J. Appl. Math. & Informatics Vol. **29***(2011), No. 3 - 4, pp. 603 - 619*
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A GENERAL ITERATIVE METHOD BASED ON THE HYBRID STEEPEST DESCENT SCHEME FOR VARIATIONAL INCLUSIONS, EQUILIBRIUM PROBLEMS†

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Abstract. To the best of our knowledge, it would probably be the first time in the literature that we clarify the relationship between Yamada's method and viscosity iteration correctly. We design iterative methods based on the hybrid steepest descent algorithms for solving variational inclusions, equilibrium problems. Our results unify, extend and improve the corresponding results given by many others.

AMS Mathematics Subject Classification : 47H09, 47J20, 47J25.
Key words and phrases : variational inclusion, equilibrium problem, strict pseudo-contractions, fixed point, variational inequality.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively. $T : H \to H$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in H$. The set of fixed points of T is the set $Fix(T) := \{x \in H : Tx = x\}$. Let C be nonempty closed convex subset of H, and 2^Hdenotes the family of all the nonempty subsets of H.

Let $A : H \to H$ be a single-valued nonlinear mapping, and let $M : H \to 2^H$ be a set-valued mapping. We consider the following variational inclusion, which is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{1}$$

where θ is the zero vector in H. The set of solutions of problem (1) is denoted by $I(A, M)$. If $A = 0$, then problem (1) becomes the inclusion problem introduced

Received July 20, 2010. Revised September 23, 2010. Accepted October 4, 2010.
*Corresponding author. †This research is Supported in part by the Science Research Foundation of Civil Aviation University of China(No.2010kys02) and supported by Fundamental Research Funds for the Central Universities (GRANT:ZXH2009D021).

by Rockafellar [18]. If $H = \mathbb{R}^m$, then problem (1) becomes the generalized equation introduced by Robinson [19]. It is known that (1) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, and so forth. Also various types of variational inclusions problems have been extended and generalized (see [1] and the references therein.)

Let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2}$$

The set of solutions of (2) is denoted by $EP(F)$. The problem (2) is very general in the sense that it inclues, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; for more details (see [3]).

Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\beta \in (0, 1)$ such that

$$\|fx - fy\| \leq \beta\|x - y\|, \quad \forall x, y \in C.$$

Moudafi [12] introduced the viscosity approximation method for nonexpansive mapping. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \ n \geq 0, \tag{3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Xu [23] proved that under certain appropriate condition on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (3) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \ for \ x \in C$$

where $C = Fix(T)$.

In [22], sequence $\{x_n\}$ defined by the iterative method below with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \ n \geq 0,$$

where A is strongly positive bounded linear operator. That is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Marino and Xu [13] consider the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \ n \geq 0,$$

it is proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (\gamma f - A)\tilde{x}, x - \tilde{x} \rangle \leq 0, \ x \in C,$$

Some methods have been proposed to solve the variational inclusion, equilibrium problem; see, for instance, [14] [17] and the references therein. Recently, S. Plubtieng and W. Sriprad [17] introduced the following iterative scheme for finding a common element of the set of solutions to the problem (1), the set of solutions of an equilibrium problem, and the set of fixed points problem of nonexpansive mappings in Hilbert space. Starting with $x_1 \in H$, define sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)S_n y_n, \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \quad \forall n \geq 0, \end{cases} \tag{4}$$

for all $n \in \mathbb{N}$, B is a strongly positive bounded linear operator, where $\lambda \in (0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1]$, and $\{r_n\} \subset (0, \infty)$, $\{S_n\}$ is a sequence of nonexpansive mappings on H. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$,the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ generated by (4) converge strongly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \bigcap I(A, M) \bigcap EP(F)$, where $z = P_\Omega(I - B + \gamma f)(z)$.

A mapping $T : C \to H$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Note that the class of k-strictly pseudo-contractions strictly includes the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. It is also said to be pseudo-contractive if $k = 1$. Clearly, the class of k-strictly pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions.

Lots of methods have been proposed to solve the equilibrium problem and fixed point problem; see, for instance, [8] [9] [11] [15] [16] and the references therein. In [11], Ying Liu introduced the following iterative scheme. Let S be a k-strictly pseudo-contractive nonself mapping, B be a strongly positive bounded linear operator, $x_1 \in C$ and let $\{x_n\}$ be generated by

$$\begin{cases} F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n u_n + (1 - \beta_n)S u_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, \quad \forall n \in \mathbb{N}. \end{cases}$$

Under suitable conditions, they proved that the sequence $\{x_n\}$ converge strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)q, p - q\rangle \geq 0, \quad \forall p \in F(S) \cap EP(F).$$

On the other hand, Yamada [24] introduced the following hybrid iterative method for solving the variational inequality

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n), \ n \geq 0,$$

where F be a k-Lipschitz and η-strongly monotone operator on H with $k > 0$, $\eta > 0$ and $0 < \mu < 2\eta/k^2$, then he proved that if $\{\lambda_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle F\tilde{x}, x-\tilde{x}\rangle \geq 0,\ x\in C.$$

Very recently, Tian [20] proposed a general iterative method for nonexpansive mappings that contains algorithms defined by Marino, Xu and Yamada:

$$x_{n+1}=\alpha_n\gamma f(x_n)+(I-\mu\alpha_n F)Tx_n,\ n\geq 0.$$

To the best of our knowledge, it would probably be the first time in the literature that we clarify the relationship between Yamada's method and viscosity iteration correctly. Then extend and generalized the iterative method introduced by Tian [20] and consider the following general iterative method:

$$\begin{cases} F'(u_n,y)+\frac{1}{\lambda_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall y\in C\\ y_n=J_{M,\lambda}(u_n-\lambda Au_n),\\ z_n=\beta_n y_n+(1-\beta_n)Ty_n,\\ x_{n+1}=\alpha_n\gamma g(x_n)+(I-\mu\alpha_n F)z_n, \quad \forall n\in\mathbb{N}, \end{cases}$$

for all $n\in\mathbb{N}$, where $\lambda\in(0,2\alpha]$, $\{\alpha_n\}\subset[0,1]$, and $\{\lambda_n\}\subset(0,\infty)$; where g is K-Lipschitz mapping on H with coefficient $K>0$, F is L-Lipschitzian and η-strongly monotone operator, and T be a k-strictly pseudo-contractive nonself mapping. Under suitable conditions, some strong convergence theorems for approximating to this common elements are proved.

