J. Appl. Math. & Informatics Vol. **29***(2011), No. 3 - 4, pp. 587 - 601*
Website: http://www.kcam.biz

CONTROLLABILITY OF STOCHASTIC FUNCTIONAL INTEGRODIFFERENTIAL EVOLUTION SYSTEMS

J. KOKILA* AND K. BALACHANDRAN

ABSTRACT. In this paper, we prove the existence and uniqueness of mild solution for stochastic functional integrodifferential evolution equations and derive sufficient conditions for the controllability results. As an illustration we consider the controllability for a system governed by a random motion of a string.

AMS Mathematics Subject Classification : 93B05, 60H15.
Key words and phrases : Controllability, Stochastic evolution equations, Integrodifferential equations.

1. Introduction

The purpose of this paper is to study the controllability of stochastic functional integrodifferential evolution equations of the form

$$\begin{aligned} dX_t = \left[AX_t + F\left(t, X_t, \int_0^t h(t,s,X_s)ds\right) + Bu(t)\right] dt \\ + G\left(t, X_t, \int_0^t k(t,s,X_s)ds\right) d\mathcal{W}_t, \quad t \in [0,T] := J \end{aligned} \tag{1}$$

$$X_0 = x_0 \tag{2}$$

defined on a Hilbert space H and where A is a linear unbounded operator, F, G, h, k are nonlinear functions defined later. $\mathcal{W}_t$ is a cylindrical Wiener process and $x_0 \in H$. The state variable $x(\cdot)$ takes its values in the Hilbert space H. The control function $u(\cdot)$ is in $L_2(J; U)$, the Hilbert space of admissible control functions with U a Hilbert space. B is a bounded linear operator from U into H.

Received July 30, 2010. Revised September 30, 2010. Accepted October 12, 2010.
*Corresponding author.

The present work may be regarded as a direct attempt to extend the recent paper Jentzen and Kloeden [11]. In that paper, the authors proved the existence and uniqueness of the following stochastic evolution equations in a unified way

$$\begin{aligned} dX_t &= [AX_t + F(X_t)]dt + B(X_t)dW_t, \\ X_0 &= x_0. \end{aligned}$$

A complete understanding of stochastic differential equations theory requires familiarity with advanced probability and stochastic processes. By incorporating random elements in the differential equation (either in the initial or boundary conditions for the problem or in the function describing the physical system) a stochastic differential equation arises. Physical systems are often modelled by ordinary differential equations, however such models may represent idealised situations as they ignore stochastic effects. Stochastic differential equation models play a prominent role in a range of application areas, including biology, chemistry, epidemology, mechanics, microelectronics, economics and finance.

For example, consider the velocity of a particle X_t in a direction represented by the Langevin equation

$$\frac{dX_t}{dt} = -aX_t + b\xi_t,$$

where aX_t denotes the velocity depending force and $b\xi_t$ the molecular force with intensity b driven by a white noise process ξ_t. In this stochastic model, it is assumed that external forces do not depend on the state X_t of the system. Symbolically, the above equation is written as stochastic differential equation of the form

$$dX_t = -aX_t dt + b\mathcal{W}_t,$$

which is a short-hand notation of the integral equation

$$X_t = X_0 - \int_0^t aX_s ds + \int_0^t bd\mathcal{W}_s.$$

The mild solution of such integral equations are in the form of stochastic integral equations. The solution of a stochastic differential equations however, inherits the nondifferentiability of sample paths from the stochastic process. Mathematical modelling of real life problems usually results in functional equations, such as ordinary or partial differential equations, integrodifferential equations and stochastic equations. The functional integrodifferential equations serve as an abstract formulation of partial functional integrodifferential equations which arise in heat flow in material with memory (see, [12]). The theory of stochastic evolution equations encounter all of difficulties due to the infinite dimensional nature of the noise processes. In many problems in almost all areas of science and engineering there are real phenomena depending on the effect of white noise random forces. These problems are intrinsically nonlinear, complex in nature and atleast mathematically modelled and described by various generalized stochastic differential and integrodifferential equations.

The stochastic control theory is a generalization of the classical control theory. Controllability of nonlinear stochastic systems has been a well-known problem and frequently discussed in the literature (Aström [1], Wonham [17], Zabczyk [18], Balasubramaniam et. al [7]. Dauer and Mahmudov [10], Mahmudov [13] investigated various controllability questions for semilinear stochastic differential equations. Stochastic controllability results for both semilinear and quasilinear evolution equations has been studied by Balasubramaniam [5, 6]. Much effort has been devoted to the study of controllability for stochastic integrodifferential equations in both finite and infinite dimensional settings. Balachandran and Karthikeyan [2], Balachandran et. al [3,4] derived sufficient conditions for the controllability of stochastic integrodifferential systems in finite dimensional spaces whereas, Subalakshmi and Balachandran [16] studied controllability of the following functional integrodifferential system in infinite dimensional spaces

$$\begin{aligned} d(X(t)) &= \left[-AX(t) + Bu(t) + f\left(t, X_t, \int_0^t g(t,s,X_s)ds\right)\right]dt \\ &\quad +\sigma\left(t, X_t, \int_0^t g(t,s,X_s)ds\right)dW(t), \quad t \in [0,T] \\ X_0 &= \Phi. \end{aligned}$$

This paper deals with the K-valued Wiener process with a finite trace nuclear operator $Q \geq 0$ and the results are obtained by using Banach fixed point theorem.

In the present work, we deal with the cylindrical Q-Wiener process with the co-variance operator $Q = I$, the identity (see, for example [8, 11, 14]).

