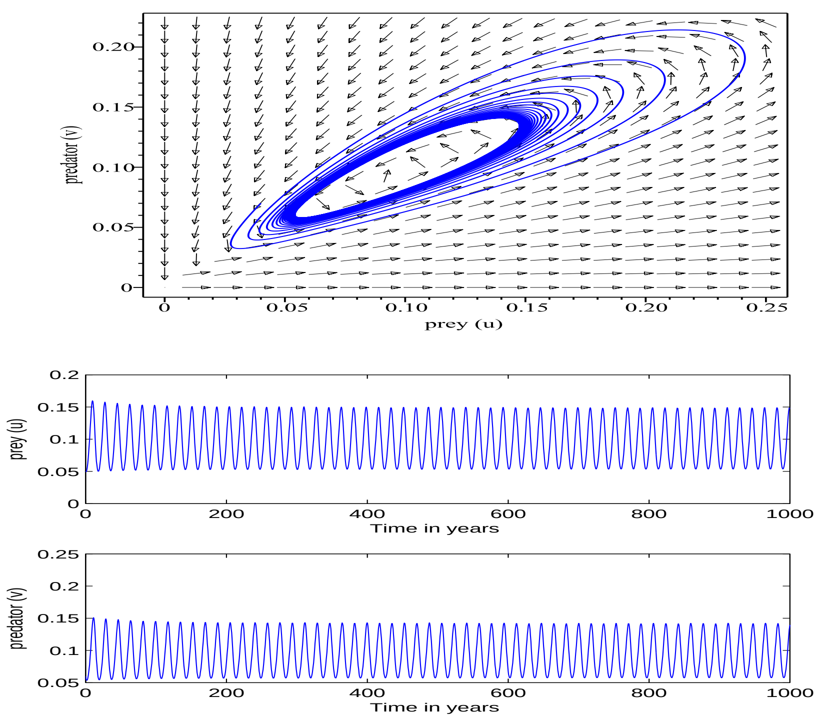

FIGURE 2. The Hopf-bifurcating periodic solution

For the numerical experiments in stochastic environment (cf. [11, 20]) with parametric white noise, the parameter estimates involved in the stochastic differential equations (4.1) - (4.2) are taken as $\alpha = 0.4$; $\beta = 0.9$; $\gamma = 0.3$; $\delta = 0.55$ and $(u(0), v(0)) = (0.6, 0.8)$. These parameters shows a stable nature of the system shown in **Figure 3**, with the value of noise intensities $\sigma_1 = 0.07$ and $\sigma_2 = 0.11$, satisfying the restrictions in (4.19).

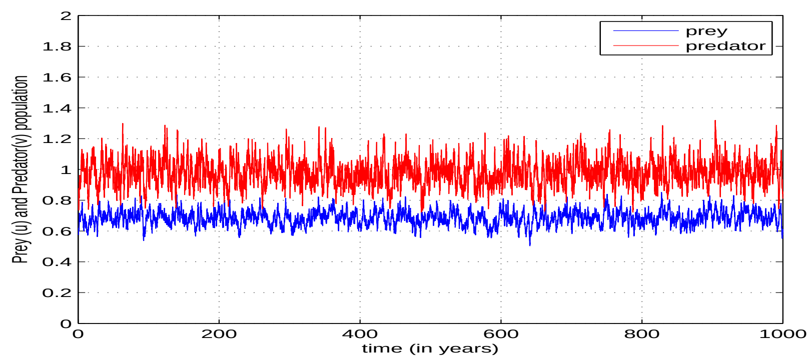

FIGURE 3. Stochastically stable population distribution for prey and predator with parametric white noise.

Now if we gradually increase the intensities of fluctuation σ_1 and σ_2 for which the system loses its stability. For $\sigma_1 = 0.9$ and $\sigma_2 = 1.1$ keeping the remaining parameters unchanged, we see a large amount of fluctuation in both prey and predator population as depicted clearly in **Figure 4**. For these parameter values with $\omega_1 = \omega_2 = 1$, we have

$$\begin{aligned}(2\,a_{11} - {\sigma_1}^2)\omega_1 + \omega_1 a_{12} - \omega_2 a_{21} &= -0.0886845224 < 0,\\(2\,a_{22} - {\sigma_2}^2)\omega_2 + \omega_1 a_{12} - \omega_2 a_{21} &= -0.4789985244 < 0.\end{aligned}$$

In this case the noise intensities crosses the threshold value presented in (4.19).

