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CONVERGENCE OF MULTI-RELAXED NONSTATIONARY MULTISPLITTING METHODS†

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ABSTRACT. Recently, Cheng et al. [3] introduced new nonstationary multisplitting methods with multi-relaxed parameters. In this paper, we first provide correct proofs for convergence results of the multi-relaxed nonstationary multisplitting method which have not been proved completely by Cheng et al., and then we provide new convergence results for the multi-relaxed nonstationary two-stage multisplitting method.

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1. Introduction

In this paper, we consider the nonstationary multisplitting methods with multi-relaxed parameters which were recently introduced by Cheng et al. [3] for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is a large sparse H-matrix. Multisplitting method was introduced by O'Leary and White [11] and was further studied by many authors [2, 4, 6, 9, 10, 12, 13, 14].

A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. A collection of triples (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, is called a *multisplitting* of A if $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's, called weighting matrices, are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$. The *multi-relaxed nonstationary multisplitting method* associated with this multisplitting and positive relaxation parameters $\omega, \omega_1, \ldots, \omega_\ell$ for solving a linear system $Ax = b$ is as follows.

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Algorithm 1: Multi-relaxed nonstationary multisplitting method
Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ
    $y_{k,0} = x_{i-1}$
    For $j = 1$ to $s(k,i)$
      $y_{k,j} = \omega_k M_k^{-1} N_k y_{k,j-1} + (1-\omega_k) y_{k,j-1} + \omega_k M_k^{-1} b$
  $x_i = \omega \sum_{k=1}^{\ell} E_k y_{k,s(k,i)} + (1-\omega) x_{i-1}$

Notice that Algorithm 1 with $\omega_k = 1$ for $k = 1, 2, \ldots, \ell$ is called the *relaxed nonstationary multisplitting method.* When (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, is a multisplitting of A and $M_k = B_k - C_k$ is a splitting of M_k for each k, the *multi-relaxed nonstationary two-stage multisplitting method* with positive relaxation parameters $\omega, \omega_1, \ldots, \omega_\ell$ for solving a linear system $Ax = b$ is as follows.

Algorithm 2: Multi-relaxed nonstationary two-stage multisplitting method
Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ
    $y_{k,0} = x_{i-1}$
    For $j = 1$ to $s(k,i)$
      $y_{k,j} = \omega_k B_k^{-1} (C_k y_{k,j-1} + N_k x_{i-1} + b) + (1-\omega_k) y_{k,j-1}$
  $x_i = \omega \sum_{k=1}^{\ell} E_k y_{k,s(k,i)} + (1-\omega) x_{i-1}$

Notice that Algorithm 2 with $\omega_1 = \cdots = \omega_\ell$ and $\omega = 1$ is called the *relaxed nonstationary two-stage multisplitting method.* Also notice that the number of inner iterations $s(k,i)$ in Algorithms 1 and 2 depends on the iteration i and the splitting $A = M_k - N_k$. Throughout the paper, it is assumed that $s(k,i) \geq 1$ for every k and i.

Cheng et al. [3] provided convergence results for both Algorithm 1 and Algorithm 2, and they showed the effectiveness of the preconditioners obtained from these methods. However, their convergence results for the multi-relaxed nonstationary multisplitting method (Algorithm 1) have not been proved completely, which is a main motivation of this paper. This paper is organized as follows. In Section 2, we present some notation, definitions and preliminary results which we refer to later. In Section 3, we first provide correct proofs for convergence results of the multi-relaxed nonstationary multisplitting method (Algorithm 1) which have not been proved completely by Cheng et al. [3], and then we provide new convergence results for the multi-relaxed nonstationary two-stage multisplitting method (Algorithm 2) for solving the linear system (1).

2. Preliminaries

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. These definitions carry immediately over to matrices. For a square matrix B, $\mathrm{diag}(B)$ denotes a diagonal matrix whose diagonal part coincides with the diagonal part of B, and $\rho(B)$ denotes the *spectral radius* of the matrix B.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j \end{cases}.$$

A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix. A splitting $A = M - N$ is called an *H-compatible splitting* if $\langle A\rangle = \langle M\rangle - |N|$. It was shown in [5] that if A is an H-matrix and $A = M - N$ is an H-compatible splitting, then M is also an H-matrix.

