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STRONG CONVERGENCE OF HYBRID METHOD FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND SEMIGROUPS

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Abstract. In this paper, some strong convergence theorems are obtained for hybrid method for modified Ishikawa iteration process of asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups in Hilbert spaces. The results presented in this article generalize and improve results of Tae-Hwa Kim and Hong-Kun Xu and others. The convergence rate of the iteration process presented in this article is faster than hybrid method of Tae-Hwa Kim and Hong-Kun Xu and others.

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1. Introduction and Preliminaries

Let X be a real Banach space, C a nonempty closed convex subset of X, and T: $C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [4] if there exists a sequence $\{k_n\} \subset [1, +\infty)$ of positive real numbers with $lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n \|x-y\|$ for all integers $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Recall also that a one-parameter family $\Im = \{T(t) : t \geq 0\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) *Lipschitzian semigroup* on C(see,e.g.,[15]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$,

(ii) $T(t+s)x = T(t)T(s)x, t, s \geq 0, x \in C$,

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(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$,

(iv) there exists a bounded measurable function $L(t) : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$,

$$\|T(t)x - T(t)y\| \leq L(t)\|x - y\|, \quad x, y \in C.$$

A Lipschitzian semigroup $\Im$ is called nonexpansive if $L(t) = 1$ for all $t > 0$, and asymptotically nonexpansive if $\limsup_{t\to\infty} L(t) \leq 1$, respectively. We use $F(\Im)$ to denote the common fixed point set of the semigroup $\Im$, that is $F(\Im) = \{x \in C : T(s)x = x, \forall s > 0\}$. Note that for an asymptotically nonexpansive semigroup $\Im$, we can always assume that the Lipschitzian constants $L(t)$ are such that $L(t) \geq 1$ for all $t > 0$, $L(t)$ is nonincreasing in t, and $lim_{t\to\infty} L(t) = 1$; otherwise we replace $L(t)$, for each $t > 0$, with $\tilde{L}(t) := max\{sup_{s\geq t} L(s),\ 1\}$.

Construction of fixed points of nonexpansive mappings (and of common fixed points of nonexpansive semigroups) is an important subject in the theory of nonexpansive mappings and finds application in a number of applied areas, in particular, in image recovery and signal processing (see,e.g.,[2,8,11,16,17]). However, the sequence $\{T^n x\}_{n=0}^{\infty}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus averaged iterations prevail. Indeed, Mann's iterations do have weak convergence. More precisely, a Mann's iteration procedure is a sequence $\{x_n\}$ which is generated in the following recursive way:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n T x_n), \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily. For example, Reich [9] proved that if X is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However we note that Mann's iterations have only weak convergence even in a Hilbert space [3].

Attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi[7] proposed the following modification of Mann iteration method (1.1) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H.

Nakajo and Takahashi[7] also propose the following iteration process for a nonexpansive semigroup $\Im = \{T(s) : 0 \leq s < \infty\}$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad chosen\ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.3}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and if $\{t_n\}$ is a positive real divergent sequence, then the sequence $\{x_n\}$ generated by (1.2)(resp.(1.3)) converges strongly to $P_{F(T)}x_0$ (resp.$P_{F(\Im)}x_0$).

The adaptation of Mann's iteration (1.1) to asymptotically nonexpansive mappings T is given below

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad n \ge 0. \tag{1.4}$$

Weak convergence of the sequence $\{x_n\}$ generated by (1.4) is proved by Schu [10] (see also Tan and Xu [14]).

Attempts to modify the Mann iteration method (1.4) so that strong convergence is guaranteed have recently been made In 2006,T.H.Kim and H.K.Xu [18] proposed the following modification of the Mann iteration method (1.4) for asymptotically non-expansive mapping in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad chosen\ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.5}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0 \ \ as\ n \to \infty$.