The outline of the paper is as follows: section 2 describes the notations, terms and conditions of the problem. Section 3 is devoted to the development of our main controllability results. Finally, section 4 concludes by giving an application to the controllability problem for a system with distributed parameters governed by the random motion of a string under the action of controls.

2. Preliminaries

Fix $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $\mathcal{F}_t$, $t \in J$. An H-valued random variable is an $\mathcal{F}$-measurable function $X_t : \Omega \to H$ and a collection of random variables $S = \{X(t,w) : \Omega \to H \, |t \in J\}$ is called a stochastic process. Usually we suppress the dependence on $w \in \Omega$ and write X_t instead of $X(t,w)$ and $X_t : J \to H$ in the place of S. Let $(H, \langle\cdot,\cdot\rangle_H)$, $(V, \langle\cdot,\cdot\rangle_V)$ be two separable $\mathbb{R}$-Hilbert spaces with norm denoted by $\|\cdot\|_H$, $\|\cdot\|_V$ respectively. Let $(D, \|\cdot\|_D)$ be a separable $\mathbb{R}$-Banach space of all bounded linear operator from V to D. $\mathcal{W}_t$, $t \in J$ is a cylindrical Q-Wiener process on V with respect to $\mathcal{F}_t$, $t \in J$ for which the co-variance operator $Q = I$ the identity on V (see [8, 14]). Let us assume the following hypothesis on the data of the problem :

$(H1)$ Let $\mathcal{I}$ be a finite or countable set. Moreover, let $(\lambda_i)_{i\in\mathcal{I}}$ be a family of positive real numbers with $\inf\limits_{i\in\mathcal{I}} \lambda_i > 0$ and let $(e_i)_{i\in\mathcal{I}}$ be an orthonormal basis of H. Then, suppose that the linear operator $A : D(A) \subset H \to H$ is given by

$$Av = \sum_{i\in\mathcal{I}} (-\lambda_i)\langle e_i, v\rangle_H e_i,$$

for all $v \in D(A)$ with $D(A) = \left\{ v \in H; \sum_{i\in\mathcal{I}} \|\lambda_i\|^2 |\langle e_i, v\rangle_H|^2 < \infty \right\}$.

$(H2)$ The mapping $F : J \times H \times H \to H$ is globally Lipschitz continuous with respect to $\|\cdot\|_H$ with the function $h : J \times J \times H \to H$ such that

$$\begin{aligned} \|F(t, v, v_1) - F(t, w, w_1)\|_H &\leq L[\|v - w\|_H + \|v_1 - w_1\|_H] \\ \left\| \int_0^t [h(t, s, v) - h(t, s, w)]ds \right\|_H &\leq L_1\|v - w\|_H \end{aligned}$$

where $v, v_1, w, w_1 \in H$ and $L, L_1 < 0$.

Let $D((-A)^m)$, $m \in \mathbb{R}$, denote the interpolation spaces of powers of the operator $-A$ (see, [9, 15]) and let $\|\cdot\|_{HS}$ denote the Hilbert-Schmidt norm for Hilbert-Schmidt operator from V to H.

$(H3)$ Suppose that $D \subset D((-A)^m)$ continuously for some $m \geq 0$ and that $G : J\times H\times H \to L(V, D)$ with $k : J\times J\times H \to H$ is a strongly measurable mapping such that $e^{At}G(t, v, w)$ is a Hilbert-Schmidt operator from V to H and

$$\begin{aligned} \|e^{At}G(t, v, w)\|_{HS} &\leq M[1 + \|v\|_H + \|w\|_H]t^{\epsilon-1/2} \\ \|e^{At}[G(t, v, v_1) - G(t, w, w_1)]\|_{HS} &\leq M[\|v - w\|_H + \|v_1 - w_1\|_H]t^{\epsilon-1/2} \\ \left\| \int_0^t [k(t, s, v) - k(t, s, w)]ds \right\|_H &\leq M_1\|v - w\|_H \end{aligned}$$

for all $v, v_1, w, w_1 \in H$, $t \in J$ and $M, M_1 > 0$ are given constants.

If $G : J\times H\times H \to L(V, D)$ is assumed to be $\mathcal{B}(J\times H\times H)\backslash\mathcal{B}(L(V, D))$-measurable, then in particular, it is strongly measurable.

$(H4)$ Let $p \in [2, \infty)$ be given and suppose that $x_0 : \Omega \to H$ is a $\mathcal{F}_0\backslash\mathcal{B}(H)$-measurable mapping with $\mathbb{E}\|x_0\|_H^p < \infty$.

$(H5)$ The linear operator $W : L^2(J, U) \to H$ is defined by

$$Wu = \int_0^T e^{A(t-s)}Bu(s)ds$$

has an invertible operator W^{-1} defined on $H\backslash KerW$ and there exists a positive constant C_1 such that

$$\|BW^{-1}\| \leq C_1.$$

Suppose the hypothesis $(H1)-(H4)$ are satisfied then the stochastic process

$$\begin{aligned} X_t &= e^{At}x_0 + \int_0^t e^{A(t-s)}F\Big(s, X_s, \int_0^s h(s,\tau,X_\tau)d\tau\Big)ds + \int_0^t e^{A(t-s)}Bu(s)ds \\ &\quad + \int_0^t e^{A(t-s)}G\Big(s, X_s, \int_0^s k(s,\tau,X_\tau)d\tau\Big)d\mathcal{W}_s, \quad \text{almost surely for } t \in J, \end{aligned}$$

defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be mild solution in H of equation (1) for a given initial value x_0.