Lemma 2.1 ([4])**.** *Let $A = D - B$ be an H-matrix with $D = diag(A)$. Then*

(a) *A and $|D|$ are nonsingular and $\rho(|D|^{-1}|B|) < 1$.*
(b) $|A^{-1}| \leq \langle A\rangle^{-1}$.

Lemma 2.2 ([1])**.** *Let T_i, $i = 1, 2, \ldots$, be a sequence of square matrices. If there exists a matrix norm $\|\cdot\|$ and a $\theta < 1$ such that $\|T_i\| \leq \theta$ for all $i = 1, 2, \ldots$, then*

$$\lim_{i\to\infty} T_i T_{i-1} \cdots T_1 = 0.$$

For a vector $v > 0$, the *weighted max norm* $\|x\|_v$ is defined by

$$\|x\|_v = \inf\{\beta > 0 \colon -\beta v \leq x \leq \beta v\}.$$

For a matrix B, $\|B\|_v$ denotes the matrix norm of B corresponding to the weighted max norm defined above. It is well-known that $\|B\|_v = \||B|v\|_v$ and $|x| \leq |y|$ implies $\|x\|_v \leq \|y\|_v$.

A general algorithm for building ILU factorization can be derived by performing Gaussian elimination and dropping some of elements in predetermined off-diagonal positions. Let S_n denote the set of all pairs of indices of off-diagonal matrix entries, that is,

$$S_n = \{(i, j) \mid i \neq j,\ 1 \leq i \leq n,\ 1 \leq j \leq n\}.$$

The following theorem shows the existence of the ILU factorization for an H-matrix A.

Theorem 2.3 ([8])**.** *Let A be an $n \times n$ H-matrix. Then, for every zero pattern set $Q \subset S_n$, there exist a unit lower triangular matrix $L = (l_{ij})$, an upper triangular matrix $U = (u_{ij})$, and a matrix $N = (n_{ij})$, with $l_{ij} = u_{ij} = 0$ if*

$(i,j) \in Q$ and $n_{ij} = 0$ if $(i,j) \notin Q$, such that $A = LU - N$. Moreover, the factors L and U are also H-matrices.

In Theorem 2.3, $A = LU - N$ is called an *ILU factorization* of A corresponding to a zero pattern set $Q \subset S_n$. The following theorem shows the relations between the ILU factorizations of an H-matrix A and $\langle A \rangle$.

Theorem 2.4 ([7, 8]). *Let A be an $n \times n$ H-matrix. Let $A = LU - N$ and $\langle A \rangle = \tilde{L}\tilde{U} - \tilde{N}$ be the ILU factorizations of A and $\langle A \rangle$ corresponding to a zero pattern set $Q \subset S_n$, respectively. Then each of the following holds:*

(a) $|L^{-1}| \le \tilde{L}^{-1}$, (b) $|U^{-1}| \le \tilde{U}^{-1}$, (c) $|N| \le \tilde{N}$.

3. Convergence of the multi-relaxed nonstationary multisplitting methods

First, we consider convergence of the multi-relaxed nonstationary multisplitting method (Algorithm 1), which can be written as

$$x_i = H_{\omega,i} x_{i-1} + P_{\omega,i} b, \ i = 1, 2, \ldots, \tag{2}$$

where

$$H_{\omega,i} = \omega \sum_{k=1}^{\ell} E_k R_k^{s(k,i)} + (1-\omega) I,$$

$$P_{\omega,i} = \omega \sum_{k=1}^{\ell} \omega_k E_k \left(\sum_{j=0}^{s(k,i)-1} R_k^j \right) M_k^{-1},$$

