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COINCIDENCE AND COMMON FIXED POINTS OF NONCOMPATIBLEMAPS

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Abstract. Fixed point theorems for two hybrid pairs of single valued and multivalued noncompatible maps under strict contractive condition are proved, without appeal to continuity of any map involved therein and completeness of underlying space. These results extend, unify and improve the earlier comparable known results.

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1. Introduction and Preliminaries

Sessa [21] introduced the concept of weakly commuting maps. Jungck [11] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps [12]. Afterwards Jungck and Rhoades [13] further extended weak compatibility to the setting of single valued and multi-valued maps. Since then, many interesting coincidence and common fixed point theorems of compatible and weakly compatible maps under various contractive conditions and assuming the continuity of at least one of the mappings, have been obtained by a number of authors. For a survey of coincidence point theory, its applications and related results, we refer to [3], [4], [5], [6],[7], [10], [22] and references contained therein. However, a study of common fixed points of non-compatible mappings is also equally interesting. Pant [19] initiated the study of noncompatible maps satisfying certain contractive conditions. In 2002, Aamri and El Moutawakil [1] defined a property (EA) for single valued maps on a metric space and obtained some common fixed point theorems for such maps under strict contractive conditions. The class of mappings satisfying (EA) property contains compatible as well as noncompatible maps. Kamran [14] extended the property (EA) for a hybrid pair of single valued and multivalued maps. Recently,

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Liu et al. [17] obtained coincidence and common fixed point results for two pairs of hybrid maps defining common (EA) property for such pairs. The aim of this paper is to obtain coincidence points of two hybrid pair of single valued and multivalued maps for which only one pair needs to satisfy (EA) property. These results don't require the continuity of any map, moreover, common fixed points of four maps are obtained under weaker condition than that, given in [17]. Our results include the results in ([5], [11], [14], [15], [17], [18], [20] and [23]) as special cases.

The following definitions and results will be needed in the sequel.

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $CB(X)$ the class of all nonempty bounded and closed subsets of X. Let H be a Hausdorff metric induced by the metric d of X, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

for $A, B \in CB(X)$.

Definition 1.1. *Let $f : X \longrightarrow X$ and $T : X \longrightarrow CB(X)$. A point x in X is said to be:*
(1) *fixed point of f if $f(x) = x$;*
(2) *fixed point of T if $x \in T(x)$;*
(3) *coincidence point of a pair (f, T) if $fx \in Tx$;*
(4) *common fixed point of a pair (f, T) if $x = fx \in Tx$.*
$F(f), C(f, T)$ and $F(f, T)$ denote set of all fixed points of f, set of all coincidence points of the pair (f, T) and the set of all common fixed points of the pair (f, T), respectively.

Definition 1.2. *Maps $f : X \longrightarrow X$, and $T : X \longrightarrow CB(X)$ are said to be:*
(5) *compatible if $fTx \in CB(X)$ for all $x \in X$ and $H(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = t \in \lim_{n\to\infty} Tx_n = A \in CB(X)$.*
(6) *noncompatible if $fTx \in CB(X)$ for all $x \in X$ and there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t \in \lim_{n\to\infty} Tx_n = A \in CB(X)$ but $\lim_{n\to\infty} H(fTx_n, Tfx_n) \neq 0$ or nonexistent.*

Definition 1.3. *Let $f : X \longrightarrow X$, $T : X \longrightarrow CB(X)$, and $fTx \in CB(X)$. The pair (f, T) is called:*
(7) *commuting if $Tfx = fTx$ for all $x \in X$;*
(8) *weakly compatible* [13] *if they commute at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f, T)$;*
(9) *$(IT)-$ commuting at $x \in X$ if $fTx \subseteq Tfx$.*

Definition 1.4. *Let $T : X \longrightarrow CB(X)$. The map $f : X \longrightarrow X$, is said to be $T-$weakly commuting at $x \in X$ if $f^2x \in Tfx$.*

