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QUASI-INTERPOLATORY APPROXIMATION SCHEME FOR MULTIVARIATE SCATTERED DATA†

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Abstract. The problem of approximation from a set of scattered data arises in a wide range of applied mathematics and scientific applications. In this study, we present a quasi-interpolatory approximation scheme for scattered data approximation problem, which reproduces a certain space of polynomials. The proposed scheme is local in the sense that for an evaluation point, the contribution of a data value to the approximating value is decreasing rapidly as the distance between two data points is increasing.

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1. Introduction

Approximation from a set of multivariate scattered data (also, referred to as unorganized data) can be found in a wide range of applied mathematics and scientific applications such as neural networks, topography and image processing. Construction a smooth surface from a larger set of scattered points (called point-cloud) is also a very important problem in computer graphics. The mathematical setting of this problem is can be written as follows: Given a set of scattered points X in $\mathbb{R}^d$ and values $f|_X$ (possibly contaminated) of some function f, the goal is to construct a function $L_X f$ such that approximates f in some sense $L_X f$. A well-known approach is radial basis function interpolation on a finite set of scattered points $X \subset \mathbb{R}^d$. This method is certainly a very useful approach towards solving the scattered data problem. However, when the data set is very large, it should solve a very large size matrix, which is usually very ill-conditioned. For the details of radial basis function interpolation, we refer the

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reader to the articles [4, 10, 11, 12, 14] and recent books which contain the core of the underlying general mathematics for radial basis functions [2, 13].

There have been intensive studies on approximation on uniform grids and these results are quite satisfactory. However, the quasi-interpolation on scattered data is much less investigated. In [15] and [16], quasi-interpolatory approximation schemes based on radial basis function with infinitely many centers were provided. Although these results provide spectral convergence order, the suggested schemes are not easy to implement and computationally expensive. For this reason, this paper provides a simple and localized approximation scheme with high accuracy. In particular, we look for an quasi-interpolatory scheme.

Definition 1. Let L_X be an approximation scheme on a set of scattered points X. Then we say that L_X is a quasi-interpolatory scheme of order k if L_X reproduces polynomials of degree k, i.e.,

$$L_X p(\mathbf{x}) = p(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \tag{1}$$

for any polynomials p of degree $\leq k$.

In this paper, on the purpose of constructing a quasi-interpolatory scheme, we first choose the basis function to be the 'shifted' thin-plate spline (see section 2). Although this function provides a high accuracy, this function has a well-known drawback; it increase polynomially fast around ∞, whence it is lack of locality property. Thus, a suitable bell-shaped function is obtained by applying a difference operator to ϕ, that is,

$$\psi := \bar{\mu} *' \phi,$$

where $\bar{\mu} := (\mu(\alpha) : \alpha \in \mathbb{Z}^d)$ is a localization sequence and $f *' g$ indicates the discrete convolution of f and g

$$f * g = \sum_{\alpha \in \mathbb{Z}^d} f(\cdot - \alpha) g(\alpha),$$

Then we look for a quasi-interpolatory approximation $L_X f$ which is of the form

$$L_X f := (\psi *' Q_h f)(\cdot/h), \tag{2}$$

with Q_h is a suitable operator (we will discuss it in section 2). The advantages of the proposed scheme are as follows. First, in contrast to the radial basis function interpolation which is a global scheme, the proposed scheme is local in the sense that for a evaluation point, the contribution of the data values to the approximating value is decreasing rapidly as the distance between two points is increasing. Practically, approximation values are determined by solving only small matrices. Next, it provides high accuracy in a way of depending on the smoothness of the approximands and the degree polynomials reproduced by the quasi-interpolation. Finally, the scheme is easy to implement and needs not to solve a heavy numerical integration problem as in the case of [15].
The following notations are used in this paper. For any $\mathbf{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$, its Euclidean norm is given by $|\mathbf{x}| := (x_1^2 + \cdots + x_d^2)^{1/2}$. The set of natural

numbers are denoted by $\mathbb{N}$. For any $k \in \mathbb{N}$, $\Pi_{\leq k}$ stands for the space of all algebraic polynomials of degree $\leq k$ in d variables. Let $\mathbb{Z}_+^d := \{\gamma \in \mathbb{Z}^d : \gamma \geq 0\}$ and for any $\alpha,\ \beta \in \mathbb{Z}_+^d$, we set $\alpha! := \alpha_1! \cdots \alpha_d!$ and $|\alpha|_1 := \sum_{k=1}^d \alpha_k$.