and

$$R_k = \omega_k M_k^{-1} N_k + (1-\omega_k) I.$$

The $H_{\omega,i}$'s are called iteration matrices for Algorithm 1. Then, it is easy to show that $P_{\omega,i} A = I - H_{\omega,i}$ for each i. Hence, the exact solution ξ of $Ax = b$ satisfies

$$\xi = H_{\omega,i} \xi + P_{\omega,i} b, \ i = 1, 2, \ldots. \tag{3}$$

From (2) and (3), the error vector $e_i = x_i - \xi$ satisfies

$$e_i = H_{\omega,i} e_{i-1} = H_{\omega,i} H_{\omega,i-1} \cdots H_{\omega,1} e_0, \ i = 1, 2, \ldots. \tag{4}$$

From (4), the sequence of vectors generated by the iteration (2) converges to the exact solution of $Ax = b$ for any initial vector x_0 if and only if

$$\lim_{i \to \infty} H_{\omega,i} H_{\omega,i-1} \cdots H_{\omega,1} = 0. \tag{5}$$

Cheng et al. [3] provided new convergence results for Algorithm 1, but their convergence proofs were done by showing $\rho(H_{\omega,i}) < 1$ for each i instead of showing the relation (5). The following example shows that their proofs are not correct.

Example 3.1. *For each* $i = 1, 2, \ldots$, *let*

$$H_{2i} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{8} & \frac{1}{2} \end{pmatrix}, \quad H_{2i-1} = \begin{pmatrix} 0 & 0 \\ 2 & \frac{1}{2} \end{pmatrix}.$$

Then, $\rho(H_{2i}) \approx 0.8536$ *and* $\rho(H_{2i-1}) = \frac{1}{2}$. *Thus* $\rho(H_i) < 1$ *for each* i. *Since*

$$H_{2i}H_{2i-1} = \begin{pmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{4} \end{pmatrix},$$

$\rho(H_{2i}H_{2i-1}) = \frac{9}{4} > 1$ *for each* i. *It follows that* $\lim_{i\to\infty} H_iH_{i-1}\cdots H_1 \neq 0$. *Hence,* $\rho(H_i) < 1$ *for each* i *does not imply that* $\lim_{i\to\infty} H_iH_{i-1}\cdots H_1 = 0$.

Theorems 3.1 and 3.3 in [3] have not been proved completely, so we provide correct proofs for these theorems which are slightly modified in what follows.

Theorem 3.2. *Let* $A = D - B$ *be an* $n \times n$ *H-matrix with* $D = diag(A)$, *and let* $J = |D|^{-1}|B|$. *For each* $k = 1, 2, \ldots, \ell$, *let* $A = M_k - N_k$ *be an H-compatible splitting of* A *with* $diag(|M_k|) \leq |D|$. *Then, the multi-relaxed nonstationary multisplitting method associated with the multisplitting* (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, *converges to the exact solution of* $Ax = b$ *for any initial vector* x_0 *if* $0 < \omega_k < \frac{2}{1+\rho}$ *for* $k = 1, 2, \ldots, \ell$ *and* $0 < \omega < \frac{2}{1+\alpha}$, *where* $\rho = \rho(J)$ *and* $\alpha = \max\{\omega_k\rho + |1-\omega_k| \mid 1 \leq k \leq \ell\}$.

Proof. From Lemma 2.2, it suffices to show that there exists a matrix norm $\|\cdot\|$ and a $\theta < 1$ such that $\|H_{\omega,i}\| \leq \theta$ for all $i = 1, 2, \ldots$. Let $e = (1, 1, \ldots, 1)^T$. Since $J \geq 0$, $J + \epsilon ee^T > 0$ for any $\epsilon > 0$ and thus there exists a Perron vector $x_\epsilon > 0$ such that

$$(J + \epsilon ee^T)x_\epsilon = \rho_\epsilon x_\epsilon, \tag{6}$$

where $\rho_\epsilon = \rho(J + \epsilon ee^T)$. Since $\rho < 1$ from Lemma 2.1 and $0 < \omega_k < \frac{2}{1+\rho}$ from the assumption, $\omega_k\rho + |1-\omega_k| < 1$ and thus $\alpha < 1$. By continuity of the spectral radius, for sufficiently small $\epsilon > 0$