Definition 1.5. *The map $f : X \longrightarrow X$ is said to coincidently idempotent with respect to $T : X \longrightarrow CB(X)$ if $f^2x = fx$ for $x \in C(f,T)$. The point x is called point of coincident idempotency.*

For example, let $X = R$ with usual metric. Define $f : X \to X$, and $T : X \to CB(X)$ by

$$fx = \begin{cases} -1, & x \leq 0 \\ -\dfrac{2}{x}, & x > 0 \end{cases}, \quad Tx = \begin{cases} \{x\}, & x \leq -1 \\ [x,1], & -1 < x \leq 1 \\ [1,x], & 1 < x < \infty \end{cases}.$$

Here, $C(f,T) = \{-1\}$. The map f is coincidently idempotent with respect to T.

Definition 1.6. *Maps $f : X \longrightarrow X$, and $T : X \longrightarrow CB(X)$ are said to satisfy property (EA) if there exists a sequence $\{x_n\}$ in X, some $t \in X$, and $A \in CB(X)$ such that $\lim\limits_{n\to\infty} fx_n = t \in A = \lim\limits_{n\to\infty} Tx_n \in CB(X)$.*

Now we present an example of hybrid pair $\{f,T\}$ which satisfies (EA) property and f is $T-$weakly commuting at some $x \in C(f,T)$.

Example 1.7. *Let $X = [0,\infty)$ with usual metric. Define $f : X \to X$, $T : X \to CB(X)$ by*

$$fx = \begin{cases} 0, & 0 \leq x < 1 \\ x+1, & 1 \leq x < \infty \end{cases} \quad \text{and } Tx = \begin{cases} \{x\}, & 0 \leq x < 1 \\ [1, x+2], & 1 \leq x < \infty \end{cases}$$

It can be easily verified that the pair $\{f,T\}$ satisfies (EA) property and f is $T-$weakly commuting at $x = 0 \in C(f,T) = [0,\infty)$. Moreover, $F(f,T) \neq \phi$.

Example 1.8. *Let $X = R$ with usual metric. Define $f, g : X \to X$, and $T, S : X \to CB(X)$ by*

$$gx = \begin{cases} \dfrac{2}{2-x}, & x < 2 \\ 0, & x \geq 2 \end{cases}, \quad fx = \begin{cases} -1, & x \leq 0 \\ -1 - \dfrac{2}{x}, & x > 0 \end{cases},$$

$$Sx = \begin{cases} \{x\}, & x \leq -1 \\ [1, x+2], & -1 < x < \infty \end{cases} \quad \text{and} \quad Tx = \begin{cases} [1, 1+x], & x > 0 \\ [0, -x], & x < 0 \end{cases}$$

Consider a sequence $\{x_n\} = \{\frac{1}{n}\}$, then $\lim\limits_{n\to\infty} gx_n = 1 \in \lim\limits_{n\to\infty} Sx_n = [1,2]$, the pair $\{g,S\}$ satisfies (EA) property. However, the pair $\{f,T\}$ does not satisfy (EA) property. Moreover, $f(X)$ and $g(X)$ are closed subsets of X.

Lemma 1.9 ([8])**.** *Let $A, B \in CB(X)$, then for any $a \in A$, $d(a,B) \leq H(A,B)$.*

2. Common fixed point

The following result extends [23, Theorem 1], [15, Theorem 3] and improves [17, Theorem 2.3].

Theorem 2.1. *Let $(X,\, d)$ be a metric space, $f, g : X \to X$, and $T, S : X \to CB(X)$ be multivalued mappings. The pair $\{g, S\}$ satisfies (EA) property, $g(X) \subseteq fX$ and there exists, $r \in [0, 1)$ such that for all $x, y \in X, x \neq y$,*

$$H(Tx, Sy) < \max\{d(fx, gy), rd(fx, Tx), rd(gy, Sy), \frac{1}{2}[d(fx, Sy) + d(gy, Tx)]\}. \quad (2.1)$$

If $f(X)$ and $g(X)$ are closed subsets of X then pairs $\{f, T\}$ and $\{g, S\}$ have coincidence points. Moreover, f, g, T and S have a common fixed point if f is $T-$weakly commuting at $x \in C(f, T)$, g is $T-$weakly commuting at $y \in C(g, T)$ and f and g are coincidently idempotent with respect to the mappings T and S respectively.