The numerical simulation in stochastic environment with parametric color noise has been performed by taking into account the same parametric values. The results on stochastic stability and unstability under the same noise intensity have been displayed in **Figures 5** and **6** respectively. It is interesting to note that the stochastic system becomes stable for $\sigma_1 = 0.07$, $\sigma_2 = 0.11$ whereas unstability occurs for $\sigma_1 = 0.9$, $\sigma_2 = 1.1$ keeping the values of the remaining parameters unchanged.

7. Discussion

In this paper, we have considered a Holling-Tanner predator-prey model with modified Leslie-Gower functional response. Our results show that under the condition $\alpha(1+\gamma) < \delta$ the model system is permanent. It is also observed that the equilibria $E_0(0,0)$ and $E_1(1,0)$ are always unstable, E_2 is conditionally stable but become unstable if the positive equilibrium E^* exists. The local asymptotical stability conditions for positive equilibrium E^* are derived and are presented in **Theorem 5**. We have also shown that our model

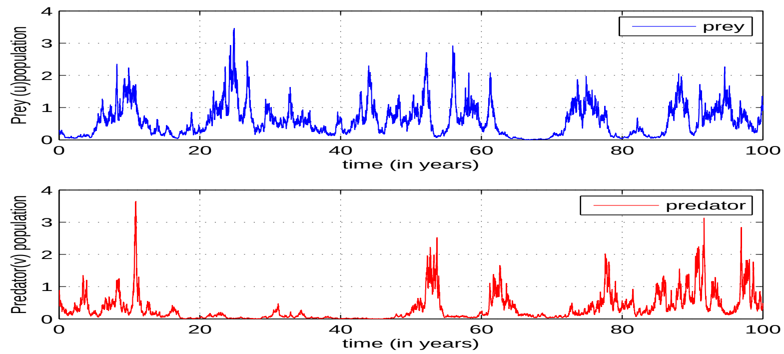

FIGURE 4. Fluctuation in prey and predator population with parametric white noise.

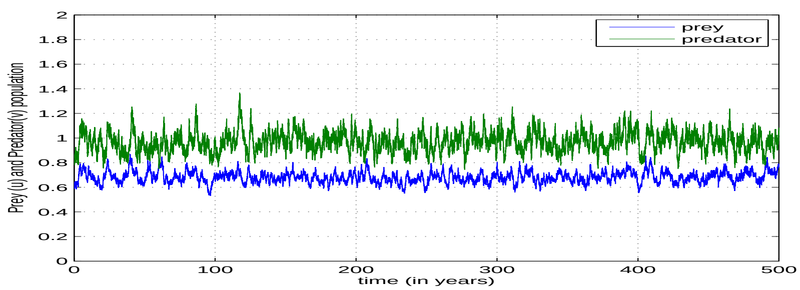

FIGURE 5. Stochastically stable population distribution for prey and predator under multiplicative colored noise

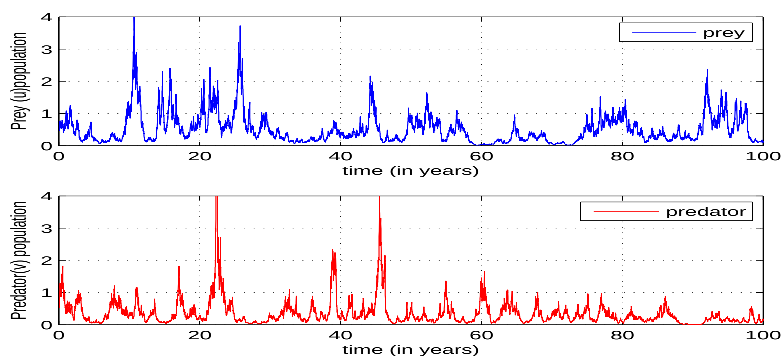

FIGURE 6. Fluctuation in prey and predator population under multiplicative colored noise

system undergoes a Hopf bifurcation, i.e., a small amplitude periodic solution emerges around E^* whenever β passes through $\beta = \beta^* = -x^* + \dfrac{\alpha\, x^* y^*}{(x^* + \delta)^2}$. The global stability analysis around E^* is carried out by using Bendixon-Dulac criterion and is presented in **Theorem 7**. Moreover, the direction and stability of the Hopf bifurcation is investigated by using center manifold and normal forms theory.