$$\rho_\epsilon < 1 \text{ and } \omega_k\rho_\epsilon + |1-\omega_k| < 1.$$

Since $A = M_k - N_k$ is an H-compatible splitting of A and $\langle A\rangle = |D|(I - J)$, one obtains

$$\begin{aligned} |R_k| &\leq \omega_k\langle M_k\rangle^{-1}|N_k| + |1-\omega_k|I \\ &\leq \omega_k\left(I - \langle M_k\rangle^{-1}|D|(I - (J + \epsilon ee^T))\right) + |1-\omega_k|I. \end{aligned} \tag{7}$$

Since $\text{diag}(|M_k|) \leq |D|$, $I \leq \langle M_k\rangle^{-1}|D|$. Using this fact and (7), one obtains

$$\begin{aligned} |R_k|x_\epsilon &\leq \omega_k\left(x_\epsilon - (1-\rho_\epsilon)\langle M_k\rangle^{-1}|D|x_\epsilon\right) + |1-\omega_k|x_\epsilon \\ &\leq (\omega_k\rho_\epsilon + |1-\omega_k|)x_\epsilon \\ &\leq \alpha_\epsilon x_\epsilon, \end{aligned} \tag{8}$$

where $\alpha_\epsilon = \max\{\omega_k \rho_\epsilon + |1-\omega_k| \mid 1 \le k \le \ell\} < 1$. Using (8), one obtains

$$\begin{aligned} |H_{\omega,i}| x_\epsilon &\le \left(\omega \sum_{k=1}^{\ell} E_k |R_k|^{s(k,i)} + |1-\omega| I\right) x_\epsilon \\ &\le (\omega \alpha_\epsilon + |1-\omega|) x_\epsilon. \end{aligned} \tag{9}$$

Since $0 < \omega < \frac{2}{1+\alpha}$ and $\alpha < 1$, $\omega\alpha + |1-\omega| < 1$. By continuity of the spectral radius, $\lim\limits_{\epsilon \to 0+} \alpha_\epsilon = \alpha$ and thus $\omega\alpha_\epsilon + |1-\omega| < 1$ for sufficiently small $\epsilon > 0$. Taking the weighted max norm $\|\cdot\|_{x_\epsilon}$ to both sides of equation (9),

$$\|H_{\omega,i}\|_{x_\epsilon} = \| |H_{\omega,i}| x_\epsilon \|_{x_\epsilon} \le \omega\alpha_\epsilon + |1-\omega| \equiv \theta_\epsilon.$$

Since i is arbitrary, $\|H_{\omega,i}\|_{x_\epsilon} \le \theta_\epsilon < 1$ for all $i = 1, 2, \ldots$. Therefore, the proof is complete. □

Theorem 3.3. *Let $A = D - B$ be an $n \times n$ H-matrix with $D = diag(A)$. Let $J = |D|^{-1}|B|$ and let $Q_1, Q_2, \ldots, Q_\ell$ be zero pattern sets which are subsets of S_n. For each $1 \le k \le \ell$, let $A = L_k U_k - N_k$ be the ILU factorization of A corresponding to Q_k. Then, the multi-relaxed nonstationary multisplitting method associated with the multisplitting $(L_k U_k, N_k, E_k)$, $k = 1, 2, \ldots, \ell$, converges to the exact solution of $Ax = b$ for any initial vector x_0 if $0 < \omega_k < \frac{2}{1+\rho}$ for $k = 1, 2, \ldots, \ell$ and $0 < \omega < \frac{2}{1+\alpha}$, where $\rho = \rho(J)$ and $\alpha = \max\{\omega_k \rho + |1-\omega_k| \mid 1 \le k \le \ell\}$.*