Definition 1. *The system (1) - (2) is said to be controllable on the interval J if for every $x_1 \in H$ there exists a control $u \in L^2(J,U)$ such that the mild solution X_t of (1) satisfies $X_T = x_1$.*

Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space and denote the L^q norm for $q \in [1,\infty)$ of a $\mathcal{F}\backslash(\mathcal{V})$-measurable mapping $z : \Omega \to \mathcal{V}$ by $\|z\|_{L^q} := (\mathbb{E}\|z\|^q)^{1/q}$. We need the following version of the Burkholder-Davis-Gundy inequality in infinite dimensions.

Lemma 1. *Let $X_t : \Omega \to HS(V,H)$ be a predictable stochastic process with $\mathbb{E}\int_0^T \|X_s\|_{HS}^2 ds < \infty$. Then, we obtain*

$$\Big\| \int_0^t X_s d\mathcal{W}_s \Big\|_{L^q} \leq q\Big(\int_0^t \| \|X_s\|_{HS} \|_{L^q}^2 ds \Big)^{1/2}$$

for every $t \in J$ and every $q \in [2,\infty)$. Both sides could be infinite.

3. Controllability results

Theorem 1. *Let $(H1)-(H5)$ be satisfied. Then, there is a unique (upto modifications) predictable stochastic process $X_t : \Omega \to H$ with $\sup_{0\leq t\leq T} \mathbb{E}\|X_t\|_H^p < \infty$, where $p \geq 2$ is given in Assumption $(H4)$ such that*

$$\mathbb{P}\left\{ \begin{aligned} X_t = e^{At}x_0 &+ \int_0^t e^{A(t-s)}F\Big(s, X_s, \int_0^s h(s,\tau,X_\tau)d\tau\Big)ds \\ &+ \int_0^t e^{A(t-s)}Bu(s)ds + \int_0^t e^{A(t-s)}G\Big(s, X_s, \int_0^s k(s,\tau,X_\tau)d\tau\Big)d\mathcal{W}_s \end{aligned} \right\} = 1 \quad (3)$$

for all $t \in J$. Furthermore, X_t is the unique mild solution of (1) in this sense and hence the system (1)-(2)is controllable.

Proof. Let $p \geq 2$ be given by the assumption $(H4)$. First, we introduce the $\mathbb{R}$-vector space γ_p of all equivalence classes of predictable stochastic processes $X_t : \Omega \to H$ with $\sup_{0\leq t\leq T} \|X_t\|_{L^p} < \infty$, where all stochastic processes that are modifications of each other lie in one equivalence class. Then, we equip this space with the norm $\|X\|_\mu := \sup_{0\leq t\leq T} e^{\mu t}\|X_t\|_{L^p}$ for every $X \in \gamma_p$ and every

$\mu \in \mathbb{R}$. Note that the pair $(\gamma_p, \|\cdot\|_\mu)$ is a Banach space for every $\mu \in \mathbb{R}$. Now, we consider the mapping $\Phi : \gamma_p \to \gamma_p$ given by

$$\begin{aligned}(\Phi X)_t &= e^{At}x_0 + \int_0^t e^{A(t-s)}F\Big(s, X_s, \int_0^s h(s,\tau,X_\tau)d\tau\Big)ds + \int_0^t e^{A(t-s)}Bu(s)ds \\ &\quad + \int_0^t e^{A(t-s)}G\Big(s, X_s, \int_0^s k(s,\tau,X_\tau)d\tau\Big)d\mathcal{W}_s,\ t \in J,\ X \in \gamma_p.\end{aligned}$$

Define the control,

$$\begin{aligned}u(t) &= W^{-1}\Big[x_1 - e^{AT}x_0 - \int_0^T e^{A(T-s)}F\Big(s, X_s, \int_0^s h(s,\tau,X_\tau)d\tau\Big)ds \\ &\quad - \int_0^T e^{A(T-s)}G\Big(s, X_s, \int_0^s k(s,\tau,X_\tau)d\tau\Big)d\mathcal{W}_s\Big](s).\end{aligned}$$

We have

$$\begin{aligned}(\Phi X)_t &:= e^{At}x_0 + \int_0^t e^{A(t-s)}F\Big(s, X_s, \int_0^s h(s,\tau,X_\tau)d\tau\Big)ds \\ &\quad + \int_0^t e^{A(t-s)}G\Big(s, X_s, \int_0^s k(s,\tau,X_\tau)d\tau\Big)d\mathcal{W}_s \\ &\quad + \int_0^t e^{A(t-s)}BW^{-1}\Big[x_1 - e^{AT}x_0 \\ &\qquad - \int_0^T e^{A(T-s)}F\Big(s, X_s, \int_0^s h(s,\tau,X_\tau)d\tau\Big)ds \\ &\qquad - \int_0^T e^{A(T-s)}G\Big(s, X_s, \int_0^s k(s,\tau,X_\tau)d\tau\Big)d\mathcal{W}_s\Big](s)ds.\end{aligned}$$

Step 1. Φ well-defined.