2. Preliminaries

Throughout this paper, we always assume that C is a nonempty closed subset of a Hilbert space H. We write $x_n\rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n\to x$ imply that $\{x_n\}$ converges strongly to x. We denoted by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integer and real numbers, respectively.

In a Hilbert space H, it is well known that for all $x,y\in H$ and $\lambda\in[0,1]$, there holds

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2.$$

For any $x\in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\| x-P_C x\|\leq\| x-y\|, \forall y\in C$. Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Furthermore, for $x\in H$ and $u\ inC$,

$$u=P_C x\Longleftrightarrow\langle x-u,u-y\rangle\geq 0, \quad \forall y\in C. \tag{5}$$

It is easy to see that (5) is equivalent to

$$\|x-y\|^2\geq\|x-u\|^2+\|y-u\|^2, \quad \forall x\in H, \quad y\in C.$$

It is also known that H satisfies Opial condition [21], that is, for any sequence $\{x_n\}\subset H$ with $x_n\rightharpoonup x$, as $n\to\infty$, the inequality

$$\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|,$$

holds for every $y\in H$ with $y\neq x$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0, \quad \forall x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0, \quad \forall x,\ y \in C$;

(A3) for each $x,\ y,\ z \in C$,

$$\lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C,\ \ y \mapsto F(x, y)$ is convex and low semicontinuous.

Recall that a mapping $A : H \to H$ is called α-inverse strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x,\ y \in H.$$

Let I be the identity mapping on H. It is well known that if $A : H \to H$ is an α-inverse strongly monotone, then A is $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping. In addition, if $0 < \lambda < 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping.

A set-valued $M : H \to 2^H$ is called monotone if for all $x,\ y \in H,\ f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if its graph $G(M) := \{(x, f) \in H \times H \mid f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Let the set-valued mapping $M : H \to 2^H$ be maximal monotone. We defined the resolvent operator $J_{M,\lambda}$ associated with M and λ as follows:

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H, \tag{6}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpasive, and 1-inverse strongly monotone, see for example [5] and that a solution of problem (1) is a fixed point of the operator $J_{M,\lambda}(I - \lambda A)$ for all $\lambda > 0$, see for instance [10].

The following lemmas are useful for our paper.

Lemma 1. ([6]) *Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, if

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0,\ \forall y \in C\},$$

then the following hold:

(1) *T_r is single-valued;*

(2) T_r *is firmly nonexpansive, that is, for any* $x,\ y \in H$,

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle, \quad \forall x,\ y \in H;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Lemma 2. ([25]) *If* $S : C \to H$ *is a* k*-strict pseudo-contraction, then the fixed point set* $F(S)$ *is closed convex so that the projection* $P_{F(S)}$ *is well defined.*

Lemma 3. ([4]) *Let* $S : C \to H$ *be* k*-strict pseudo-contraction. Define* $T : C \to H$ *by* $Tx = \lambda x + (1-\lambda)Sx$ *for each* $x \in C$*. Then, as* $\lambda \in [k, 1)$, T *is a nonexpansive mapping such that* $F(T) = F(S)$.

Lemma 4. ([7]) *In a Hilbert space* H*, there holds the inequality:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Lemma 5. ([5]) *Let* $M : H \to 2^H$ *be a maximal monotone mapping and* $A : H \to H$ *be a Lipschitz continuous mapping. Then the mapping* $S = M + A : H \to 2^H$ *is a maximal monotone mapping.*

Remark ([14]) Lemma 2.5 implies that $I(A, M)$ is closed and convex if $M : H \to 2^H$ is a maximal monotone mapping and $A : H \to H$ be an inverse strongly monotone mapping.

Lemma 6. ([21]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \ge 0,$$

where γ_n *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\ \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} \mid \delta_n \mid < \infty$.

Then $\lim_{n\to\infty}\ a_n = 0$.

Lemma 7. ([20]) *Let* H *be a Hilbert space,* $f : H \to H$ *a contraction with coefficient* $0 < \alpha < 1$*, and* F *is* k*-Lipschitz continuous operators and* η*-strongly monotone operator with* $k > 0$, $\eta > 0$*. Then for* $0 < \gamma < \mu\eta/\alpha$,

$$\langle x - y, (\mu F - \gamma f)x - (\mu F - \gamma f)y\rangle \ge (\mu\eta - \gamma\alpha)\|x - y\|^2, \quad \forall x,\ y \in H.$$

That is, $\mu F - \gamma f$ *is strongly monotone with coefficient* $\mu\eta - \gamma\alpha$.

Lemma 8. ([2]) *Let* H *be a Hilbert space,* K *be a closed convex subset of* H*. For any integer* $N \ge 1$*, assume that, for each* $1 \le i \le N$, $T_i : K \to H$ *is a* k_i*-strictly pseudo-contractive mapping for some* $0 \le k_i < 1$*. Assume that* $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$*. Then* $\sum_{i=1}^N \eta_i T_i$ *is a non-self-*k*-strictly pseudo-contractive mapping with* $k = \max\{k_i : 1 \le i \le N\}$.

Lemma 9. ([2]) *Let* $\{T_i\}_{i=1}^N$ *and* $\{\eta_i\}_{i=1}^N$ *be given as in Lemma 2.9. Suppose that* $\{T_i\}_{i=1}^N$ *has a common fixed point in* K*. Then* $F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^N F(T_i)$.