Proof. For each $1 \le k \le \ell$, let $\langle A \rangle = \tilde{L}_k \tilde{U}_k - \tilde{N}_k$ be the ILU factorization of $\langle A \rangle$ corresponding to Q_k. By some manipulation, it can be shown that $|D^{-1}| \le (\tilde{L}_k \tilde{U}_k)^{-1}$ for all $k = 1, 2, \ldots, \ell$. It follows that for all $k = 1, 2, \ldots, \ell$

$$I \le (\tilde{L}_k \tilde{U}_k)^{-1} |D|. \tag{10}$$

Using (10) and Theorem 2.4, this theorem can be proved in a similar way as was done for Theorem 3.2. □

Next, we consider convergence of the multi-relaxed nonstationary two-stage multisplitting method (Algorithm 2), which can be written as

$$x_i = H^*_{\omega,i} x_{i-1} + P^*_{\omega,i} b, \ i = 1, 2, \ldots, \tag{11}$$

where

$$H^*_{\omega,i} = \omega \sum_{k=1}^{\ell} E_k \left((R^*_k)^{s(k,i)} + \omega_k \left(\sum_{j=0}^{s(k,i)-1} (R^*_k)^j \right) B_k^{-1} N_k \right) + (1-\omega) I,$$

$$P^*_{\omega,i} = \omega \sum_{k=1}^{\ell} \omega_k E_k \left(\sum_{j=0}^{s(k,i)-1} (R^*_k)^j \right) B_k^{-1},$$

and

$$R^*_k = \omega_k B_k^{-1} C_k + (1-\omega_k) I.$$

The $H^*_{\omega,i}$'s are called iteration matrices for Algorithm 2. It is easy to show that $P^*_{\omega,i} A = I - H^*_{\omega,i}$ for each i and the sequence of vectors generated by the iteration (11) converges to the exact solution of $Ax = b$ for any initial vector x_0 if and only if

$$\lim_{i\to\infty} H^*_{\omega,i} H^*_{\omega,i-1} \cdots H^*_{\omega,1} = 0. \tag{12}$$

Lemma 3.4. *Let A be an $n\times n$ H-matrix. For each $1 \le k \le \ell$, let $A = M_k - N_k$ be an H-compatible splitting of A and $M_k = B_k - C_k$ be an H-compatible splitting of M_k. Let $R^*_k = \omega_k B_k^{-1} C_k + (1-\omega_k)I$ and $H^*_{i,k} = (R^*_k)^{s(k,i)} + \omega_k \left(\sum_{j=0}^{s(k,i)-1} (R^*_k)^j\right) B_k^{-1} N_k$ for $k = 1,2,\ldots,\ell$ and $i = 1,2,\ldots$. If $\omega_k \in (0,1]$, then there exists a $\theta_k \in [0,1)$ such that $|H^*_{i,k}|v \le \theta_k v$ for all i, where $v = \langle A\rangle^{-1} e > 0$ and $e = (1,1,\ldots,1)^T$.*

Proof. Let

$$\tilde{R}_k = \omega_k \langle B_k\rangle^{-1} |C_k| + (1-\omega_k)I,$$

$$\tilde{H}_{i,k} = (\tilde{R}_k)^{s(k,i)} + \omega_k \left(\sum_{j=0}^{s(k,i)-1} (\tilde{R}_k)^j\right) \langle B_k\rangle^{-1} |N_k|.$$

Since $A = M_k - N_k$ and $M_k = B_k - C_k$ are H-compatible splittings and $0 < \omega_k \le 1$, it can be easily shown that

$$|R^*_k| \le \tilde{R}_k \text{ and thus } |H^*_{i,k}| \le \tilde{H}_{i,k}. \tag{13}$$

Since $I - \tilde{R}_k = \omega_k \langle B_k\rangle^{-1} \langle M_k\rangle$ and $\langle A\rangle = \langle M_k\rangle - |N_k|$, one obtains

$$\tilde{H}_{i,k} = I - \omega_k \left(\sum_{j=0}^{s(k,i)-1} (\tilde{R}_k)^j\right) \langle B_k\rangle^{-1} \langle A\rangle. \tag{14}$$