Proof. Since the pair $\{g, S\}$ satisfies property (EA), there exists a sequence $\{x_n\}$ in X, $t \in X$ and $D \in CB(X)$ such that $\lim_{n\to\infty} gx_n = t \in D = \lim_{n\to\infty} Sx_n$. Since, $g(X) \subseteq fX$, for each x_n, there exists y_n in X such that $fy_n = gx_n$. Therefore, $\lim_{n\to\infty} fy_n = t \in D = \lim_{n\to\infty} Sx_n$. Now closedness of $f(X)$ and $g(X)$ implies that $t \in f(X) \cap g(X)$, there exists elements u and v in X such that $t = fu$ and $t = gv$. We claim that $fu \in Tu$. If not, then condition (2.1) implies that,

$$H(Tu, Sx_n) < \max\{d(fu, gx_n), rd(fu, Tu), rd(gx_n, Sx_n), \frac{1}{2}[d(fu, Sx_n) + d(gx_n, Tu)]\}.$$

Taking limit $n \to \infty$, we have

$$\begin{aligned} H(Tu, D) &\leq \max\{d(fu, t), rd(fu, Tu), rd(t, D), \frac{1}{2}[d(fu, D) + d(t, Tu)]\} \\ &\leq \max\{rd(fu, Tu), \frac{1}{2}d(fu, Tu)\}. \end{aligned}$$

It further implies that

$$d(fu, Tu) \leq \max\{rd(fu, Tu), \frac{1}{2}d(fu, Tu)\},$$

which is a contradiction. Thus $fu \in Tu$. Now we show that $\lim_{n\to\infty} Ty_n = D$. Otherwise, there exists a positive real number ε, positive integer N, and a subsequence $\{Ty_{n_k}\}$ of $\{Ty_n\}$ such that $H(Ty_{n_k}, D) \geq \varepsilon$, for $n_k \geq N$. Now,

$$\begin{aligned} H(Ty_{n_k}, D) \leq &H(Ty_{n_k}, Sx_{n_k}) + H(Sx_{n_k}, D) \\ &< \max\{d(fy_{n_k}, gx_{n_k}), rd(fy_{n_k}, Ty_{n_k}), rd(gx_{n_k}, Sx_{n_k}), \\ &\quad \frac{1}{2}[d(fy_{n_k}, Sx_{n_k}) + d(gx_{n_k}, Ty_{n_k})]\} + H(Sx_{n_k}, D). \end{aligned}$$

Applying limit $n \to \infty$,

$$\lim_{n\to\infty} d(Ty_{n_k}, t) \leq \lim_{n\to\infty} H(Ty_{n_k}, D) \leq \max\{r \lim_{n\to\infty} d(t, Ty_{n_k}), \frac{1}{2} \lim_{n\to\infty} d(t, Ty_{n_k})\},$$

that is,

$$\lim_{n\to\infty} d(Ty_{n_k}, t) \leq \max\{r \lim_{n\to\infty} d(t, Ty_{n_k}), \frac{1}{2} \lim_{n\to\infty} d(t, Ty_{n_k})\},$$

which is a contradiction. Hence $\lim_{n\to\infty} Ty_n = D$. Now we show that $gv \in Sv$, if not, then condition (2.1) implies that,

$$H(Ty_n, Sv) < \max\{d(fy_n, gv), rd(fy_n, Ty_n), rd(gv, Sv), \frac{1}{2}[d(fy_n, Sv) + d(gv, Ty_n)]\}.$$