They also proposed the following modification of the Mann iteration method (1.4) for asymptotically nonexpansive semigroup in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad chosen\ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \tilde{\theta}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.6}$$

where

$$\tilde{\theta}_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1](diamC)^2 \to 0 \ \ as\ n \to \infty.$$

It is purpose of this paper to generalized iteration process (1.5) and (1.6) to Ishikawa type hybrid iteration processes for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ D_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + \phi_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n \bigcap Q_n}(x_0) \end{cases} \tag{1.7}$$

where

$$\theta_n = (1-\alpha_n)(\beta_n k_n^2 + (1-\beta_n)k_n^4 - 1)(diamC)^2 \to 0 \ \ as\ n \to \infty.$$
$$\phi_n = (1-\beta_n)(k_n^2 - 1)(diamC)^2 \to 0 \ \ as\ n \to \infty.$$

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds, \\ z_n = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \tilde{\theta}_n\}, \\ D_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + \tilde{\phi}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n \bigcap Q_n}(x_0) \end{cases} \tag{1.8}$$

where

$$\tilde{\theta}_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2(\beta_n + (1-\beta_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2) - 1](diam(C))^2$$

$$\tilde{\phi}_n = (1-\beta_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1](diam(C))^2$$

We shall prove that both iteration processes (1.7) and (1.8) converges strongly to a fixed point of T and a common fixed point of $\Im$, respectively, provided the sequence $\{\alpha_n\}$ is bounded from above.

We will use the notation $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

2. Convergence of asymptotically nonexpansive mappings

Before presenting the main result of this section, we include the following lemma which is well known as the demiclosedness principle for saymptotically nonexpansive mappings and which is a special case of [6,Theorem3.1].

Lemma 2.1. [6] *Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties (i) $\{x_n\}$ converges weakly to a point z, (ii) $Tx_n - x_n$ converges strongly to 0, then $z \in F(T)$.*

Theorem 2.2. *Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] such that $\alpha_n \leq a,\ b \leq \beta_n$ for all n and for some $a, b \in (0,1)$. Define a sequence $\{x_n\}$ in C by (1.7). Then $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$.*

Proof. By the result of K.Goebel and K.Kirk[4], we known that, $F(T)$ is nonempty. Now observe that C_n is convex. Indeed, the defining inequality in C_n is equivalent to the following inequality

$$\langle 2(x_n - y_n), z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \theta_n$$

which is affine(and hence convex) in z.

Now, we show that $F(T) \subset C_n$ for all n. Indeed, for all $p \in F(T)$ we have

$$\|y_n - p\|^2 \leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2\|z_n - p\|^2. \tag{2.1}$$
$$\|z_n - p\|^2 \leq \beta_n\|x_n - p\|^2 + (1-\beta_n)k_n^2\|x_n - p\|^2. \tag{2.2}$$

Substituting (2.2) into (2.1), we get

$$\begin{aligned}
&\|y_n - p\|^2 \leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2[\beta_n + (1-\beta_n)k_n^2]\|x_n - p\|^2\\
&\leq \|x_n - p\| - (1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)k_n^2[\beta_n + (1-\beta_n)k_n^2]\|x_n - p\|^2\\
&\leq \|x_n - p\| + (1-\alpha_n)[k_n^2[\beta_n + (1-\beta_n)k_n^2] - 1]\|x_n - p\|^2\\
&= \|x_n - p\|^2 + (1-\alpha_n)(\beta_n k_n^2 + (1-\beta_n)k_n^4 - 1)\|x_n - p\|^2\\
&\leq \|x_n - p\|^2 + (1-\alpha_n)(\beta_n k_n^2 + (1-\beta_n)k_n^4 - 1)(diamC)^2\\
&\leq \|x_n - p\|^2 + \theta_n.
\end{aligned}$$

So $p \in C_n$ for all n.

Now we prove that D_n is also convex and $F(T) \subset D_n$ for all n, indeed, the defining inequality in D_n is equivalent to the following inequality

$$\langle 2(x_n - z_n), z\rangle \leq \|x_n\|^2 - \|z_n\|^2 + \phi_n$$

which is affine(and hence convex) in z.

Now, we show that $F(T) \subset D_n$ for all n. Indeed, for all $p \in F(T)$ we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)k_n^2\|x_n - p\|^2\\
&\leq \|x_n - p\|^2 + (1-\beta_n)(k_n^2 - 1)\|x_n - p\|^2\\
&\leq \|x_n - p\|^2 + (1-\beta_n)(k_n^2 - 1)(diamC)^2\\
&\leq \|x_n - p\|^2 + \phi_n.
\end{aligned}$$

So $p \in D_n$ for all n.