Since $v > 0$ and $\langle B_k\rangle^{-1} e > 0$, from (13) and (14) one obtains

$$\begin{aligned} |H^*_{i,k}|v \le \tilde{H}_{i,k} v &= v - \omega_k \langle B_k\rangle^{-1} e - \cdots \\ &\le v - \omega_k \langle B_k\rangle^{-1} e < v \end{aligned} \tag{15}$$

for all i. From relation (15), there exists a $0 \le \theta_k < 1$ such that $|H^*_{i,k}|v \le \theta_k v$ for all i. Therefore, the proof is complete. □

Theorem 3.5. *Let A be an $n\times n$ H-matrix. For each $1 \le k \le \ell$, let $A = M_k - N_k$ be an H-compatible splitting of A and $M_k = B_k - C_k$ be an H-compatible splitting of M_k. Then, the multi-relaxed nonstationary two-stage multisplitting method with $A = M_k - N_k$ as outer splittings and $M_k = B_k - C_k$ as inner splittings converges to the exact solution of $Ax = b$ for any initial vector x_0 if $0 < \omega_k \le 1$ for $k = 1,2,\ldots,\ell$ and $0 < \omega < \frac{2}{1+\theta}$, where $\theta = \max\{\theta_k \mid 1 \le k \le \ell\}$ and θ_k is defined as in Lemma 3.4.*

Proof. Let $H^*_{i,k}$ be defined as in Lemma 3.4. Then $H^*_{\omega,i} = \omega \sum_{k=1}^{\ell} E_k H^*_{i,k} + (1-\omega)I$. Using this equation and Lemma 3.4, one obtains

$$\begin{aligned} |H^*_{\omega,i}|v &\le \omega \sum_{k=1}^{\ell} E_k |H^*_{i,k}|v + |1-\omega|v \\ &\le \omega \sum_{k=1}^{\ell} E_k \theta_k v + |1-\omega|v \\ &\le (\omega\theta + |1-\omega|)v. \end{aligned} \tag{16}$$

From (16), $\|H^*_{\omega,i}\|_v \le (\omega\theta + |1-\omega|)$ for all $i = 1, 2, \ldots$. Since $0 < \omega < \frac{2}{1+\theta}$, $(\omega\theta + |1-\omega|) < 1$. Hence, Lemma 2.2 implies (12), which completes the proof. □

Lemma 3.6. *Let A be an $n \times n$ H-matrix. Let $Q_1, Q_2, \ldots, Q_\ell$ be zero pattern sets which are subsets of S_n. For each $1 \le k \le \ell$, let $A = M_k - N_k$ be an H-compatible splitting and $M_k = L_k U_k - C_k$ be the ILU factorization of M_k corresponding to Q_k. Let $R^*_k = \omega_k (L_k U_k)^{-1} C_k + (1-\omega_k)I$ and $H^*_{i,k} = (R^*_k)^{s(k,i)} + \omega_k \left(\sum_{j=0}^{s(k,i)-1} (R^*_k)^j \right) (L_k U_k)^{-1} N_k$ for $k = 1, 2, \ldots, \ell$ and $i = 1, 2, \ldots$. If $\omega_k \in (0, 1]$, then there exists a $\theta_k \in [0, 1)$ such that $|H^*_{i,k}|v \le \theta_k v$ for all i, where $v = \langle A \rangle^{-1} e > 0$ and $e = (1, 1, \ldots, 1)^T$.*

Proof. For each $1 \le k \le \ell$, let $\langle M_k \rangle = \tilde{L}_k \tilde{U}_k - \tilde{C}_k$ be the ILU factorization of $\langle M_k \rangle$ corresponding to Q_k. Let

$$\tilde{R}_k = \omega_k (\tilde{L}_k \tilde{U}_k)^{-1} \tilde{C}_k + (1-\omega_k)I,$$

$$\tilde{H}_{i,k} = (\tilde{R}_k)^{s(k,i)} + \omega_k \left(\sum_{j=0}^{s(k,i)-1} (\tilde{R}_k)^j \right) (\tilde{L}_k \tilde{U}_k)^{-1} |N_k|.$$