Taking limit $n \to \infty$, we have

$$H(D, Sv) \leq \max\{rd(gv, Sv), \frac{1}{2}(d(gv, Sv)\},$$

which further implies that

$$d(gv, Sv) \leq \max\{rd(gv, Sv), \frac{1}{2}d(gv, Sv)\},$$

which is a contradiction. Hence $gv \in Sv$. Now, we show that f, g, T and S have a common fixed point. By assumption, $f^2u \in Tfu$ and $g^2v \in Sgv$. Next, we claim that $u = fu$. If not, then condition (2.1) implies that

$$\begin{aligned} H(Tu, Sfu) &= H(Tu, Sgv) \\ &< \max\{d(fu, g^2v),\ rd(fu, Tu), rd(g^2v, Sgv), \frac{1}{2}[d(fu, Sgv) + d(g^2v, Tu)]\} \\ &= \frac{1}{2}d(fu, Sgv) \leq \frac{1}{2}H(Tu, Sgv), \end{aligned}$$

which is a contradiction and the claim follows. Similarly we obtain $v = gv$. If not, then from condition (2.1), we obtain

$$\begin{aligned} H(Tgv, Sv) &= H(Tfu, Sv) \\ &< \max\{d(f^2u, gv),\ rd(f^2u, Tfu), rd(gv, Sv), \frac{1}{2}[d(f^2u, Sv) + d(gv, Tgv)]\} \\ &= \frac{1}{2}d(gv, Tfu) \leq \frac{1}{2}H(Tfu, Sv), \end{aligned}$$

a contradiction. Thus f, g, T and S have a common fixed point. □

Example 2.2. *Let $X = [0, \infty)$ with usual metric. Define $f, g : X \to X$ and $T, S : X \to CB(X)$ by*

$$fx = \begin{cases} 0, & x = 0 \\ \frac{x}{4}, & 0 < x < \infty \end{cases}$$

$$gx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq \frac{1}{2} \\ \frac{x}{3}, & \frac{1}{2} < x < \infty \end{cases}$$

and

$$Sx = [0, \frac{x}{10}],\ 0 \leq x < \infty,$$

$$Tx = [0, \frac{x}{12}],\ 0 \leq x < \infty.$$

Note that $g(X) \subseteq fX$ and the pair $\{g, S\}$ satisfies (EA) property. Obviously

$$H(Tx, Sy) = \frac{1}{2}\left|\frac{x}{6} - \frac{y}{5}\right|.$$

If $x, y \in [0, \frac{1}{2}]$ *with* $x \neq y$ *we have*

$$d(fx, gy) = \frac{1}{2}\left|y - \frac{x}{2}\right|, \;\; d(fx, Tx) = \frac{1}{6}x, \;\; d(gy, Sy) = \frac{2y}{5}$$
$$\text{and} \;\; \frac{1}{2}\left[d(fx, Sy) + d(gy, Tx)\right] \leq \frac{1}{2}\left[\left|\frac{x}{4} - \frac{y}{10}\right| + \left|\frac{y}{2} - \frac{x}{12}\right|\right].$$

For $x, y \in (\frac{1}{2}, \infty)$ *with* $x \neq y$ *we obtain*

$$d(fx, gy) = \frac{1}{3}\left|\frac{3x}{4} - y\right|, \;\; d(fx, Tx) = \frac{1}{6}x, \;\; d(gy, Sy) = \frac{7}{30}y$$
$$\text{and} \;\; \frac{1}{2}\left[d(fx, Sy) + d(gy, Tx)\right] \leq \frac{1}{2}\left[\left|\frac{x}{4} - \frac{y}{10}\right| + \left|\frac{y}{3} - \frac{x}{12}\right|\right].$$

If $x \in [0, \frac{1}{2}]$ *and* $y \in (\frac{1}{2}, \infty)$ *then*

$$d(fx, gy) = \frac{1}{3}\left|y - \frac{3x}{4}\right|, \;\; d(fx, Tx) = \frac{1}{6}x, \;\; d(gy, Sy) = \frac{7}{30}y$$
$$\text{and} \;\; \frac{1}{2}[d(fx, Sy) + d(gy, Tx)] \leq \frac{1}{2}\left[\left|\frac{x}{4} - \frac{y}{10}\right| + \left|\frac{y}{3} - \frac{x}{12}\right|\right].$$