Next we show that $F(T) \subset C_n \bigcap D_n \bigcap Q_n$, for all $n \geq 0$. It suffices to show that $F(T) \subset Q_n$, for all $n \geq 0$. We prove this by induction. For $n = 0$, we have

$F(T) \subset Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1} \rangle \geq 0, \quad \forall z \in Q_n \cap C_n,$$

as $F(T) \subset C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence the $F(T) \subset C_n \bigcap Q_n$ holds for all n.

Next, we show that $\|x_{n+1} - x_n\| \to 0$, indeed , by the definition of Q_n , we have $x_n = P_{Q_n}(x_0)$ which together with the fact that $x_{n+1} \in C_n \cap Q_n$ implies that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

This shows that the sequence $\{\|x_n - x_0\|\}$ is increasing, since C is bounded then $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Noticing again that $x_n = P_{Q_n}(x_0)$ and $x_{n+1} \in Q_n$ which implies that $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$, and noticing the identity

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v \rangle, \quad \forall u, v \in H.$$

we have

$$\|x_{n+1} - x_n\|^2 = \|(x_{n+1} - x_0) - (x_n - x_0)\|^2$$

$$\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle$$

$$\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \ n \to \infty.$$

We now prove $\|Tx_n - x_n\| \to 0$, we first prove $\|Tz_n - x_n\| \to 0$, indeed,

$$\|T^n z_n - x_n\| = \frac{1}{1 - \alpha_n}\|y_n - x_n\| \leq \frac{1}{1 - \alpha_n}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|)$$

Since $x_{n+1} \in C_n$, then

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n.$$

Because $\theta_n \to 0$, and we have proved $\|x_{n+1} - x_n\| \to 0$, so that $\|y_n - x_{n+1}\| \to 0$, therefore, which leads to

$$\|T^n z_n - x_n\| \to 0. \tag{2.3}$$

On the other hand, we have

$$\|T^n x_n - x_n\| \leq \|T^n x_n - T^n z_n\| + \|T^n z_n - x_n\|$$

$$\leq k_n\|x_n - z_n\| + \|T^n z_n - x_n\|$$

$$= k_n(1 - \beta_n)\|T^n x_n - x_n\| + \|T^n z_n - x_n\|.$$

This result implies

$$\|T^n x_n - x_n\| \le \frac{1}{1 - k_n(1-\beta_n)} \|T^n z_n - x_n\|.$$

By using condition $0 < b \le \beta_n$ and (2.3) we obtain that

$$\|T^n x_n - x_n\| \to 0, \ as\ n \to \infty.$$

Putting $k = \sup\{k_n : n \ge 1\} < \infty$, we deduce that

$$\|Tx_n - x_n\| \le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\|$$

$$+\|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|$$

$$\le k\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+k)\|x_{n+1} - x_n\| \to 0.$$

We claim that $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$, if not, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a real number $\varepsilon > 0$ such that $\|x_{n_k} - P_{F(T)}(x_0)\| \ge \varepsilon$. Without loss generality, we can assume $\|x_n - P_{F(T)}(x_0)\| \ge \varepsilon$ for all n.

It is well-known that, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $\widetilde{x} \in C$, by Lemma 2.1, $\widetilde{x} \in F(T)$. We now prove $\widetilde{x} = P_{F(T)(x_0)}$ and the convergence is strong. Put $x' = P_{F(T)(x_0)}$ and consider the sequence $\{x_0 - x_{n_i}\}$, then we have $x_0 - x_{n_i} \rightharpoonup x_0 - \widetilde{x}$ and by the weak lower semicontinuity of the norm and by the fact that $\|x_0 - x_{n+1}\| \le \|x_0 - x'\|$ for all $n \ge 0$ which is implied by the fact that $x_{n+1} = P_{F(T)(x_0)}$, we have

$$\|x_0 - x'\| \le \|x_0 - \widetilde{x}\|$$

$$\le \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \le \limsup_{i\to\infty} \|x_0 - x_{n_i}\|$$

$$\le \|x_0 - x'\|.$$

This implies $\|x_0 - x'\| = \|x_0 - \widetilde{x}\|$, by the uniqueness of the nearest point projection of x_0 onto $P_F(T)(x_0)$ hence $\widetilde{x} = x'$ and

$$\|x_0 - x_{n_i}\| \to \|x_0 - x'\|, \ as\ i \to \infty.$$

It follows that $x_0 - x_{n_i} \to x_0 - x'$, hence $x_{n_i} \to x'$. This is a contradiction with assume $\|x_n - x'\| \ge \varepsilon$. This completes the proof. □