2. Localization Process

In this paper, on the purpose of constructing a quasi-interpolatory scheme, we choose the basis function to be the 'shifted' thin-plate spline

$$\phi := \begin{cases} (|\cdot|^2 + \lambda^2)^{m-d/2}, & m,\ d \text{ odd}, \\ (|\cdot|^2 + \lambda^2)^{m-d/2} \log(|\cdot|^2 + \lambda^2)^{1/2}, & m, d \text{ even}. \end{cases} \tag{3}$$

From the definition (3), we see that the 'shifted' thin-plate spline ϕ increase at some polynomial degree fast around ∞. Hence, it has the Fourier transform in the sense of tempered distribution [8] such that $\hat{\phi}$ is of the form

$$\hat{\phi} = c(m, d)\tilde{K}_{m/2}(c\cdot)|\cdot|^{-2m}$$

where $c(m, d)$ is a positive constant depending on m and d, and $\tilde{K}_\nu(|t|) := |t|^\nu K_\nu(|t|) \neq 0$, $t \geq 0$, with $K_\nu(|t|)$ the modified Bessel function of order ν. Due to the fact that the basis function ϕ grows at some polynomial degree away from zero, it may lose local property of the approximation. In order to overcome this limitation, we apply a difference operator to ϕ to the a bell-shaped function

$$\psi = \sum_{\alpha \in \mathbb{Z}^d} \mu(\alpha)\phi(\cdot - \alpha) \tag{4}$$

so that it satisfies at least the partition of unity

$$\sum_{\alpha \in \mathbb{Z}^d} \psi(\cdot - \alpha) = 1. \tag{5}$$

The coefficient sequence $(\mu(\alpha) : \alpha \in \mathbb{Z}^d)$ is assumed to have finite support (generally a milder condition is imposed on μ). Further, the localized function ψ is assumed to satisfy the condition

$$\sup_{x \in \mathbb{R}^d} (1 + |x|)^{m_\psi} \psi(x) < \infty, \quad \hat{\psi}(0) \neq 0 \tag{6}$$

for some $m_\psi > d$. Recalling that $\hat{\phi}$ is continuous on $\mathbb{R}^d \backslash \{0\}$ and has a singularity of order $2m$ at the origin for some positive integer m, the following lemma is useful for our further error analysis.

Lemma 1. [5] *There exists a localization sequence* $\bar{\mu} := (\mu(\alpha) : \alpha \in \mathbb{Z}^d)$ *such that the bell-shape function* ψ *in* (4) *satisfies the relation* (6) *with* $m_\psi > 2m + d$ *and the polynomial reproducing property*

$$\psi *' p = p, \tag{7}$$

for any polynomial $p \in \Pi_{\leq 2m-1}$.

3. Bounded Operator Q_h

In order to discuss the accuracy of approximation, we measure the 'fill-distance' of X in $\mathbb{R}^d$ by

$$h := h(X) := \sup_{\mathbf{x}\in\mathbb{R}^d} \min_{\mathbf{x}_n\in X} |\mathbf{x}-\mathbf{x}_n|. \tag{8}$$

Towards the construction of our approximation scheme, we define an operator Q_h as follows. For the given $h\alpha \in h\mathbb{Z}^d$, we choose a local $X_{h\alpha} \subset X$ around $h\alpha$ and an ℓ_1-bounded coefficient vector $(a(\cdot,\mathbf{x}_n))_{\mathbf{x}_n\in X_{h\alpha}}$, satisfying the polynomial reproduction property of degree $2m-1$:

$$\begin{aligned} &(1) \sum_{\mathbf{x}_n\in X_{h\alpha}} a(h\alpha,\mathbf{x}_n))p(\alpha) = p(h\alpha), \quad p\in\Pi_{\le 2m-1} \\ &(2) \sum_{\mathbf{x}_n\in X_{h\alpha}} |a(h\alpha,\mathbf{x}_n))| \le c, \end{aligned} \tag{9}$$

for some constant $c>0$ independent of $h\alpha$. Then, we define by Q_hf

$$Q_hf(h\alpha) := \sum_{\mathbf{x}_n\in X_{h\alpha}} a(h\alpha,\mathbf{x}_n)f(\mathbf{x}_n) \tag{10}$$