Finally if $x \in (\frac{1}{2}, \infty)$ *and* $y \in [0, \frac{1}{2}]$ *then*

$$d(fx, gy) = \frac{1}{2}\left|\frac{x}{2} - y\right|, \;\; d(fx, Tx) = \frac{3x}{20}, \;\; d(gy, Sy) = \frac{5y}{12}$$
$$\text{and} \;\; \frac{1}{2}[d(fx, Sy) + d(gy, Tx)] \leq \frac{1}{2}\left[\left|\frac{x}{4} - \frac{y}{10}\right| + \left|\frac{y}{2} - \frac{x}{10}\right|\right].$$

In each case, it can be verified that (2.1) is satisfied for $r = \frac{9}{10}$.
Thus all the axioms of Theorem 2.1 are satisfied. Moreover 0 *is the common fixed point of* f, g, S *and* T.

Corollary 2.3. *Let* $(X,\ d)$ *be a metric space,* $f, g : X \to X$ *and* $T, S : X \to CB(X)$ *be multivalued mappings. The pair* $\{g, S\}$ *is noncompatible,* $g(X) \subseteq fX$ *and there exists* $0 \leq r < 1$ *such that for all* $x, y \in X, x \neq y$,

$$H(Tx, Sy) < \max\left\{d(fx, gy), rd(fx, Tx), rd(gy, Sy), \frac{1}{2}[d(fx, Sy) + d(gy, Tx)]\right\}.$$

If $f(X)$ *and* $g(X)$ *are closed subset of* X*, then pairs* $\{f, T\}$ *and* $\{g, S\}$ *have a coincidence point. Moreover, if* f *is* $T-$*weakly commuting at* $x \in C(f, T)$*,* g *is* $T-$*weakly commuting at* $y \in C(g, T)$*, and* f *and* g *are coincidently idempotent with respect to the mappings* T *and* S *respectively, then* f, g, T *and* S *have a common fixed point.*

If we take, $H = S$ and $f = g$, then corollary 2.3 extends [23, Theorem 3], to set valued mappings.

Corollary 2.4. *Let* $(X,\ d)$ *be a metric space,* $f, g : X \to X$ *and* $T, S : X \to CB(X)$ *be multivalued mappings. The pair* $\{g, S\}$ *satisfies (EA) property,* $g(X) \subseteq$

fX and

$$H(Tx,Sy) < h\max\left\{d(fx,gy),\ d(fx,Tx), d(gy,Sy), \frac{1}{2}[d(fx,Sy)+d(gy,Tx)]\right\}, \quad (2.2)$$

for all $x, y \in X$ for which $x \neq y$ and $h \in [0,1)$. If $f(X)$ and $g(X)$ are closed subset of X, then pairs $\{f,T\}$ and $\{g,S\}$ have coincidence points. Moreover, f, g, T and S have a common fixed point if f is $T-$weakly commuting at $x \in C(f,T)$, g is $T-$weakly commuting at $y \in C(g,T)$ and f and g are coincidently idempotent with respect to the mappings T and S respectively.

Proof. Since (2.2) is a special case of (2.1), the result follows immediately from Theorem 2.1. □

Corollary 2.4. can be viewed as an extension of [2, Theorem 1], [9, Theorem 2]. Moreover [15, Theorem 2] become special case of Corollary, by setting $S = T$ and $f = g$.

Corollary 2.5. *Let (X, d) be a metric space, $f, g : X \to X$ and $T, S : X \to CB(X)$ be multivalued mappings, the pair $\{g,S\}$ satisfies (EA) property, $g(X) \subseteq fX$ and*

$$H(Tx,Sy) < \max\left\{d(fx,gy), \frac{1}{2}[d(fx,Tx)+d(gy,Sy)], \frac{1}{2}[d(fy,Sx)+d(gx,Sy)]\right\}. \quad (2.3)$$

If $f(X)$ and $g(X)$ are closed subset of X then pairs $\{f,T\}$ and $\{g,S\}$ have coincidence points. Moreover, if f is $T-$weakly commuting at $x \in C(f,T)$, g is S weakly commuting at $y \in C(g,T)$ and f and g are coincidently idempotent with respect to the mappings T and S respectively, then f, g, T and S have a common fixed point.