Given $t \in J$ and $X \in \gamma_p$, the mapping from $J \times \Omega$ to $HS(V, H)$ defined by $(s, \omega) \to e^{A(t-s)}G(s, X_s(\omega), \int_0^s k(s,\tau,X_\tau)d\tau)$ for every $s \in J$, $\omega \in \Omega$ is a predictable stochastic process, since e^{As} is continuous in $L(D, H)$ for $s \in (0, T]$ and $G(\cdot, v, w)$ is strongly measurable by $(H3)$. Hence

$$\begin{aligned}&\Big\| \int_0^t e^{A(t-s)}G(s, X_s, X'_s)d\mathcal{W}_s \Big\|_{L^p} \\ &\qquad \le p\Big(\int_0^t \|\|e^{A(t-s)}G(s, X_s, X'_s)\|_{HS}\|^2_{L^p} ds\Big)^{1/2} \\ &\qquad \le p\Big(\int_0^t \|M(1+\|X_s\|_H+\|X'_s\|_H)(t-s)^{\epsilon-1/2}\|^2_{L^p} ds\Big)^{1/2} \\ &\qquad \le Mp\Big(\int_0^t [1+\|X_s\|_H(1+M_1)]^2(t-s)^{2\epsilon-1}ds\Big)^{1/2}\end{aligned}$$

due to Lemma(1) and $(H3)$, we obtain

$$\begin{aligned}\left\| \int_0^t e^{A(t-s)} G(s, X_s, X_s') d\mathcal{W}_s \right\|_{L^p} &\leq Mp\Big(1+ \sup_{0\leq s\leq T} \|X_s\|_{L^P}(1+M_1)\Big)\Big(\int_0^t s^{2\epsilon-1ds}\Big)^{1/2} \\ &\leq \frac{Mp}{\sqrt{2\epsilon}} t^{\epsilon}\left(1+\left[\sup_{0\leq s\leq T}\|X_s\|_{L^p}\right](1+M_1)\right) \\ &< \infty.\end{aligned}$$

By Lebesgue's theorem, $\int_0^t e^{A(t-s)} G(s, X_s, X_s') d\mathcal{W}_s$ for $t \in [0, T]$ is mean square continuous. Hence, it has a predictable version. Similarly,

$$\begin{aligned}&\left\| \int_0^t e^{A(t-s)} F(s, X_s, X_s') ds \right\|_{L^p} \\ &\leq \int_0^t \|e^{A(t-s)} F(s, X_s, X_s')\|_{L^p} ds \\ &\leq \sup_{0\leq s\leq T} \|e^{As}\|_{L(H,H)} \int_0^t \|F(s, X_s, X_s')\|_{L^p} ds \\ &\leq \sup_{0\leq s\leq T} \|e^{As}\|_{L(H,H)} \int_0^t L[1+ \|X_s\|_{L^p} + \|X_s'\|_{L^p})] ds \\ &\leq \int_0^t L[1 + \|X_s\|_{L^p} + L_1\|X_s\|_{L^p})] ds \\ &\leq L\left[1+\left(\sup_{0\leq s\leq T}\|X_s\|_{L^p}\right)(1+L_1)\right] t \\ &< \infty.\end{aligned}$$

And hence,

$$\begin{aligned}&\|(\Phi X)_t\|_{L^p} \\ &\leq \frac{Mp}{\sqrt{2\epsilon}}\left[1+\left(\sup_{0\leq s\leq T}\|X_s\|_{L^p}\right)(1+M_1)\right]t^{\epsilon} + L\left[1+\left(\sup_{0\leq s\leq T}\|X_s\|_{L^p}\right)(1+L_1)\right]t \\ &\quad + \int_0^t \left\| e^{A(T-s)} BW^{-1}\left\{x_1 - e^{AT}x_0 - L\left[1+\left(\sup_{0\leq s\leq T}\|X_s\|_{L^p}\right)(1+L_1)\right]t \right.\right. \\ &\quad \left.\left. -\frac{Mp}{\sqrt{2\epsilon}}\left[1+\left(\sup_{0\leq s\leq T}\|X_s\|_{L^p}\right)(1+M_1)\right]t^{\epsilon}\right\}(s) ds \right. \\ &< \infty.\end{aligned}$$

Hence Φ is well defined.
Step 2. Φ is contraction. For $X, Y \in \gamma_p,\ t \in [0, T]$ we obtain,

$$\begin{aligned}&(\Phi X)_t - (\Phi Y)_t \\ &= \int_0^t e^{A(t-s)}[F(s, X_s, X_s') - F(s, Y_s, Y_s')] ds\end{aligned}$$

$$+\int_0^t e^{A(t-s)}[G(s,X_s,X_s')-G(s,Y_s,Y_s')]d\mathcal{W}_s$$
$$+\int_0^t e^{A(t-s)}BW^{-1}\Big[\int_0^T e^{A(T-s)}[F(s,Y_s,Y_s')-F(s,X_s,X_s')]ds$$
$$+\int_0^T e^{A(T-s)}[G(s,Y_s,Y_s')-G(s,X_s,X_s')]d\mathcal{W}_s\Big](s)ds$$