such that $f(h\alpha)\approx Q_h(\alpha)$. Here, in order to make the above polynomial reproducing property meaningful, it is basically assumed that

$$\#X_{h\alpha} \ge \dim\Pi_{\le 2m-1}.$$

In what follows, we estimate the accuracy of the operator Q_h. Our analysis is performed on the space Sobolev space $W^k_\infty(\mathbb{R}^d)$ which consists of all functions whose derivatives of orders $\le k$ are bounded, that is,

$$W^k_\infty(\mathbb{R}^d) := \Big\{ f : \|f\|_{W^m_\infty(\mathbb{R}^d)} := \sum_{|\alpha|_1\le m} \|D^\alpha f\|_{L_\infty(\mathbb{R}^d)} < \infty \Big\}.$$

The following lemma estimates the approximation property of Q_hf to $f \in W^k_\infty(\mathbb{R}^d)$.

Lemma 2. *Let $f\in W^{2m}_\infty(\mathbb{R}^d)$ and Q_hf be defined as in* (10)*. Then, as $h\to 0$, we have an estimate of the form*

$$|f(h\alpha) - Q_hf(\alpha)| \le ch^{2m}$$

with a constant $c>0$ independent of α and h.

Proof. Note that for any $\alpha\in\mathbb{Z}^d$,

$$\sum_{\mathbf{x}_n\in X_{h\alpha}} a(h\alpha,\mathbf{x}_n) = 1.$$

Then, it follows that

$$f(h\alpha) - Q_hf(\alpha) = \sum_{\mathbf{x}_n\in X_{h\alpha}} a(h\alpha,\mathbf{x}_n)(f(h\alpha)-f(\mathbf{x}_n)). \tag{11}$$

Now, to estimate the right-hand side of the above equation, let us represent the function f as $f = T_{h\alpha}f + R_{h\alpha}f$ with $T_{h\alpha}f$ the mth degree Taylor polynomial of f around $h\alpha$, that is,

$$T_{h\alpha}f := T_{h\alpha}^{2m-1}f := \sum_{|\ell|_1<2m} \frac{(\cdot - h\alpha)^\ell}{\ell!} f^{(\ell)}(h\alpha), \tag{12}$$

and $R_{h\alpha}f$ the remainder of $T_{h\alpha}$. It is obvious that

$$f(h\alpha) - T_{h\alpha}f(\mathbf{x}_n) = \sum_{0<|\ell|_1<2m} \frac{(\mathbf{x}_n - h\alpha)^\ell}{\ell!} f^{(\ell)}(h\alpha).$$

Due to the polynomial reproducing property of the vector $(a(\cdot, \mathbf{x}_n))_{\mathbf{x}_n \in X_{h\alpha}}$ up to the degree $2m-1$, it is immediate that

$$\sum_{\mathbf{x}_n \in X_{h\alpha}} a(h\alpha, \mathbf{x}_n)(f(h\alpha) - T_{h\alpha}f(\mathbf{x}_n)) = 0. \tag{13}$$

Next, invoking that

$$f = T_{h\alpha}f + R_{h\alpha}f,$$

we need to estimate the remainder $R_{h\alpha}f$. In fact, there exists some number ξ_n between $\mathbf{x}_n$ and $h\alpha$ so that $R_{h\alpha}f$ can be expressed as

$$R_{h\alpha}f(\mathbf{x}_n) := \sum_{|\ell|_1=2m} \frac{(\mathbf{x}_n - h\alpha)^\ell}{\ell!} f^{(\ell)}(\xi_n). \tag{14}$$

Combining (11) and (13), we have

$$\begin{aligned}|f(h\alpha) - Q_h f(\alpha)| &\le \sum_{\mathbf{x}_n \in X_{h\alpha}} |a(h\alpha, \mathbf{x}_n)||R_{h\alpha}f(\xi_n)| \\ &\le ch^{2m}\|f\|_{W_\infty^{2m}(\mathbb{R}^d)} \sum_{\mathbf{x}_n \in X_{h\alpha}} |a(h\alpha, \mathbf{x}_n)|\end{aligned}$$

for some constant $c > 0$ independent of α and h. Invoking the fact that the vector $(a(\cdot, \mathbf{x}_n))_{\mathbf{x}_n \in X_{h\alpha}}$ is ℓ_1-bounded, this lemma is proved immediately. □

4. Quasi-interpolatory Schemes and Error Analysis

For a given data $f|_X$, our approximation scheme on X is constructed as follows.