We then extend the deterministic model into stochastic environment. Undoubtedly, there are some limitations in biology for the deterministic approach to the model system because fluctuations are always presented in environment. Also it is a very difficult task to determine the eventual size of a population above which the deterministic approximations are reasonable. On the other hand, a stochastic model supplies a realistic illustration to a great extent of a natural system than its deterministic counterpart. We, in the present manuscript include fluctuations to our deterministic model system (2.2) in the form of multiplicative white and colored noises.

In presence of multiplicative white noises, We have obtained conditions for stochastic stability of the positive equilibrium E^* in the sense of mean square. Our mathematical findings indicate that the stochastic stability of the positive equilibrium point depends upon the magnitude of intensities of the environmental driving forces (σ_i, i=1,2). This condition is presented in the equation (4.19). In case of stochastic stability of population models, it intuitively seems appropriate to refer the systems characterized by large fluctuations in population numbers as '*unstable*' and to those with relatively small fluctuations as '*stable*'. If the intensities of environmental fluctuations do not cross their threshold values defined in (4.19) then it is possible to find a dense smoke cloud of population distribution within a hypothetical circular shell centered at E^* with a small radius.

The condition of stochastic stability of E^* in presence of multiplicative colored noises is derived in **Theorem 11**. The result presented in (5.13) shows that the relaxation of $E\{|u_1(t)|^p\}$ and $E\{|u_2(t)|^p\}$ to zero is accelerated in presence of colored noise with compare to white noise, which means colored noise has a stabilizing effect with respect to white noise. The necessary numerical simulations of the desired quantities in support of our analytical findings have been performed by using MATLAB software and are presented through their graphical representations in order to substantiate the applicability of the proposed model under consideration.

8. Appendix

For the sake of simplicity of notation, $\beta = \beta^* + \theta$, so $\theta = 0$ is the Hopf bifurcation value for the system (2.2). Let $x_1 = x - x^*$, $x_2 = y - y^*$. Thus, the equilibrium (x^*, y^*) of system (2.2) is translated to the origin. Now we expand the right side of system(3.2) in Taylor series expansion and system (2.2) reduces to:

$$\frac{dx_1}{dt} = -a_{11}x_1 - a_{12}x_2 + \sum_{m+n\geq 2} \frac{\Omega_{mn}}{m!n!} x_1^m x_2^n, \tag{8.1a}$$

$$\frac{dx_2}{dt} = a_{21}x_1 - a_{22}x_2 + \sum_{m+n\geq 2} \frac{\Gamma_{mn}}{m!n!} x_1^m x_2^n, \tag{8.1b}$$

where $m, n \geq 0$; $\Omega_{mn} = \left.\dfrac{\partial^{m+n} F_1}{\partial^m x \partial^n y}\right|_{E^*}$; $\Gamma_{mn} = \left.\dfrac{\partial^{m+n} F_2}{\partial^m x \partial^n y}\right|_{E^*}$; F_1, F_2 have the same expression as given in (2.2) and a_{ij}'s are given in (4.8).

Now we consider the equivalent system (8.1) of the system (2.2). The eigenvector v associated with eigenvalue $\lambda = \rho + i\omega$ is

$$v = \begin{bmatrix} 1 \\ \frac{-a_{11}-\rho-i\omega}{a_{12}} \end{bmatrix},$$

where

$$\rho = -\frac{1}{2}(a_{11} + a_{22}) \text{ and } \omega = \frac{1}{2}\sqrt{4(a_{12}a_{21} + a_{11}a_{22}) - (a_{11} + a_{22})^2}.$$