$$\begin{aligned}
&\|(\Phi X)_t-(\Phi Y)_t\|_{L^p}\\
&\le \int_0^t \|e^{A(t-s)}[F(s,X_s,X_s')-F(s,Y_s,Y_s')]\|_{L^p}ds\\
&\quad+\int_0^t \|e^{A(t-s)}[G(s,X_s,X_s')-G(s,Y_s,Y_s')]\|_{L^p}d\mathcal{W}_s\\
&\quad+\int_0^t \Big\|e^{A(t-s)}BW^{-1}\Big[\int_0^T e^{A(T-s)}[F(s,Y_s,Y_s')-F(s,X_s,X_s')]ds\\
&\quad+\int_0^T e^{A(T-s)}[G(s,Y_s,Y_s')-G(s,X_s,X_s')]d\mathcal{W}_s\Big](s)\Big\|_{L^p}ds\\
&\le L\sup_{0\le s\le T}\|e^{A(t-s)}\|_{L(H,H)}\Big(\int_0^t(\|X_s-X_s'\|)\|_{L^p}+(\|Y_s-Y_s'\|)\|_{L^p})ds\Big)\\
&\quad+p\Big(\int_0^t\Big\|\|e^{A(t-s)}[G(s,X_s,X_s')-G(s,Y_s,Y_s')]\|_{HS}\Big\|_{L^p}^2ds\Big)^{1/2}\\
&\quad+\Big(\sup_{0\le s\le T}\|e^{A(t-s)}\|_{L(H,H)}\Big)\\
&\quad\Big(\int_0^t\Big\|BW^{-1}\Big[\int_0^T e^{A(T-s)}[F(s,Y_s,Y_s')-F(s,X_s,X_s')]ds\\
&\quad+\int_0^T e^{A(T-s)}[G(s,Y_s,Y_s')-G(s,X_s,X_s')]d\mathcal{W}_s\Big](s)\Big\|_{L^p}ds\Big)\\
&\le L\Big(\int_0^t e^{-\mu s}(1+L_1)\|X_s-Y_s\|_{L^p}e^{\mu s}ds\Big)\\
&\quad+p\Big(\int_0^t\Big\|M(t-s)^{\epsilon-1/2}[\|X_s-Y_s\|_H+\|X_s'-Y_s'\|_H]\Big\|_{L^p}^2ds\Big)^{1/2}\\
&\quad+\|BW^{-1}\|\int_0^T\|e^{A(t-s)}[F(s,X_s,X_s')-F(s,Y_s,Y_s')]\|_{L^p}ds\\
&\quad+\|BW^{-1}\|\int_0^T\|e^{A(t-s)}[G(s,X_s,X_s')-G(s,Y_s,Y_s')]\|_{L^p}d\mathcal{W}_s\\
&\le \Big(L(1+L_1)\|X-Y\|_\mu+C_1L(1+L_1)\|X-Y\|_\mu\Big)\Big(\int_0^T e^{-\mu s}ds\Big)
\end{aligned}$$

$$+p\left(\int_0^t \left\|M(1+M_1)(t-s)^{\epsilon-1/2}\|X_s-Y_s\|_H\right\|_{L^p}^2 ds\right)^{1/2}$$
$$+pC_1\left(\int_0^T \left\|M(1+M_1)(t-s)^{\epsilon-1/2}\|X_s-Y_s\|_H\right\|_{L^p}^2 ds\right)^{1/2}.$$

For $\mu \in \mathbb{R}$ by Lemma(1), this yields

$$\begin{aligned}
&e^{\mu t}\|(\Phi X)_t-(\Phi Y)_t\|_{L^p}\\
&\leq L(1+L_1)\|X-Y\|_\mu\left(\int_0^t e^{\mu(t-s)}ds\right)\\
&\quad+C_1L(1+L_1)\|X-Y\|_\mu\left(\int_0^T e^{\mu(t-s)}ds\right)\\
&\quad+Mp(1+M_1)\left(\int_0^t (t-s)^{2\epsilon-1}e^{2\mu(t-s)}ds\right)^{1/2}\|X-Y\|_\mu\\
&\quad+MpC_1(1+M_1)\left(\int_0^T (t-s)^{2\epsilon-1}e^{2\mu(t-s)}ds\right)^{1/2}\|X-Y\|_\mu\\
&\leq L(1+L_1)\|X-Y\|_\mu\left(\int_0^t e^{\mu s}ds\right)\\
&\quad+C_1L(1+L_1)\|X-Y\|_\mu\left(\int_0^T e^{\mu s}ds\right)\\
&\quad+Mp(1+M_1)\|X-Y\|_\mu\sqrt{\int_0^t (t-s)^{2\epsilon-1}e^{2\mu(t-s)}ds}\\
&\quad+MpC_1(1+M_1)\|X-Y\|_\mu\sqrt{\int_0^T (t-s)^{2\epsilon-1}e^{2\mu(t-s)}ds},
\end{aligned}$$

for every $\mu > 0$, $t \in J$. Hence,

$$\begin{aligned}
\|\Phi X-\Phi Y\|_\mu &\leq \frac{L}{|\mu|}(1+L_1)(1+C_1)\|X-Y\|_\mu\\
&\quad+\left(Mp(1+M_1)(1+C_1)\sqrt{\int_0^T (t-s)^{2\epsilon-1}e^{2\mu(s)}ds}\right)\|X-Y\|_\mu\\
\|\Phi X-\Phi Y\|_\mu &\leq \left[\frac{L}{|\mu|}(1+L_1)(1+C_1)\right.\\
&\quad\left.+Mp(1+M_1)(1+C_1)\sqrt{\int_0^T (t-s)^{2\epsilon-1}e^{2\mu(s)}ds}\right]\|X-Y\|_\mu,
\end{aligned}$$

for every $\mu > 0$.

Finally, for $\mu \to -\infty$, we see that Φ is contraction with respect to $\|\cdot\|_\mu$, so there is a unique element $X \in \gamma_p$ with $X = \Phi X$. Hence X_t is the unique mild solution of (1) and the system(1)-(2) is controllable. □

Finally, we also obtain the following regularity result of the solution if further assumption on $e^{At}G(\cdot,\cdot,\cdot)$ are satisfied.

Theorem 2. *Let Assumption $(H1)-(H5)$ be satisfied and let $\nu \in (0,1)$ be such that $\mathbb{E}\|(-A)^{\nu}x_0\|_H^p < \infty$. Furthermore, suppose that $(-A)^{\nu}e^{At}G(t,v,w)$ is a Hilbert-Schmidt operator from V to H with*

$$\|(-A)^{\nu}e^{At}G(t,v,w)\|_{HS} \leq L(1+\|v\|_H+\|w\|_H)t^{\epsilon-1/2}$$

for all $v,w \in H$ and all $t \in J$ with constants $L,\epsilon > 0$. Then, the unique solution process $X_t : \Omega \to H$ of the equation(1) given by the Theorem(1) satisfies $\sup_{0\leq t\leq T} \mathbb{E}\|(-A)^{\nu}X_t\|_H^p < \infty$.