Step 1: For the given $\alpha \in \mathbb{Z}^d$, choosing an ℓ_1-bounded coefficient vector $(a(\cdot, \mathbf{x}_n))_{\mathbf{x}_n \in X_{h\alpha}}$, we approximate $f(h\alpha)$ by $Q_h(\alpha)$ in (10).
Step 2: With the localized function ψ in (4), the final approximation $L_X f$ is defined by

$$L_X f := \sum_{\alpha \in \mathbb{Z}^d} \psi_h(\cdot/h - \alpha) Q_h f(\alpha). \tag{15}$$

We now provide the approximation order of L_X. Error analysis on the Sobolev space is a common approach in approximation theory. Thus, we estimate the difference between f and $L_X f$ for f in the Sobolev space $W_\infty^{2m}(\mathbb{R}^d)$. The following lemma is useful for our error analysis.

Lemma 3. *Let $f \in W_\infty^{2m}(\mathbb{R}^d)$ and put $f_h = f(h\cdot)$. Then, we have*

$$\|f - \psi * f_h(\cdot/h)\|_{L_\infty(\mathbb{R}^d)} \le ch^{2m}. \tag{16}$$

Proof. Applying (5) induces the identity

$$f(\mathbf{x}) - \psi * f_h(\mathbf{x}/h) = \sum_{\alpha\in\mathbb{Z}^d} \psi(\mathbf{x}/h - \alpha)(f(\mathbf{x}) - f(h\alpha)).$$

Write $f = T_\mathbf{x} f + R_\mathbf{x} f$ with $T_\mathbf{x} f$ the Taylor polynomial of f of degree $2m-1$ around $\mathbf{x}$ (see (12)) and $R_\mathbf{x} f$ the remainder of $T_\mathbf{x} f$. Then,

$$f(\mathbf{x}) - T_\mathbf{x}(h\alpha) = \sum_{0<|\ell|_1<2m} \frac{(h\alpha - \mathbf{x})^\ell}{\ell!} f^{(\ell)}(\mathbf{x}),$$

Due to the polynomial reproducing property of ψ, it is immediate that

$$\sum_{\alpha\in\mathbb{Z}^d} \psi(\mathbf{x}/h - \alpha)(f(\mathbf{x}) - T_\mathbf{x}(h\alpha)) = 0.$$

Thus, from the explicit formula of the remainder $R_\mathbf{x}$ (see (14)), we can induce the required result. □

Theorem 1. *Let ϕ be the 'shifted' thin-plate spline in* (3) *so that its Fourier transform has a singularity of order $2m$, $m \in \mathbb{N}$, at the origin. Then, for every $f \in W_\infty^{2m}(\mathbb{R}^d)$, we have*

$$\|f - L_X f\|_{L_\infty(\mathbb{R}^d)} \le c_f h^{2m} \tag{17}$$

for some constant $c_f > 0$.

Proof. For convenience, put $f_h := f(h\cdot)$. It is useful to divide the error $f - L_X f$ by using the term $\psi *' f_h$ as follows:

$$f - L_X f = (f - \psi *' f_h(\cdot/h)) + (\psi *' f_h(\cdot/h) - \psi *' Q_h f(\cdot/h)).$$

In view of Lemma 3, it suffices to estimate the term $\psi *' f_h(\cdot/h) - \psi *' Q_h f(\cdot/h)$. For this proof, we see that

$$\begin{aligned}|\psi *' f_h(\cdot/h)) - (\psi *' Q_h f)(\cdot/h)| &\le |\psi *' (f_h - Q_h f)(\cdot/h)| \\ &\le \|f_h - Q_h f\|_{L_\infty(\mathbb{R}^d)} \sum_{\alpha\in\mathbb{Z}^d} |\psi(\cdot/h - \alpha)|\end{aligned}$$

Therefore, by Lemma 2 and (6), this theorem is proved immediately, that is,

$$|f(x) - L_X f(x)| \le c_f h^{2m}$$

for some constant $c_f > 0$. □

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