3. Main results

Throughout the rest of this paper, we always assume that g is a K-Lipschitz mapping on H with coefficient $K > 0$, and F is L-Lipschitz and η-strongly monotone operator on H with $L > 0$, $\eta > 0$. Let $T : C \to H$ be k-strict pseudo-contractive nonself mapping. Let $A : H \to H$ be a $\alpha-$inverse strongly monotone mapping, $M : H \to 2^H$ be a maximal monotone mapping and let $J_{M,\lambda}$ be defined as in (6). Let $0 < \mu < 2\eta/L^2$, $0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/K = \tau/K$. Let $\{T_{\lambda_n}\}$ be a sequence of mappings defined as Lemma 1. Define a mapping $S_n : C \to H$ by $S_n x = \beta_n x + (1 - \beta_n)Tx,\ \forall x \in C$, where $\beta_n \in [k, 1)$. Then,by Lemma 3, S_n is nonexpansive and $F(S_n) = F(T)$.

Proposition. Let H be a real Hilbert space, let S be a nonexpansive mapping on H. Let F be a L-Lipschitz and η-strongly monotone operator on H with $L > 0$, $\eta > 0$ and $0 < \mu < 2\eta/L^2$, then Yamada's method: $x_0 \in H$ arbitrarily, $x_{n+1} = (I - \mu\alpha_n F)Sx_n$ belongs to the viscosity iteration method.

Proof. We make a transformation

$$\begin{aligned} x_{n+1} &= (I - \mu\alpha_n F)Sx_n - \alpha_n Sx_n + \alpha_n Sx_n \\ &= \alpha_n(I - \mu F)Sx_n + (1 - \alpha_n)Sx_n, \end{aligned}$$

let $T = (I - \mu F)S$, then for $\forall x,\ y \in H$, we have

$$\begin{aligned} \|Tx - Ty\| &= \|(I - \mu F)Sx - (I - \mu F)Sy\| \\ &\le [1 - \mu(\eta - \frac{\mu k^2}{2})]\|x - y\|. \end{aligned}$$

So T is a contraction, hence Yamada's method is also a viscosity iteration. □

Theorem 1. *Let H be a real Hilbert space, let F' be a bifunction from $H \times H \to R$ satisfying (A1)-(A4) and let $T : C \to H$ be a k-strict pseudo-contractive mapping. Let $A : H \to H$ be a α-inverse strongly monotone mapping, $M : H \to 2^H$ be a maximal monotone mapping such that $\Omega := F(T) \cap EP(F') \cap I(A, M) \neq \emptyset$. Let g be a K-Lipschitz mapping on H with coefficient $K > 0$ and let F be L-Lipschitz and η-strongly monotone operator on H with $L > 0$, $\eta > 0$ and $0 < \mu < 2\eta/L^2$, $0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/K = \tau/K$. Let $\{x_n\}$ be sequence generated by $x_1 \in H$ and*

$$\begin{cases} F'(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \forall y \in C \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ z_n = \beta_n y_n + (1 - \beta_n)Ty_n, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \mu\alpha_n F)z_n, \forall n \in \mathbb{N}, \end{cases} \tag{7}$$

where $u_n = T_{\lambda_n} x_n$, $z_n = S_n y_n$, $\lambda \in (0, 2\alpha]$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $0 < k < \beta_n < \lambda < 1$, $\lim_{n\to\infty} \beta_n = \lambda$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii) $\{\lambda_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then, $\{x_n\}$ *converges strongly to a point* $z \in \Omega := F(T) \cap EP(F') \cap I(A, M)$, *which solves the variational inequality:*

$$\langle(\gamma g - \mu F)z, x - z\rangle \le 0, \quad \forall x \in \Omega. \tag{8}$$

Equivalently, $z = P_\Omega(I + \gamma g - \mu F)z$.

Proof. Take $v \in F(T) \cap EP(F') \cap I(A, M) = \Omega$. Since $u_n = T_{\lambda_n} x_n$, and $v = T_{\lambda_n} v$, then, from Lemma 1, we know that, for any $n \in \mathbb{N}$,

$$\|u_n - v\| = \|T_{\lambda_n} x_n - T_{\lambda_n} v\| \le \|x_n - v\|. \tag{9}$$

We note from $v \in \Omega$ that $v = J_{M,\lambda}(v - \lambda Av)$. As $I - \lambda A$ is nonexpansive, we have

$$\begin{aligned} \|y_n - v\| &= \|J_{M,\lambda}(u_n - \lambda Au_n) - J_{M,\lambda}(v - \lambda Av)\| \\ &\le \|(u_n - \lambda Au_n) - (v - \lambda Av)\| \\ &\le \|u_n - v\| \\ &\le \|x_n - v\|. \end{aligned} \tag{10}$$

Further, since $S_n v = v$, we have

$$\|z_n - v\| = \|S_n y_n - S_n v\| \le \|y_n - v\| \le \|u_n - v\| \le \|x_n - v\|.$$

Then, we have

$$\begin{aligned} \|x_{n+1} - v\| &= \|\alpha_n \gamma g(x_n) + (I - \mu\alpha_n F)z_n - v\| \\ &= \|\alpha_n(\gamma g(x_n) - \mu Fv) + (I - \mu\alpha_n F)z_n - (I - \mu\alpha_n F)v\| \\ &\le \alpha_n\|\gamma g(x_n) - \mu Fv\| + (1 - \alpha_n\tau)\|z_n - v\| \\ &\le \alpha_n\|\gamma(g(x_n) - g(v)) + (\gamma g(v) - \mu Fv)\| + (1 - \alpha_n\tau)\|x_n - v\| \\ &\le \alpha_n\gamma K\|x_n - v\| + \alpha_n\|\gamma g(v) - \mu Fv\| + (1 - \alpha_n\tau)\|x_n - v\| \\ &= (1 - \alpha_n(\tau - \gamma K))\|x_n - v\| + \alpha_n\|\gamma g(v) - \mu Fv\| \\ &\le \max\{\|x_n - v\|, \frac{1}{\tau - \gamma K}\|\gamma g(v) - \mu Fv\|\}. \end{aligned} \tag{11}$$