We define

$$P = (Re(v)), -Im(v)) = \begin{bmatrix} 1 & 0 \\ \frac{-a_{11}-\rho}{a_{12}} & \frac{\omega}{a_{12}} \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then in terms of variables y_1 and y_2, the system (8.1) becomes

$$\frac{dy_1}{dt} = \rho y_1 - \omega y_2 + \Phi(y_1, y_2; \beta), \tag{8.2a}$$

$$\frac{dy_2}{dt} = \omega y_1 + \rho y_2 + \Psi(y_1, y_2; \beta), \tag{8.2b}$$

where

$$\begin{aligned} \Phi(y_1, y_2; \beta) &= \frac{1}{2}\Omega_{20}y_1^2 + \Omega_{11}y_1\phi(y_1, y_2) + \frac{1}{2}\Omega_{02}\phi^2(y_1, y_2) + \frac{1}{2}\Omega_{21}y_1^2\phi(y_1, y_2) \\ &+ \frac{1}{2}\Omega_{21}y_1\phi^2(y_1, y_2) + \frac{1}{6}\Omega_{30}y_1^3 + \frac{1}{6}\Omega_{03}\phi^3(y_1, y_2) + h.o.t., \end{aligned} \tag{8.3a}$$

$$\begin{aligned} \Psi(y_1, y_2; \beta) &= \frac{a_{12}}{\omega}\Big[\frac{1}{2}\Gamma_{20}y_1^2 + \Gamma_{11}y_1\phi(y_1, y_2) + \frac{1}{2}\Gamma_{02}\phi^2(y_1, y_2) + \frac{1}{6}\Gamma_{30}y_1^3 \\ &+ \frac{1}{2}\Gamma_{21}y_1^2\phi(y_1, y_2) + \frac{1}{2}\Gamma_{21}y_1\phi^2(y_1, y_2) + \frac{1}{6}\Gamma_{03}\phi^3(y_1, y_2)\Big] \\ &+ \frac{a_{11}+\rho}{\omega}\Big[\Omega_{20}y_1^2 + \Omega_{11}y_1\phi(y_1, y_2) + \frac{1}{2}\Omega_{02}\phi^2(y_1, y_2) + \frac{1}{6}\Omega_{30}y_1^3 \\ &+ \frac{1}{2}\Omega_{21}y_1^2\phi(y_1, y_2) + \frac{1}{2}\Omega_{21}y_1\phi^2(y_1, y_2) + \frac{1}{6}\Omega_{03}\phi^3(y_1, y_2)\Big] + h.o.t. \end{aligned} \tag{8.3b}$$

$$\text{and } \phi(y_1, y_2) = -\frac{a_{11}+\rho}{a_{12}}y_1 + \frac{\omega}{a_{12}}y_2. \tag{8.4}$$

Here h.o.t. stands for higher order terms. As the system (2.2) undergoes Hopf bifurcation at the positive equilibrium $E^*(x^*, y^*)$ at $\beta = \beta^*$, therefore, in the above system

(8.2),

$$\rho = 0,\ \omega = \omega_0 = \sqrt{R} = \sqrt{\beta x^* \left\{1 + \frac{\alpha(\delta - \gamma)}{(x^* + \delta)^2}\right\}}. \tag{8.5}$$