Proof. f we take $r \in (\frac{1}{2}, 1)$, in Theorem 2.1, condition (2.3) becomes a special case of condition (2.1). □

[14, Theorem 3.4] is a special case of Corollary 2.5, with $f = g$ and $T = S$. Corollary 2.5 also improves [17, Theorem 2.3] which itself generalizes many results in the existing literature.
Let $\varphi : R^+ \to R^+$ be a continuous and nondecreasing function such that $0 < \varphi(t) < t$ for each $t \in (0,\infty)$.
The following theorem improves [17, Theorem 2.10]. Also, common fixed points of four maps are obtained under much weaker conditions than those given in [17].

Theorem 2.6. *Let (X, d) be a metric space, $f, g : X \to X$ and $T, S : X \to CB(X)$ be multivalued mappings. The pair $\{g,S\}$ satisfies (EA) property, $g(X) \subseteq fX$ and for all $x, y \in X, x \neq y$,*

$$H(Tx,Sy) \leq \varphi\left(\max\{d(fx,gy), d(fx,Tx), d(gy,Sy), d(fx,Sy), d(gy,Tx)\}\right). \quad (2.4)$$

If $f(X)$ and $g(X)$ are closed subset of X then pairs $\{f,T\}$ and $\{g,S\}$ have coincidence points. Moreover, f, g, T and S have a common fixed point if f is $T-$weakly commuting at $x \in C(f,T)$, g is $T-$weakly commuting at $y \in C(g,T)$

and f and g are coincidently idempotent with respect to the mappings T and S respectively.

Proof. Since, the pair $\{g, S\}$ satisfies property (EA), there exists a sequence $\{x_n\}$ in X, $t \in X$ and $D \in CB(X)$ such that $\lim_{n\to\infty} gx_n = t \in D = \lim_{n\to\infty} Sx_n$. Also as, $g(X) \subseteq fX$, for each x_n, there exists y_n in X such that $fy_n = gx_n$. Therefore, $\lim_{n\to\infty} fy_n = t \in D = \lim_{n\to\infty} Sx_n$. As, $t \in f(X) \cap g(X)$, there exists elements u and v in X such that $t = fu$ and $t = gv$. We claim that $fu \in Tu$, if not, then condition (2.4) implies that,

$$H(Tu, Sx_n) \leq \varphi(\max\{d(fu, gx_n), d(fu, Tu), d(gx_n, Sx_n), d(fu, Sx_n), d(gx_n, Tu)\}).$$

Taking limit $n \to \infty$, we obtain

$$\begin{aligned} H(Tu, D) &\leq \varphi(\max\{d(fu, t), d(fu, Tu), d(t, D), d(fu, D), d(t, Tu)\}) \\ &\leq \varphi(d(fu, Tu)) < d(fu, Tu). \end{aligned}$$

It further implies that

$$d(fu, Tu) \leq H(Tu, D) < d(fu, Tu),$$

a contradiction. Hence $fu \in Tu$. Now we claim that $\lim_{n\to\infty} Ty_n = D$. Suppose not, then there exists a positive real number ε, positive integer N, and subsequence $\{Ty_{n_k}\}$ of $\{Ty_n\}$ such that $H(Ty_{n_k}, D) \geq \varepsilon$, for $n_k \geq N$. Now,

$$\begin{aligned} H(Ty_{n_k}, D) &\leq H(Ty_{n_k}, Sx_{n_k}) + H(Sx_{n_k}, D) \\ &< \varphi(\max\{d(fy_{n_k}, gx_{n_k}), d(fy_{n_k}, Ty_{n_k}), d(gx_{n_k}, Sx_{n_k}), \\ &\qquad d(fy_{n_k}, Sx_{n_k}), d(gx_{n_k}, Ty_{n_k})\}) + H(Sx_{n_k}, D), \end{aligned}$$