Proof. The solution process X_t satisfies

$$\mathbb{P}\left\{\begin{array}{l} X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}F\left(s,X_s,\int_0^s h(s,\tau,X_\tau)d\tau\right)ds \\ \quad + \int_0^t e^{A(t-s)}Bu(s)ds + \int_0^t e^{A(t-s)}G\left(s,X_s,\int_0^s k(s,\tau,X_\tau)d\tau\right)d\mathcal{W}_s \end{array}\right\} = 1$$

for every $t \in J$. Consider,

$$\|I_1\| = \|(-A)^{\nu}e^{At}x_0\|_{L^p} = \|e^{At}(-A)^{\nu}x_0\|_{L^p} = \|(-A)^{\nu}x_0\|_{L^p} < \infty$$

for every $t \in J$, where $p \geq 2$ is given in $(H4)$.

$$\begin{aligned} \|I_2\| &= \left\|(-A)^{\nu}\int_0^t e^{A(t-s)}F(s,X_s,X_s')ds\right\|_{L^p} \\ &\leq \int_0^t \|(-A)^{\nu}e^{A(t-s)}F(s,X_s,X_s')\|_{L^p}ds \\ &\leq \int_0^t \|(-A)^{\nu}e^{A(t-s)}\|_{L(H,H)}\|F(s,X_s,X_s')\|_{L^p}ds \\ &\leq \int_0^t (t-s)^{-\nu}L(1+\|X_s\|_{L^p}+\|X_s'\|_{L^p})ds \\ &\leq \int_0^t (t-s)^{-\nu}L(1+\|X_s\|_{L^p}(1+L_1))ds \\ &< \infty \end{aligned}$$

for $t \in J$, we used the fact that $\|F(t,v,w)\|_H \leq L(1+\|v\|+\|w\|)$ with $\left\|\int_0^t h(t,s,v)ds\right\| \leq L_1(\|v\|)$, for all $v,w \in H$ with constants $L,L_1 > 0$.

$$\begin{aligned} \|I_3\| &= \left\|(-A)^{\nu}\int_0^t e^{A(t-s)}F(s,X_s,X_s')ds\right\|_{L^p} \\ &\leq L\left(1+(1+L_1)\sup_{0\leq s\leq T}\|X_s\|_{L^p}\right)\left(\int_0^t s^{-\nu}ds\right) \end{aligned}$$

$$
\begin{aligned}
&\le L\frac{t^{1-\nu}}{1-\nu}\Big(1+(1+L_1)\sup_{0\le s\le T}\|X_s\|_{L^p}\Big)\\
&\le \frac{L(T+1)}{1-\nu}\Big(1+(1+L_1)\sup_{0\le s\le T}\|X_s\|_{L^p}\Big)\\
&< \infty,\ \text{for every } t\in J.\\
\|I_4\| &= \Big\|(-A)^{\nu}\int_0^t e^{A(t-s)}G(s,X_s,X_s')d\mathcal{W}s\Big\|_{L^p}\\
&\le p\Big(\int_0^t \Big\|\|(-A)^{\nu}e^{A(t-s)}G(s,X_s,X_s')\|_{HS}\Big\|_{L^p}^2 ds\Big)^{1/2}\\
&\le p\Big(\int_0^t \Big\|M(1+\|X_s\|_H+\|X_s'\|)\|_H(t-s)^{(\epsilon-1/2)}\Big\|_{L^p}^2 ds\Big)^{1/2}\\
&\le p\Big(\int_0^t \Big\|M(1+\|X_s\|_H+M_1\|X_s)\|_H(t-s)^{(\epsilon-1/2)}\Big\|_{L^p}^2 ds\Big)^{1/2}\\
&\le Mp\Big(1+\sup_{0\le s\le T}\|X_s\|_H(1+M_1)\Big)\Big(\int_0^t s^{(2\epsilon-1)}ds\Big)^{1/2}\\
&\le Mp\frac{T^{\epsilon}}{\sqrt{2\epsilon}}\Big(\sup_{0\le s\le T}\|X_s\|_H(1+M_1)\Big)\\
&< \infty,\text{for all } t\in J.
\end{aligned}
$$

Finally, we obtain the last term, by using $\|I_2\|-\|I_4\|$ and $(H5)$

$$\|I_5\| = \Big\|(-A)^{\nu}\int_0^t e^{A(t-s)}Bu(s)ds\Big\|_{L^p} < \infty.$$

Hence by using Lemma(1) and [8] which yields the assertation. □

4. Example

Consider a string fixed at the end points $x=0$ and $x=1$. The string moves randomly by the three forces such as, external forces, random forces driven by (Gaussian) white noise type and the elastic force. The external forces and the random forces are given by $f(X_t(x),X_t'(x)):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g(X_t(x),X_t''(x)):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and the modulus of the elasiticity is given by positive constant $\kappa>0$. x is the parameter of the string. We assume that the control influence upon the random motion is the 'absolute' control and it is defined by a bounded linear operator $B:U\to L^2[\mathbb{R}^d,\mathbb{R}],\ d\in\mathbb{N}$.
We can describe this stochastic model by the following equations

$$
\begin{aligned}
dX_t(x) = &\Big[\frac{\kappa}{2}\Delta X_t(x)+f(X_t(x),X_t'(x))+Bu(t)\Big]dt\\
&+g(X_t(x),X_t''(x))d\mathcal{W}_t,\quad t\in[0,T]:=J
\end{aligned}
\tag{4}
$$