Then the system (8.2) reduces to

$$\frac{dy_1}{dt} = -\omega_0 y_2 + \Phi_0(y_1, y_2; \beta), \tag{8.6a}$$

$$\frac{dy_2}{dt} = \omega y_1 + \Psi_0(y_1, y_2; \beta), \tag{8.6b}$$

where

$$\begin{aligned}
\Phi_0(y_1, y_2) &= \frac{1}{2}\Omega_{20} y_1^2 + \Omega_{11} y_1 \phi_0(y_1, y_2) + \frac{1}{2}\Omega_{02}\phi_0^2(y_1, y_2) + \frac{1}{6}\Omega_{30} y_1^3 \\
&\quad + \frac{1}{2}\Omega_{21} y_1^2 \phi_0(y_1, y_2) + \frac{1}{2}\Omega_{21} y_1 \phi_0^2(y_1, y_2) + \frac{1}{6}\Omega_{03}\phi^3(y_1, y_2) + h.o.t., \\
\Psi_0(y_1, y_2) &= \frac{a_{12}}{\omega_0}\Big[\frac{1}{2}\Gamma_{20} y_1^2 + \Gamma_{11} y_1 \phi_0(y_1, y_2) + \frac{1}{6}\Gamma_{30} y_1^3 + \frac{1}{2}\Gamma_{02}\phi_0^2(y_1, y_2) \\
&\quad + \frac{1}{2}\Gamma_{21} y_1^2 \phi_0(y_1, y_2) + \frac{1}{2}\Gamma_{21} y_1 \phi_0^2(y_1, y_2) + \frac{1}{6}\Gamma_{03}\phi_0^3(y_1, y_2)\Big] \\
&\quad + \frac{a_{11}}{\omega_0}\Big[\Omega_{11} y_1 \phi_0(y_1, y_2) + \frac{1}{2}\Omega_{02}\phi_0^2(y_1, y_2) + \Omega_{20} y_1^2 + \frac{1}{6}\Omega_{30} y_1^3 \\
&\quad + \frac{1}{2}\Omega_{21} y_1^2 \phi_0(y_1, y_2) + \frac{1}{2}\Omega_{21} y_1 \phi_0^2(y_1, y_2) + \frac{1}{6}\Omega_{03}\phi_0^3(y_1, y_2)\Big] + h.o.t., \\
\phi_0(y_1, y_2) &= \xi y_1 + \eta y_2,\ \xi = -\frac{a_{11}}{a_{12}},\ \eta = \frac{\omega_0}{a_{12}}.
\end{aligned}$$

Now we calculate the following quantities at

$$\beta = \beta^* \text{ and } (y_1, y_2) = (0, 0).$$

$$\begin{aligned}
g_{11} &= \frac{1}{4}\Big[\frac{\partial^2 \Phi_0}{\partial y_1^2} + \frac{\partial^2 \Phi_0}{\partial y_2^2} + i\Big(\frac{\partial^2 \Psi_0}{\partial y_1^2} + \frac{\partial^2 \Psi_0}{\partial y_2^2}\Big)\Big] \\
&= \frac{1}{4}\Bigg[\Omega_{20} + 2\xi\Omega_{11} + (\xi^2 + \eta^2)\Omega_{02} + i\Bigg(\frac{a_{12}}{\omega_0}\Big(\Gamma_{20} + 2\xi\Gamma_{11} + (\xi^2 + \eta^2)\Gamma_{02}\Big) \\
&\quad + \frac{a_{11}}{\omega_0}\Big(\Omega_{20} + 2\xi\Omega_{11} + (\xi^2 + \eta^2)\Omega_{02}\Big)\Bigg)\Bigg],
\end{aligned} \tag{8.7a}$$

$$\begin{aligned}
g_{02} &= \frac{1}{4}\Big[\frac{\partial^2 \Phi_0}{\partial y_1^2} - \frac{\partial^2 \Phi_0}{\partial y_2^2} - 2\frac{\partial^2 \Psi_0}{\partial y_1 \partial y_2} + i\Big(\frac{\partial^2 \Psi_0}{\partial y_1^2} - \frac{\partial^2 \Psi_0}{\partial y_2^2} + 2\frac{\partial^2 \Phi_0}{\partial y_1 \partial y_2}\Big)\Big] \\
&= \frac{1}{4}\Bigg[\Omega_{20} + 2\xi\Omega_{11} + (\xi^2 - \eta^2)\Omega_{02} - 2\eta\frac{a_{12}}{\omega_0}(\Gamma_{11} + \xi\Gamma_{02}) - 2\eta\frac{a_{11}}{\omega_0}(\Omega_{11} + \xi\Omega_{02})
\end{aligned}$$

$$+ i\Bigg(\frac{a_{12}}{\omega_0}\Big(\Gamma_{20} + 2\xi\Gamma_{11} + (\xi^2 - \eta^2)\Gamma_{02}\Big) + \frac{a_{11}}{\omega_0}\Big(\Omega_{20} + 2\xi\Omega_{11} + (\xi^2 - \eta^2)\Omega_{02}\Big)$$