Using Theorem 2.4, one can easily obtain

$$|R^*_k| \le \tilde{R}_k \text{ and thus } |H^*_{i,k}| \le \tilde{H}_{i,k}. \tag{17}$$

The remaining part of the proof can be done in the similar way as was done for Lemma 3.4. □

Theorem 3.7. *Let A be an $n \times n$ H-matrix. Let $Q_1, Q_2, \ldots, Q_\ell$ be zero pattern sets which are subsets of S_n. For each $1 \le k \le \ell$, let $A = M_k - N_k$ be an H-compatible splitting and $M_k = L_k U_k - C_k$ be the ILU factorization of M_k corresponding to Q_k. Then, the multi-relaxed nonstationary two-stage multisplitting method with $A = M_k - N_k$ as outer splittings and $M_k = L_k U_k - C_k$ as inner splittings converges to the exact solution of $Ax = b$ for any initial vector x_0 if $0 < \omega_k \le 1$ for $k = 1, 2, \ldots, \ell$ and $0 < \omega < \frac{2}{1+\theta}$, where $\theta = \max\{\theta_k \mid 1 \le k \le \ell\}$ and θ_k is defined as in Lemma 3.6.*

Proof. Using Lemma 3.6, this theorem can be proved as in Theorem 3.5. □

In Theorems 3.5 and 3.7, it can be seen that $\theta < 1$ and thus the upper bound of ω is greater than 1 even if it is difficult to estimate θ a priori. Notice that Theorems 3.5 and 3.7 are convergence results which are different from Theorem 3.5 in [3].

In Algorithm 2, if $s(k,i) = s(k)$ for all i, then one obtains the *multi-relaxed two-stage multisplitting method*, which can be written as

$$x_i = H^*_\omega x_{i-1} + P^*_\omega b,\ \ i = 1, 2, \ldots, \tag{18}$$

where

$$H^*_\omega = \omega \sum_{k=1}^{\ell} E_k \left((R^*_k)^{s(k)} + \omega_k \left(\sum_{j=0}^{s(k)-1} (R^*_k)^j \right) B_k^{-1} N_k \right) + (1-\omega)I,$$

$$P^*_\omega = \omega \sum_{k=1}^{\ell} \omega_k E_k \left(\sum_{j=0}^{s(k)-1} (R^*_k)^j \right) B_k^{-1},$$

and

$$R^*_k = \omega_k B_k^{-1} C_k + (1-\omega_k)I.$$

Theorem 3.8. *Let A be an $n \times n$ H-matrix. For each $1 \le k \le \ell$, let $A = M_k - N_k$ be an H-compatible splitting of A and $M_k = B_k - C_k$ be an H-compatible splitting of M_k. Then, the multi-relaxed two-stage multisplitting method with $A = M_k - N_k$ as outer splittings and $M_k = B_k - C_k$ as inner splittings converges to the exact solution of $Ax = b$ for any initial vector x_0 if $0 < \omega_k \le 1$ for $k = 1, 2, \ldots, \ell$ and $0 < \omega < \frac{2}{1+\rho(H^*_1)}$, where H^*_1 denotes H^*_ω with $\omega = 1$.*

Proof. Notice that $H^*_\omega = \omega H^*_1 + (1-\omega)I$. Since $0 < \omega_k \le 1$ for $k = 1, 2, \ldots, \ell$, $\rho(H^*_1) < 1$ can be shown in the similar way as was done for Theorem 3.4 in [2]. Since $0 < \omega < \frac{2}{1+\rho(H^*_1)}$, $\rho(H^*_\omega) < 1$ is obtained. Hence, the proof is complete. □

In Theorems 3.8, notice that the upper bound of ω is greater than 1 since $\rho(H^*_1) < 1$.

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