It follows from (11) and induction that

$$\|x_n - v\| \le \max\{\|x_1 - v\|, \frac{1}{\tau - \gamma K}\|\gamma g(v) - \mu Fv\|\}, \ n \in \mathbb{N}.$$

Hence $\{x_n\}$ is bounded and therefore $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{g(x_n)\}$, $\{Au_n\}$, $\{Fz_n\}$ are also bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. We have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n \gamma g(x_n) + (I - \mu\alpha_n F)z_n - \alpha_{n-1}\gamma g(x_{n-1}) \\
&\quad - (I - \mu\alpha_{n-1}F)z_{n-1}\| \\
&= \|\alpha_n \gamma g(x_n) - \alpha_n \gamma g(x_{n-1}) + \alpha_n \gamma g(x_{n-1}) - \alpha_{n-1}\gamma g(x_{n-1}) \\
&\quad + (I - \mu\alpha_n F)z_n - (I - \mu\alpha_n F)z_{n-1} + (I - \mu\alpha_n F)z_{n-1} \\
&\quad - (I - \mu\alpha_{n-1}F)z_{n-1}\| \\
&\le \alpha_n\gamma\|g(x_n - g(x_{n-1}))\| + |\alpha_n - \alpha_{n-1}|\gamma\|g(x_{n-1})\| \\
&\quad + \|(I - \mu\alpha_n F)z_n - (I - \mu\alpha_n F)z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\mu\|Fz_{n-1}\| \\
&\le \alpha_n\gamma K\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K_1 + (1 - \alpha_n\tau)\|z_n - z_{n-1}\|,
\end{aligned} \tag{12}$$

where $K_1 = \sup\{\gamma\|g(x_n)\| + \mu\|Fz_n\| : n \in \mathbb{N}\} < \infty$. On the other hand, we note that

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \|S_n y_n - S_{n-1}y_{n-1}\| \\
&\le \|S_n y_n - S_n y_{n-1}\| + \|S_n y_{n-1} - S_{n-1}y_{n-1}\| \\
&\le \|y_n - y_{n-1}\| + \|S_n y_{n-1} - S_{n-1}y_{n-1}\| \\
&\le \|\beta_n y_{n-1} + (1 - \beta_n)Ty_{n-1} - (\beta_{n-1}y_{n-1} + (1 - \beta_{n-1})Ty_{n-1})\| \\
&\quad + \|y_n - y_{n-1}\| \\
&\le \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|K_2,
\end{aligned} \tag{13}$$

where $K_2 = \sup\{\|y_n - Ty_n\| : n \in \mathbb{N}\} < \infty$. Since $I - \lambda A$ is nonexpansive, it follows that

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|J_{M,\lambda}(u_n - \lambda Au_n) - J_{M,\lambda}(u_{n-1} - \lambda Au_{n-1})\| \\
&\le \|(u_n - \lambda Au_n) - (u_{n-1} - \lambda Au_{n-1})\| \\
&\le \|u_n - u_{n-1}\|.
\end{aligned} \tag{14}$$

From $u_n = T_{\lambda_n}x_n$ and $u_{n+1} = T_{\lambda_{n+1}}x_{n+1}$, we have

$$F(u_{n+1}, y) + \frac{1}{\lambda_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in C, \tag{15}$$

and

$$F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C. \tag{16}$$

Putting $y = u_n$ in (15) and $y = u_{n+1}$ in (16), we have

$$F(u_{n+1}, u_n) + \frac{1}{\lambda_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0,$$

and

$$F(u_n, u_{n+1}) + \frac{1}{\lambda_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0.$$

So, from (A2) we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{\lambda_n} - \frac{u_{n+1} - x_{n+1}}{\lambda_{n+1}}\rangle \ge 0.$$

Since $\lim_{n\to\infty}\lambda_n > 0$, we assume that there exists a real number a such that $\lambda_n > a > 0$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\le \langle u_{n+1}-u_n, x_{n+1}-x_n + (1-\frac{\lambda_n}{\lambda_{n+1}})(u_{n+1}-x_{n+1})\rangle \\ &\le \|u_{n+1}-u_n\|\{\|x_{n+1}-x_n\| + |1-\frac{\lambda_n}{\lambda_{n+1}}|\|u_{n+1}-x_{n+1}\|\},\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\| + \frac{1}{\lambda_{n+1}}|\lambda_{n+1}-\lambda_n|\|u_{n+1}-x_{n+1}\| \\ &\le \|x_{n+1}-x_n\| + \frac{1}{a}|\lambda_{n+1}-\lambda_n|K_3,\end{aligned} \tag{17}$$

where $K_3 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\} < \infty$. From (12) we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \alpha_n\gamma K\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K_1 + |\beta_n-\beta_{n-1}|K_2 \\ &\quad + (1-\alpha_n\tau)\|y_n-y_{n-1}\| \\ &\le \alpha_n\gamma K\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K_1 + |\beta_n-\beta_{n-1}|K_2 \\ &\quad + (1-\alpha_n\tau)\|u_n-u_{n-1}\| \\ &\le \alpha_n\gamma K\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K_1 + |\beta_n-\beta_{n-1}|K_2 \\ &\quad + \frac{K_3}{a}|\lambda_n-\lambda_{n-1}| + (1-\alpha_n\tau)\|x_n-x_{n-1}\| \\ &\le (1-\alpha_n(\tau-\gamma K))\|x_n-x_{n-1}\| \\ &\quad + K_4(|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}| + |\lambda_n-\lambda_{n-1}|),\end{aligned}$$

where $K_4 = \max\{K_1, K_2, \frac{K_3}{a}\}$. Hence, by Lemma 7, we have $\|x_{n+1}-x_n\| \to 0$, as $n \to \infty$. From (17) and $|\lambda_{n+1}-\lambda_n| \to 0$, as $n \to \infty$ we have $\|u_{n+1}-u_n\| \to 0$, as $n \to \infty$. Moreover, we have from (14) that $\|y_{n+1}-y_n\| \to 0$, as $n \to \infty$. From (13) we have $\|z_{n+1}-z_n\| \to 0$, as $n \to \infty$.