which, on taking limit $n \to \infty$ implies that,

$$\lim_{n\to\infty} d(Ty_{n_k}, t) \leq \lim_{n\to\infty} H(Ty_{n_k}, D) \leq \varphi(\lim_{n\to\infty} d(t, Ty_{n_k}) < \lim_{n\to\infty} d(t, Ty_{n_k}),$$

which is a contradiction. Hence $\lim_{n\to\infty} Ty_n = D$. Now we show that $gv \in Sv$, if not, then condition (2.4) gives,

$$H(Ty_n, Sv) < \varphi(\max\{d(fy_n, gv), d(fy_n, Ty_n), d(gv, Sv), d(fy_n, Sv), d(gv, Ty_n)\}).$$

Taking limit $n \to \infty$, we have

$$H(D, Sv) \leq \varphi(d(gv, Sv)).$$

It further implies that

$$d(gv, Sv) \leq \varphi(d(gv, Sv)) < d(gv, Sv),$$

which is a contradiction. Hence $gv \in Sv$. Thus, there exists points u, v in X such that $fu \in Tu, gv \in Sv$. Next, we claim that $u = fu$. If not, then condition (2.4) implies that

$$\begin{aligned} H(Tu, Sfu) &= H(Tu, Sgv) \\ &< \varphi(\max\{d(fu, g^2v),\ d(fu, Tu), d(g^2v, Sgv), d(fu, Sgv), d(g^2v, Tu)\} \\ &\leq \varphi(d(fu, Sgv)) \leq \varphi(H(Tu, Sgv)) < H(Tu, Sgv), \end{aligned}$$

which is a contradiction. Hence the claim follows. Similarly we show that $v = gv$. If not, then condition (2.4) implies that

$$\begin{aligned} H(Tgv, Sv) &= H(Tfu, Sv) \\ &< \varphi(\max\{d(f^2u, gv),\ d(f^2u, Tfu), d(gv, Sv), d(f^2u, Sv), d(gv, Tfu)\} \\ &= \varphi(d(fu, Sv)) \leq \varphi(H(Tfu, Sv)) < H(Tfu, Sv), \end{aligned}$$

again a contradiction. Thus f, g, T and S have a common fixed point. □

Example 2.7. *Let $X = [0, \infty)$ with usual metric. Define $f, g : X \to X$, and $T, S : X \to CB(X)$ by*

$$fx = \frac{x}{3},\ \ gx = \frac{x}{2},\ \ Sx = [0, \frac{x}{8}] \ \text{and}\ Tx = [0, \frac{x}{10}].$$

Define $\varphi : R^+ \to R^+$ by $\varphi(t) = \dfrac{9t}{10}$. Note that $g(X) \subseteq fX$ and the pair $\{g, S\}$ satisfies (EA) property.
Now for $x, y \in [0, \infty)$ with $x \neq y$ we have

$$H(Tx, Sy) = \frac{1}{2}\left|\frac{x}{5} - \frac{y}{4}\right|,\ \ d(fx, gy) = \left|\frac{x}{3} - \frac{y}{2}\right|,\ \ d(fx, Tx) = \frac{7}{30}x,$$
$$d(gy, Sy) = \frac{3}{8}y,\ \ d(fx, Sy) = \left|\frac{x}{3} - \frac{y}{8}\right|,\ \ \text{and}\ d(gy, Tx) = \frac{1}{2}\left|y - \frac{x}{5}\right|.$$

If $x > y$, then

$$H(Tx, Sy) < \frac{9}{10}\left(\frac{7}{30}x\right) = \varphi\left(\frac{7}{30}x\right).$$

And, if $x < y$,

$$H(Tx, Sy) < \frac{1}{8}y < \frac{9}{10}\left(\frac{3}{8}y\right) = \varphi\left(\frac{3}{8}y\right).$$

Thus, in each case

$$H(Tx, Sy) \leq \varphi(\max\{d(fx, gy), d(fx, Tx), d(gy, Sy), d(fx, Sy), d(gy, Tx)\})$$

is satisfied. Hence all the axioms of Theorem 2.6 are satisfied. Moreover 0 *is the common fixed point of f, g, S and T.*

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