Remark The iteration process $\{x_{n+1}\}$ defined by (1.7) of this paper is the projection of x_0 into the subset $C_n \bigcap D_n \bigcap Q_n$ of C. It is obvious that, the convergence rate of $\{x_{n+1}\}$ which converges in norm to $P_{F(T)}(x_0)$ is faster than the iteration process $\{x_{n+1}\}$ defined by (1.5) which is only the projection of x_0 into the subset $C_n \bigcap Q_n$ of C.

3. Strong convergence theorem of asymptotically nonexpansive semigroups

Assume in this section that $\Im = \{T(t) : t \geq 0\}$ is an asymptotically nonexpansive semigroup defined on a nonempty closed convex bounded subset C of a Hilbert space H. Recall that we use $L(t)$ to denote the Lipschitzian constant of the mapping $T(t)$ and assume that $L(t)$ is bounded and measurable so that the integral $\int_0^t L(s)ds$ exists for all $t > 0$. Recall also that $L(t) \geq 1$ for all $t > 0$, and $L(t)$ is nonincreasing in t, and $\lim_{t\to\infty} L(t) = 1$. In the rest of this section, we put $L = \sup\{L(t) : 0 < t < \infty\} < \infty$. Recall furthermore that we use $F(\Im)$ to denote the common fixed point set of $\Im$. Note that the boundedness of C implies that $F(\Im)$ is nonempty.

In order to prove our strong convergence theorem, we first establish some technical lemmas.

Lemma 3.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties*
(i) $x_n \rightharpoonup z$;
(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$,
where $x_n \rightharpoonup z$ denote that $\{x_n\}$ converges weakly to z, then $z \in F(\Im)$.

Proof. This lemma is the continuous version of Lemma 2.3 of Tan and Xu[12]. The proof given in [12] is easily extended to the continuous case. This complete the proof. □

Lemma 3.2 [18]. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t) : 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Then it holds that*

$$\limsup_{s\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(u)xdu - T(s)(\frac{1}{t}\int_0^t T(u)xdu)\| = 0$$

Now we present the strong convergence of an asymptotically nonexpansive semigroup on C in a Hilbert space.

Theorem 3.3. *Let $\Im$ be an asymptotically nonexpansive semigroup defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] such that $\alpha_n \leq a,\ b \leq \beta_n$ for all n and for some $a, b \in (0, 1)$. Define a sequence $\{x_n\}$ in C by the algorithm (1.8). The $\{x_n\}$ converges strongly to $P_{F(\Im)}(x_0)$.*

Proof. First observe that $F(\Im) \subset C_n$ for all $n \geq 0$. Indeed, we have for all $p \in F(\Im)$,

$$\|y_n - p\|^2 = \|\alpha_n x_n + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)z_n du - p\|^2$$

$$\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\frac{1}{t_n}\int_0^{t_n} T(u)z_n du - p\|^2$$

$$\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n} \|T(u)z_n - p\|du)^2$$

$$\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2\|z_n - p\|^2. \tag{3.1}$$

We have also for all $p \in F(\Im)$,

$$\|z_n - p\|^2 \leq \beta_n\|x_n - p\|^2 + (1-\beta_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2\|x_n - p\|^2 \tag{3.2}$$

Substituting (3.2) into (3.1) we have that

$$\|y_n - p\|^2 \leq \alpha_n\|x_n - p\|^2 +$$

$$(1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2(\beta_n\|x_n - p\|^2 + (1-\beta_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2\|x_n - p\|^2)$$

$$\leq \|x_n - p\|^2 - (1-\alpha_n)\|x_n - p\|^2 +$$

$$(1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2(\beta_n + (1-\beta_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2)\|x_n - p\|^2$$

$$= \|x_n - p\|^2 + (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2(\beta_n + (1-\beta_n)(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2) - 1](diam(C))^2$$

$$\leq \|x_n - p\|^2 + \tilde{\theta_n}$$

So that $p \in C_n$ for all $n \geq 0$.