$$X_0(x)=x_0. \tag{5}$$

The question of controllability for the above system now can be formulated as follows: is it possible to steer the string to the motionless state at time T.
Let us consider $O := (0,1)^d = \mathbb{R}^d$ with $d \in \mathbb{N}$ and $H = L^2[O, \mathbb{R}]$, the Hilbert space with scalar product and the norm $\langle v, w\rangle_H = \int_O v(x)w(x)dx$ for every $v, w \in \mathbb{R}$. We also define $V := H$, moreover $A = \nu\Delta$ where $\nu = \kappa/2$ with $\nu > 0$ (i.e., $\kappa/2 > 0$) be constant times the Laplacian with Dirichlet boundary conditions. For $\mathcal{I} = \mathbb{N}^d$,

$$\begin{aligned} e_i(x) &= 2^{d/2}\prod_{j=1}^{d}\sin(i_j\pi x_j) \\ \lambda_i &= \nu\pi^2\sum_{j=1}^{d} i_j^2 \end{aligned}$$

for all $x = (x_1, x_2, ..., x_d) \in O$ and $i = (i_1, i_2, ..., i_d) \in \mathcal{I}$.
Then the operator A is given by $Af = \sum_{i\in\mathcal{I}} -\lambda_i\langle e_i, f\rangle_H e_i$, for all $f \in D(A)$ with

$$D(A) = \left\{ f \in H : \sum_{i\in\mathcal{I}} \lambda_i^2 |\langle e_i, f\rangle_H|^2 < \infty \right\}.$$

Hence Assumption $(H1)$ holds.
Now, let $f, g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be two globally Lipschitz continuous functions in the sense that

$$\begin{aligned} |f(x, x_1) - f(y, y_1)| &\le L[|x - y| + |x_1 - y_1|] \\ |g(x, x_2) - g(y, y_2)| &\le M[|x - y| + |x_2 - y_2|] \end{aligned}$$

for all $x, x_1, x_2, y, y_1, y_2 \in \mathbb{R}$ with a constant $L, M > 0$. Now define $F(v, w) : H \times H \to H$ and $\zeta(v, w_1) : H \times H \to H$ by

$$F(v, w)(x) = f(v(x), w(x)) \tag{6}$$

$$\zeta(v, w_1)(x) = g(v(x), w_1(x)) \tag{7}$$

for all $v, w, w_1 \in H, x \in (0, 1)^d$. Hence F and ζ are globally Lipschitz continuous with respect to $\|.\|_H$ with the function $h, k : J \times J \times H \to H$ such that

$$\begin{aligned} \|F(v, v_1) - F(w, w_1)\|_H &\le L[\|v - w\|_H + \|v_1 - w_1\|_H] \text{ ; with} \\ \left\|\int_0^t [h(t, s, v) - h(t, s, w)]ds\right\| &\le L_1\|v - w\|_H \\ \|\zeta(v, v_2) - \zeta(w, w_2)\|_H &\le M[\|v - w\|_H + \|v_2 - w_2\|_H] \text{ ; with} \\ \left\|\int_0^t [k(t, s, v) - k(t, s, w)]ds\right\| &\le M_1\|v - w\|_H \end{aligned}$$

where $v, w, v_1, v_2, w_1, w_2 \in H$ and $L, L_1, M, M_1 > 0$. Hence Assumption $(H2)$ holds.

Let $d \in \mathcal{I}$ and $D := H$. Let $\{f_i\}_{i\in I}$ be another orthonormal basis in H with the property that $f_i : O \to \mathbb{R}$ are continuous functions which satisfy

$$\sup_{i\in I} \sup_{x\in O} |f_i(x)| < \infty. \tag{8}$$

Then, we define G by

$$G : H \times H \to L(H, D),\ (G(v, v_1)(w))(x)) := (\zeta(v, v_1))(x).(w)(x) \tag{9}$$

for every $x \in O$ and $v, v_1, w \in H$. Indeed, G is well defined, since, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|G(v, v_1)(w)\|_D &\leq \int_O |\zeta(v, v_1)(x).w(x)|dx \\
&\leq \left(\int_O |\zeta(v, v_1)(x)|^2 dx\right)^{1/2} \left(\int_O |w(x)|^2 dx\right)^{1/2} \\
&= \|\zeta(v, v_1)\|_H \|w\|_H
\end{aligned}$$

for all $v, v_1, w \in H$, therefore $G(v, v_1)$ is a bounded linear operator from H to D with the property

$$\|G(v, v_1)\|_{L(H,D)} \leq \|\zeta(v, v_1)\|_H, \quad \text{for all } v, v_1 \in H.$$

Since ζ is globally Lipschitz continuous, we have G is also a globally Lipschitz continuous from H to $L(H, D)$ and is measurable, in the sense that

$$\begin{aligned}
\|G(v, v_1) - G(w, w_1)\| &\leq \|\zeta(v, v_1) - \zeta(w, w_1)\|_H \\
&\leq M[\|v - w\|_H + \|v_1 - w_1\|_H].
\end{aligned}$$