$$+ 2\eta(\Omega_{11} + \xi\Omega_{02})\Bigg)\Bigg], \tag{8.7b}$$

$$g_{20} = \frac{1}{4}\Big[\frac{\partial^2\Phi_0}{\partial y_1^2} - \frac{\partial^2\Phi_0}{\partial y_2^2} + 2\frac{\partial^2\Psi_0}{\partial y_1\partial y_2} + i\Big(\frac{\partial^2\Psi_0}{\partial y_1^2} - \frac{\partial^2\Psi_0}{\partial y_2^2} - 2\frac{\partial^2\Phi_0}{\partial y_1\partial y_2}\Big)\Big]$$

$$= \frac{1}{4}\Bigg[\Omega_{20} + 2\xi\Omega_{11} + (\xi^2 - \eta^2)\Omega_{02} + 2\eta\frac{a_{12}}{\omega_0}(\Gamma_{11} + \xi\Gamma_{02}) + 2\eta\frac{a_{11}}{\omega_0}(\Omega_{11} + \xi\Omega_{02})$$

$$+ i\Bigg(\frac{a_{12}}{\omega_0}\Big(\Gamma_{20} + 2\xi\Gamma_{11} + (\xi^2 - \eta^2)\Gamma_{02}\Big) + \frac{a_{11}}{\omega_0}\Big(\Omega_{20} + 2\xi\Omega_{11} + (\xi^2 - \eta^2)\Omega_{02}\Big)$$

$$- 2\eta(\Omega_{11} + \xi\Omega_{02})\Bigg)\Bigg], \tag{8.7c}$$

$$g_{21} = \frac{1}{8}\Big[\frac{\partial^3\Phi_0}{\partial y_1^3} + \frac{\partial^3\Phi_0}{\partial y_1\partial y_2^2} + \frac{\partial^3\Psi_0}{\partial y_1^2\partial y_2} + \frac{\partial^3\Psi_0}{\partial y_2^3} + i\Big(\frac{\partial^3\Psi_0}{\partial y_1^3} + \frac{\partial^3\Psi_0}{\partial y_1\partial y_2^2} - \frac{\partial^3\Phi_0}{\partial y_1^2\partial y_2} - \frac{\partial^3\Phi_0}{\partial y_2^3}\Big)\Big]$$

$$= \frac{1}{8}\Bigg[\Omega_{30} + 3\xi\Omega_{21} + (3\xi^2 + \eta^2)\Omega_{12} + \xi^2\Omega_{03}(\xi + \eta) + \eta\frac{a_{12}}{\omega_0}\Big(\Gamma_{21} + 2\xi\Gamma_{12}$$

$$+ (\xi^2 + \eta^2)\Gamma_{03}\Big) + \eta\frac{a_{11}}{\omega_0}\Big(\Omega_{21} + 2\xi\Omega_{12} + (\xi^2 + \eta^2)\Omega_{03}\Big) + i\Bigg\{\frac{a_{12}}{\omega_0}\Big(\Gamma_{30} + 3\xi\Gamma_{21}$$

$$+ (3\xi^2 + \eta^2)\Gamma_{12} + \xi^2(\xi + \eta)\Gamma_{03}\Big) + \eta\frac{a_{11}}{\omega_0}\Big(\Omega_{30} + 3\xi\Omega_{21} + (3\xi^2 + \eta^2)\Omega_{12}$$

$$+ \xi^2(\xi + \eta)\Omega_{03}\Big) - \eta\Big(\Omega_{21} + 2\xi\Omega_{12} + (\xi^2 + \eta^2)\Omega_{03}\Big)\Bigg\}\Bigg]. \tag{8.7d}$$

Again

$$\rho'(\beta) = -\frac{1}{2}, \tag{8.8a}$$

$$\omega'(\beta) = \frac{1}{4}\frac{\left(4\frac{\alpha x^*}{x^*+\delta} - 2(\beta^* + \beta)\right)}{\sqrt{4\frac{\alpha x^*\beta}{x^*+\delta} - 4\beta\beta^* - (\beta - \beta^*)^2}}, \tag{8.8b}$$

$$\text{where} \quad \beta^* = -x^* + \frac{\alpha x^* y^*}{(x^* + \delta)^2}.$$

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