It from follows (3.2) that

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 + (1-\beta_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1]\|x_n - p\|^2$$

$$\leq \|x_n - p\|^2 + \widetilde{\phi_n}.$$

So that $p \in D_n$ for all $n \geq 0$.

As in the proof of Theorem 2.2, $\{x_n\}$ is well defined and $F(\Im) \subset C_n \bigcap Q_n$ for all $n \geq 0$. Also, similar to the proof of Theorem 2.2, we can show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Since

$$\|\frac{1}{t_n}\int_0^{t_n} \|T(u)z_n du - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\|$$

$$\leq \frac{1}{1-a}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|)$$

Note that the fact $x_{n+1} \in C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \tilde{\theta_n}$$

Thus, $\|x_n - x_{n+1}\| \to 0$ implies $\|y_n - x_{n+1}\| \to 0$, therefore, we have

$$\|\frac{1}{t_n}\int_0^{t_n} T(u)z_n du - x_n\| \to 0.$$

Because that

$$\begin{aligned}
&\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \\
&\leq \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - \frac{1}{t_n}\int_0^{t_n} T(u)z_n du\| + \|\frac{1}{t_n}\int_0^{t_n} T(u)z_n du - x_n\| \\
&\leq \frac{1}{t_n}\int_0^{t_n} \|T(u)x_n - T(u)z_n\| du \ + \|\frac{1}{t_n}\int_0^{t_n} T(u)z_n du - x_n\| \\
&\leq (\frac{1}{t_n}\int_0^{t} L(u)du)\|x_n - z_n\| + \|\frac{1}{t_n}\int_0^{t_n} T(u)z_n du - x_n\| \qquad (3.4)
\end{aligned}$$

Since

$$\|z_n - x_n\| = (1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \qquad (3.5)$$

Substituting (4.0) into (3.9) we have

$$\begin{aligned}
&\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \\
&\leq \frac{1}{1-(\frac{1}{t_n}\int_0^{t_n} L(u)du)(1-\beta_n)}\|\frac{1}{t_n}\int_0^{t_n} T(u)z_n du - x_n\|
\end{aligned}$$

Because $\lim_{u\to\infty} L(u) = 1$, it is easy to see

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n} L(u)du = 1.$$

It follows from condition $b < \beta_n$ and above result that

$$\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \to 0 \qquad (3.6)$$

Now, we consider that

$$\|T(s)x_n - x_n\| \le \|T(s)x_n - T(s)(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du)\|$$

$$+\|T(s)(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du) - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| + \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\|$$

$$(L+1)\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\|$$

$$+\|T(s)(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du) - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| \tag{3.7}$$

Combining (3.6),(3.7) and by lemma3.3 we have that

$$\limsup_{s\to\infty}\limsup_{n\to\infty}\|T(s)x_n - x_n\| = 0.$$

An application of Lemma 3.1 implies that every weak limit point of $\{x_n\}$ is a member of $F(\Im)$. Repeating the last part of the proof of Theorem 2.2, we can prove that $P_{F(\Im)}(x_0)$ is the only weak limit point of $\{x_n\}$, hence $\{x_n\}$ weakly converges to $P_{F(\Im)}(x_0)$, and that the convergence is moreover in the strong topology. This completes the proof. □

Remark. The iteration process $\{x_{n+1}\}$ defined by (1.8) of this paper is the projection of x_0 into the subset $C_n \bigcap D_n \bigcap Q_n$ of C. It is obvious that, the convergence rate of $\{x_{n+1}\}$ which converges in norm to $P_{F(T)}(x_0)$ is faster than the iteration process $\{x_{n+1}\}$ defined by (1.6) which is only the projection of x_0 into the subset $C_n \bigcap Q_n$ of C.

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