Now, we prove that for any given $v \in \Omega$, $\|Au_n - Av\| \to 0$, as $n \to \infty$. It follows from Lemma 1 that

$$\begin{aligned}\|u_n-v\|^2 &= \|T_{\lambda_n}x_n - T_{\lambda_n}v\|^2 \\ &\le \langle T_{\lambda_n}x_n - T_{\lambda_n}v, x_n - v\rangle \\ &= \langle u_n - v, x_n - v\rangle \\ &= \frac{1}{2}(\|u_n-v\|^2 + \|x_n-v\|^2 - \|x_n-u_n\|^2),\end{aligned}$$

So, we have

$$\|u_n-v\|^2 \le \|x_n-v\|^2 - \|x_n-u_n\|^2. \tag{18}$$

Therefore, we have

$$\begin{aligned}
\|x_{n+1}-v\|^2 &= \|\alpha_n(\gamma g(x_n)-\mu Fv)+(I-\mu\alpha_n F)z_n-(I-\mu\alpha_n F)v\|^2\\
&\le (1-\alpha_n\tau)^2\|z_n-v\|^2+2\alpha_n\langle\gamma g(x_n)-\mu Fv, x_{n+1}-v\rangle\\
&\le (1-\alpha_n\tau)^2\|y_n-v\|^2+2\alpha_n\langle\gamma g(x_n)-\mu Fv, x_{n+1}-v\rangle\\
&\le (1-\alpha_n\tau)^2\|y_n-v\|^2+2\alpha_n\gamma\langle g(x_n)-g(v), x_{n+1}-v\rangle\\
&\quad+2\alpha_n\langle\gamma g(v)-\mu Fv, x_{n+1}-v\rangle\\
&\le (1-\alpha_n\tau)^2\|y_n-v\|^2+2\alpha_n\gamma K\|x_n-v\|\|x_{n+1}-v\|\\
&\quad+2\alpha_n\|\gamma g(v)-\mu Fv\|\|x_{n+1}-v\| \qquad (19)\\
&\le (1-\alpha_n\tau)^2\|u_n-v\|^2+2\alpha_n\gamma K\|x_n-v\|\|x_{n+1}-v\|\\
&\quad+2\alpha_n\|\gamma g(v)-\mu Fv\|\|x_{n+1}-v\|\\
&\le (1-\alpha_n\tau)^2(\|x_n-v\|^2-\|x_n-u_n\|^2)+2\alpha_n\gamma K\|x_n-v\|\|x_{n+1}-v\|\\
&\quad+2\alpha_n\|\gamma g(v)-\mu Fv\|\|x_{n+1}-v\|\\
&\le (1-\alpha_n\tau)^2(\|x_n-v\|^2-\|x_n-u_n\|^2)+\alpha_n\gamma K\|x_n-v\|^2\\
&\quad+\alpha_n\gamma K\|x_{n+1}-v\|^2+2\alpha_n\|\gamma g(v)-\mu Fv\|\|x_{n+1}-v\|,
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n\tau)^2\|x_n-u_n\|^2 &\le \alpha_n\tau^2\|x_n-v\|^2+2\alpha_n\|\gamma g(v)-\mu Fv\|\|x_{n+1}-v\|\\
&\quad+\alpha_n\gamma K\|x_n-v\|+(\|x_n-v\|^2-\|x_{n+1}-v\|^2)\\
&\le \alpha_n\tau^2\|x_n-v\|^2+2\alpha_n\|\gamma g(v)-\mu Fv\|\|x_{n+1}-v\|\\
&\quad+\|x_{n+1}-x_n\|(\|x_n-v\|+\|x_{n+1}-v\|)\\
&\quad+\alpha_n\gamma K\|x_n-v\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded, $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$, as $n\to\infty$, it follows that $\|x_n-u_n\|\to 0$, as $n\to\infty$.

Put $M_1=\sup\{\|\gamma g(v)-\mu Fv\|\|x_n-v\| : n\in\mathbb{N}\}$, it follows from (19), the nonexpansive of $J_{M,\lambda}$ and the inverse strongly monotone of A that

$$\begin{aligned}
\|x_{n+1}-v\|^2 &\le (1-\alpha_n\tau)^2\|y_n-v\|^2+2\alpha_n\gamma K\|x_n-v\|\|x_{n+1}-v\|+2\alpha_n M_1\\
&\le (1-\alpha_n\tau)^2\|(u_n-\lambda Au_n)-(v-\lambda Av)\|^2+2\alpha_n M_1\\
&\quad+\alpha_n\gamma K(\|x_n-v\|^2+\|x_{n+1}-v\|^2)\\
&\le (1-\alpha_n\tau)^2(\|u_n-v\|^2+\lambda(\lambda-2\alpha)\|Au_n-Av\|^2)+2\alpha_n M_1\\
&\quad+\alpha_n\gamma K(\|x_n-v\|^2+\|x_{n+1}-v\|^2)\\
&\le (1-\alpha_n\tau)^2\|x_n-v\|^2+(1-\alpha_n\tau)^2\lambda(\lambda-2\alpha)\|Au_n-Av\|^2\\
&\quad+\alpha_n\gamma K(\|x_n-v\|^2+\|x_{n+1}-v\|^2)+2\alpha_n M_1\\
&\le (1-\alpha_n(2\tau-\gamma K)+(\alpha_n\tau)^2)\|x_n-v\|^2+\alpha_n\gamma K\ \|x_{n+1}-v\|^2\\
&\quad+2\alpha_n M_1+(1-\alpha_n\tau)^2\lambda(\lambda-2\alpha)\|Au_n-Av\|^2\\
&\le \|x_n-v\|^2+(1-\alpha_n\tau)^2\lambda(\lambda-2\alpha)\|Au_n-Av\|^2\\
&\quad+2\alpha_n M_1+\alpha_n\gamma K\ \|x_{n+1}-v\|^2+\alpha_n\tau^2\|x_n-v\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_n\tau)^2\lambda(2\alpha-\lambda)\|Au_n-Av\|^2 &\le \alpha_n\tau^2\|x_n-v\|^2+\alpha_n\gamma K\ \|x_{n+1}-v\|^2\\ &\quad+(\|x_n-v\|^2-\|x_{n+1}-v\|^2)+2\alpha_nM_1\\ &\le \alpha_n\tau^2\|x_n-v\|^2+\alpha_n\gamma K\ \|x_{n+1}-v\|^2\\ &\quad+\|x_{n+1}-x_n\|(\|x_n-v\|+\|x_{n+1}-v\|)\\ &\quad+2\alpha_nM_1.\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1}-x_n\| \to 0$, as $n \to \infty$, we have $\|Au_n - Av\| \to 0$, as $n \to \infty$.

Since $J_{M,\lambda}$ is 1-inverse-strongly monotone and $I-\lambda A$ is nonexpansive, we have

$$\begin{aligned}\|y_n-v\|^2 &= \|J_{M,\lambda}(u_n-\lambda Au_n)-J_{M,\lambda}(v-\lambda Av)\|^2\\ &\le \langle (u_n-\lambda Au_n)-(v-\lambda Av), y_n-v\rangle\\ &= \frac{1}{2}\{\|(u_n-\lambda Au_n)-(v-\lambda Av)\|^2+\|y_n-v\|^2\}\\ &\quad-\frac{1}{2}\|(v-\lambda Av)-(y_n-v)\|\\ &\le \frac{1}{2}\{\|u_n-v\|^2+\|y_n-v\|^2-\|u_n-y_n-\lambda(Au_n-Av)\|^2\}\\ &= \frac{1}{2}\{\|u_n-v\|^2+\|y_n-v\|^2-\|u_n-y_n\|^2\}\\ &\quad+2\lambda\langle u_n-y_n, Au_n-Av\rangle-\frac{1}{2}\lambda^2\|Au_n-Av\|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|y_n-v\|^2 \le \|u_n-v\|^2-\|u_n-y_n\|^2+2\lambda\langle u_n-y_n, Au_n-Av\rangle\\ -\lambda^2\|Au_n-Av\|^2.\end{aligned} \tag{20}$$

From (9), (20) and (19), we have

$$\begin{aligned}\|x_{n+1}-v\|^2 &\le (1-\alpha_n\tau)^2\|y_n-v\|^2+2\alpha_n\gamma K\|x_n-v\|\|x_{n+1}-v\|+2\alpha_nM_1\\ &\le (1-\alpha_n\tau)^2\{\|u_n-v\|^2-\|u_n-y_n\|^2+2\lambda\langle u_n-y_n, Au_n-Av\rangle\}\\ &\quad-(1-\alpha_n\tau)^2\lambda^2\|Au_n-Av\|^2+2\alpha_n\gamma K\|x_n-v\|\|x_{n+1}-v\|\\ &\quad+2\alpha_nM_1\\ &\le (1-\alpha_n\tau)^2\|x_n-v\|^2-(1-\alpha_n\tau)^2\|u_n-y_n\|^2+2\alpha_nM_1\\ &\quad-(1-\alpha_n\tau)^2\lambda^2\|Au_n-Av\|^2\\ &\quad+\alpha_n\gamma K(\|x_n-v\|^2+\|x_{n+1}-v\|^2)\\ &\quad+2(1-\alpha_n\tau)^2\lambda\langle u_n-y_n, Au_n-Av\rangle\\ &= (1-\alpha_n(2\tau-\gamma K)+(\alpha_n\tau)^2)\|x_n-v\|^2-(1-\alpha_n\tau)^2\|u_n-y_n\|^2\\ &\quad-(1-\alpha_n\tau)^2\lambda^2\|Au_n-Av\|^2+\alpha_n\gamma K\|x_{n+1}-v\|^2+2\alpha_nM_1\\ &\quad+2(1-\alpha_n\tau)^2\lambda\langle u_n-y_n, Au_n-Av\rangle\\ &\le \|x_n-v\|^2+\alpha_n\tau^2\|x_n-v\|^2-(1-\alpha_n\tau)^2\|u_n-y_n\|^2+2\alpha_nM_1\\ &\quad-(1-\alpha_n\tau)^2\lambda^2\|Au_n-Av\|^2+\alpha_n\gamma K\|x_{n+1}-v\|^2\\ &\quad+2(1-\alpha_n\tau)^2\lambda\langle u_n-y_n, Au_n-Av\rangle.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}(1-\alpha_n\tau)^2\|u_n-y_n\|^2 &\le \alpha_n\tau^2\|x_n-v\|^2+2(1-\alpha_n\tau)^2\lambda\langle u_n-y_n,Au_n-Av\rangle\\ &\quad-(1-\alpha_n\tau)^2\lambda^2\|Au_n-Av\|^2+2\alpha_nM_1\\ &\quad+\alpha_n\gamma K\|x_{n+1}-v\|^2+(\|x_n-v\|^2-\|x_{n+1}-v\|^2)\\ &\le \alpha_n\tau^2\|x_n-v\|^2+2(1-\alpha_n\tau)^2\lambda\langle u_n-y_n,Au_n-Av\rangle\\ &\quad-(1-\alpha_n\tau)^2\lambda^2\|Au_n-Av\|^2+2\alpha_nM_1\\ &\quad+\|x_{n+1}-x_n\|(\|x_n-v\|+\|x_{n+1}-v\|)\\ &\quad+\alpha_n\gamma K\|x_{n+1}-v\|^2.\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$ and $\|Au_n-Av\|\to 0$, as $n\to\infty$, we have $\|u_n-y_n\|\to 0$, as $n\to\infty$.

Notice that

$$x_{n+1}=\alpha_n\gamma g(x_n)+(I-\mu\alpha_nF)S_ny_n,$$

we have

$$\begin{aligned}\|x_n-S_ny_n\| &\le \|x_n-S_ny_{n-1}\|+\|S_ny_{n-1}-S_ny_n\|\\ &\le \alpha_{n-1}\|\gamma g(x_{n-1})-\mu FS_{n-1}y_{n-1}\|+\|S_{n-1}y_{n-1}-S_ny_{n-1}\|\\ &\quad+\|y_{n-1}-y_n\|\\ &\le \alpha_{n-1}\|\gamma g(x_{n-1})-\mu FS_{n-1}y_{n-1}\|+|\beta_n-\beta_{n-1}|\|y_{n-1}+Ty_{n-1}\|\\ &\quad+\|y_{n-1}-y_n\|.\end{aligned}$$

Since $\alpha_n\to 0$, $|\beta_n-\beta_{n-1}|\to 0$ and $\|y_{n+1}-y_n\|\to 0$, as $n\to\infty$, we have $\|x_n-S_ny_n\|\to 0$, as $n\to\infty$. Moreover, we note that

$$\|S_ny_n-y_n\|\le\|S_ny_n-x_n\|+\|x_n-u_n\|+\|u_n-y_n\|.$$

From $\|x_n-u_n\|\to 0$ and $\|u_n-y_n\|\to 0$, as $n\to\infty$, we have $\|S_ny_n-y_n\|\to 0$, as $n\to\infty$.

Next we show that

$$\limsup_{n\to\infty}\langle(\gamma g-\mu F)z,x_n-z\rangle\le 0,$$

where $z=P_\Omega(I+\gamma g-\mu F)z$ is a unique solution of the variational inequlity (8), we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle(\gamma g-\mu F)z,x_{n_i}-z\rangle=\limsup_{n\to\infty}\langle(\gamma g-\mu F)z,x_n-z\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to x. Without loss of generality, we can assume that $u_{n_i}\rightharpoonup x$, as $i\to\infty$. From $\|u_n-y_n\|\to 0$, as $n\to\infty$, we obtain $y_{n_i}\rightharpoonup x$, as $i\to\infty$.

Let we show $x \in F(T)$. Assume $x \notin F(S_n)$. Since $y_{n_i} \rightharpoonup x$, as $i \to \infty$, and $x \neq S_n x$, it follows from the Opial condition that

$$\begin{aligned}\liminf_{n\to\infty} \|y_{n_i} - x\| &< \liminf_{n\to\infty} \|y_{n_i} - S_n x\| \\ &\leq \liminf_{n\to\infty}(\|y_{n_i} - S_n y_{n_i}\| + \|S_n y_{n_i} - S_n x\|) \\ &\leq \lim_{n\to\infty} \|y_{n_i} - x\|.\end{aligned}$$

This is a contradiction. So, we get $x \in F(S_n)$ and hence $x \in F(T)$.

Let us show $x \in EP(F')$. It follows by (7) and (A2) that

$$\frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq F'(y, u_n).$$

Replacing n by n_i, we have

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \geq F'(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup x$, as $i \to \infty$, it follows from (A4) that $0 \geq F'(y, x)$ for all $y \in C$. Put $z_t = ty + (1-t)x$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$ and hence $F'(z_t, x) \leq 0$. So, from (A1) and (A4) we have

$$0 = F'(z_t, z_t) \leq tF'(z_t, y) + (1-t)F'(z_t, x) \leq tF'(z_t, y).$$

and hence $0 \leq F'(z_t, y)$. From (A3), we have $0 \leq F'(x, y)$ for all $y \in C$ and hence $x \in EP(F')$.

Next we show $x \in I(A, M)$. In fact, since A is α-inverse-strongly monotone, A is an $1/\alpha$-Lipschiz continuous monotone mapping and $D(A) = H$. It follows from Lemma 5 that $M + A$ is maximal monotone. Let $(p, f) \in G(M, A)$, that is, $f - Ap \in M(P)$. Again since $y_n = J_{M,\lambda}(u_n - \lambda A u_n)$, we have $u_n - \lambda A u_n \in (I + \lambda M)(y_n)$, that is

$$\frac{1}{\lambda}(u_n - y_n - \lambda A u_n) \in M(y_n).$$

By the maximal monotonicity of M, we have

$$\langle p - y_n, f - Ap - \frac{1}{\lambda}(u_n - y_n - \lambda A u_n)\rangle \geq 0,$$

and so

$$\begin{aligned}\langle p - y_n, f\rangle &\geq \langle p - y_n, Ap + \frac{1}{\lambda}(u_n - y_n - \lambda A u_n)\rangle \\ &= \langle p - y_n, Ap - Ay_n + Ay_n - Au_n + \frac{1}{\lambda}(u_n - y_n)\rangle \\ &\geq 0 + \langle p - y_n, Ay_n - Au_n\rangle + \langle p - y_n, \frac{1}{\lambda}(u_n - y_n)\rangle.\end{aligned}$$

It follows from $\|u_n - y_n\| \to 0$, $\|Au_n - Ay_n\| \to 0$ and $y_n \rightharpoonup x$, as $n \to \infty$, that

$$\lim_{n\to\infty} \langle p - y_n, f\rangle = \langle p - x, f\rangle \geq 0.$$

Since $A+M$ is maximal monotone, this implies that $\theta \in (M+A)(x)$, that is, $x \in I(A,M)$. Hence, $x \in \Omega := F(T) \cap EP(F') \cap I(M,A)$.

This implies that

$$\limsup_{n\to\infty}\langle(\gamma g-\mu F)z, x_n - z\rangle = \lim_{i\to\infty}\langle(\gamma g-\mu F)z, x_{n_i} - z\rangle = \langle(\gamma g-\mu F)z, x - z\rangle \le 0.$$

Finally we prove that $x_n \to z$, as $n \to \infty$. From (7), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n\gamma g(x_n) + (I-\mu\alpha_n F)S_n y_n - z\|^2\\ &= \|\alpha_n(\gamma g(x_n) - \mu F z) + (I-\mu\alpha_n F)S_n y_n - (I-\mu\alpha_n F)z\|^2\\ &\le (1-\alpha_n\tau)^2\|y_n - z\|^2 + 2\alpha_n\langle\gamma g(x_n) - \mu F z, x_{n+1}-z\rangle\\ &\le (1-\alpha_n\tau)^2\|y_n - z\|^2 + 2\alpha_n\gamma\langle g(x_n) - g(z), x_{n+1}-z\rangle\\ &\quad + 2\alpha_n\langle\gamma g(z) - \mu F z, x_{n+1}-z\rangle\\ &\le (1-\alpha_n\tau)^2\|x_n - z\|^2 + 2\alpha_n\gamma K\|x_n - z\|\|x_{n+1}-z\|\\ &\quad + 2\alpha_n\langle\gamma g(z) - \mu F z, x_{n+1}-z\rangle\\ &\le (1-\alpha_n\tau)^2\|x_n - z\|^2 + \alpha_n\gamma K(\|x_n - z\|^2 + \|x_{n+1}-z\|^2)\\ &\quad + 2\alpha_n\langle\gamma g(z) - \mu F z, x_{n+1}-z\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \frac{1-2\alpha_n\tau + (\alpha_n\tau)^2 + \alpha_n\gamma K}{1-\alpha_n\gamma K}\|x_n - z\|^2\\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma K}\langle\gamma g(z) - \mu F z, x_{n+1}-z\rangle\\ &\le [1 - \frac{2(\tau-\gamma K)\alpha_n}{1-\alpha_n\gamma K}]\|x_n - z\|^2 + \frac{(\alpha_n\tau)^2}{1-\alpha_n\gamma K}\|x_n - z\|^2\\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma K}\langle\gamma g(z) - \mu F z, x_{n+1}-z\rangle\\ &\le (1-r_n)\|x_n - z\|^2 + \delta_n,\end{aligned}$$

where $\gamma_n := (2\alpha_n(\tau-\gamma K))/(1-\alpha_n\gamma K)$ and $\delta_n := \alpha_n/(1-\alpha_n\gamma K)\{\alpha_n\tau^2\|x_n - z\|^2 + 2\langle\gamma g(z) - \mu F z, x_{n+1} - z\rangle\}$. It easily verified that $\gamma_n \to 0$, as $n\to\infty$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$. Hence, by Lemma 7, the sequence $\{x_n\}$ converges strongly to z.

□

Theorem 2. *Let H be a real Hilbert space, let F' be a bifunction from $H\times H \to R$ satisfying (A1)-(A4) and let $T_i : C \to H$ be a k_i-strict pseudo-contractive mapping for some $0 \le k_i < 1$ and $\cap_{i=1}^N F(T_i) \ne \emptyset$. Assume that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Let $A : H \to H$ be a $\alpha-$inverse strongly monotone mapping, $M : H \to 2^H$ be a maximal monotone mapping such that $\Omega := F(T) \cap EP(F') \cap I(A,M) \ne \emptyset$. Let g be a K-Lipschitz mapping on H with* *coefficient $K > 0$ and let F be L-Lipschitz and η-strongly monotone operator on H with $L > 0$, $\eta > 0$ and $0 < \mu < 2\eta/L^2$, $0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/K = \tau/K$. Let $\{x_n\}$ be sequence generated by $x_1 \in H$ and*

$$\begin{cases} F'(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ z_n = \beta_n y_n + (1 - \beta_n)\sum_{i=1}^N \eta_i T_i y_n, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \mu\alpha_n F) z_n, \forall n \in \mathbb{N}, \end{cases}$$

where $u_n = T_{\lambda_n} x_n$, $\lambda \in (0, 2\alpha]$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $0 < k < \beta_n < \lambda < 1$, $\lim_{n\to\infty} \beta_n = \lambda$ *and* $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$;

(iii) $\{\lambda_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$.

Then, $\{x_n\}$ converges strongly to a point $z \in \Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F') \cap I(A, M)$, which solves the variational inequality (8).

Proof. Define a mapping $T : C \to H$ by $Tx = \sum_{n=1}^N \eta_i T_i x$. By Lemma 9 and Lemma 10, we conclude that $T : C \to H$ is a k-strict pseudo-contractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$ and $F(T) = F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^N F(T_i)$. From Theorem 1, we obtain desired conclusion easily. This completes the proof. □

Remark 1. If T is nonexpansive, F is strongly positive bounded linear operator, $\beta_n = 0$, $\mu = 1$, $F = I$ and g is a contractive mapping, then Theorem 1 reduces to Theorem 3.1 of Peng, Wang, Shyu and Yao [14].

Remark 2. If F is strongly positive bounded linear operator, $\mu = 1$, $A \equiv 0$, $M \equiv 0$ and g is a contractive mapping, then Theorem 1 reduces to Theorem 3.2 of Liu [11].

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