Combining the definitions in (7)-(9), we obtain

$$G : H \times H \to L(H, D), (G(v, v_1)(w))(x) := g(v(x), v_1(x)).w(x)$$

for all $x \in (0, 1)^d$ and $v, v_1, w \in H$.
Let $\nu \in [0, 1)$, we have $(-A)^\nu e^{At} G(v, v_1)$ is a linear bounded operator from H to $D = H$ and hence

$$\begin{aligned}
\|(-A)^\nu e^{At} G(v, v_1)\|_{HS} &\leq \|(-A)e^{At}\|_{L(H,H)} \|G(v, v_1)\|_{HS} \\
&\leq t^{-\nu} \left(\sup_{i\in I} \|G(v, v_1)(f_i)\|_H^2\right)^{1/2} \\
&\leq t^{-\nu} \left(\sup_{i\in I} \|\zeta(v, v_1)\|\right) \left(\sup_{i\in I} \sup_{x\in O} |f_i(x)|\right) \\
&\leq C\|\zeta(v, v_1)\|_H t^{-\nu}
\end{aligned}$$

for all $v, v_1 \in H$ and all $t \in [0, T]$ and by using equation(8), where the constant $C > 0$. Hence, we obtain

$$\|(-A)^\nu e^{At} G(v, v_1)\|_{HS} \leq C\|\zeta(v, v_1)\|_H t^{-\nu}$$

for every $v, v_1 \in H$, $t \in (0, T]$ and $\nu \in [0, 1)$. In the same way, we can show that

$$\|(-A)^\nu e^{At}[G(v, v_1) - G(w, w_1)]\|_{HS} \leq C\|\zeta(v, v_1) - \zeta(w, w_1)\|_H t^{-\nu}.$$

Thus, Assumption $(H3)$ holds.
Now we can formulate our controlled system governed by the stochastic functional integrodifferential evolution equation in the Hilbert space H of the form

$$\begin{aligned} dX_t = &\left[AX_t + F\left(X_t, \int_0^t h(t, s, X_s)ds\right) + Bu(t)\right]dt \\ &+ G\left(X_t, \int_0^t k(t, s, X_s)\right)d\mathcal{W}_t, \quad t \in [0, T] := J \end{aligned} \tag{10}$$

$$X_0 = x_0 \tag{11}$$

where $\mathcal{W}_t$ is cylindrical Wiener process.9
(A) The control operator $B \in L(U, H)$ defined on the Hilbert space U is surjective then the corresponding linear system is controllable and hence there exists the inverse W^{-1} for the control operator $W : L^2(J : U) \to H$ by

$$W(u) = \int_0^T e^{A(t-s)}Bu(s)ds$$

Finally, if the initial value satisfies the condition $(H4)$, then the system (4)-(5) has a unique solution by Theorem(1) and with the choice of (A), the given system (4)-(5) is controllable.

References

1. K.J.Aström, *Introduction to Stochastic Control Theory*, Academic Press, New York, 1970.
2. K.Balachandran, S.Karthikeyan, *Controllability of stochastic integrodifferential systems*, Int. J. Control. **80**(2007), 486-491.
3. K.Balachandran, J.H.Kim, S.Karthikeyan, *Controllability of semilinear stochastic integrodifferential equations*, Kybernetika, **43**(2007), 31-44.
4. K.Balachandran, J.H.Kim, S.Karthikeyan, *Complete controllability of stochastic integrodifferential systems*, Dyn. Syst. Appl. **17**(2008), 43-52.
5. P.Balasubramaniam, *Controllability of quasilinear stochastic evolution equations in Hilbert spaces*, J. Appl. Math. Stoch. Anal. **14**(2001), 151-159.
6. P.Balasubramaniam, J.P.Dauer, *Controllability of semilinear stochastic evolution equations in Hilbert spaces*, J. Appl. Math. Stoch. Anal. **14**(2001), 329-339.
7. P.Balasubramaniam, J.Y.Park, P.Muthukumar, *Approximate controllability of neutral stochastic functional differential systems with infinite delay*, Stoch. Anal. Appl. **28**(2010), 389-400.
8. G.Da Prato, J.Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.
9. G.Da Prato, J.Zabczyk, *Ergodicity for infinite Dimensional Systems*, Cambridge University Press, Cambridge, 1996.
10. J.P. Dauer, N.I. Mahmudov, *Controllability of stochastic semilinear functional differential equations in Hilbert spaces*, J. Math. Anal. Appl. **290**(2004), 373-394.
11. A.Jentzen, P.E.Kloeden, *A unified existence and uniqueness theorem for stochasic evolution equations*, Bull. Aust. Math. Soc. **81**(2010), 33-46.

12. R.MacCamy, *An integrodifferential equation with applications in heat flow*, Q. Appl. Math. **35**(1977/78), 1-99.
13. N.I.Mahmudov, *Controllability of semilinear stochastic systems in Hilbert spaces*, J. Math. Anal. Appl. **288**(2003), 197-211.
14. C.Prèvot, M.Rockner, *A Concise Course on Stochastic Partial Differential Equations*, Springer, Berlin, 2007.
15. G.R.Sell, Y.You, *Dynamics of Evolutionary Equations*, Springer, New York, 2002.
16. R.Subalakshmi, K.Balachandran, J.Y.Park, *Controllability of semilinear stochastic functional integrodifferential systems in Hilbert spaces*, Nonlinear Analysis: Hybrid Systems, **3**(2009), 39-50.
17. W.M.Wonham, *Random differential equations in control theory*, Probab. Methods Appl. Math. **2**(1970), 131-212.
18. J.Zabczyk, *Controllability of Stochastic Linear Systems*, Syst. Control Lett. **1**(1982), 25-31.

J. Kokila received M.Sc from Kongu Arts and Science College affiliated to Bharathiar University, in 2008 and her M.Phil from Bharathiar University, in 2009. Now, she is persuing her Ph.D study at Bharathiar University. Her research area is controllability of stochastic systems.

Department of Mathematics, Bharathiar University, Coimbatore, India.
e-mail: kokimaths@gmail.com

K. Balachandran received M.Sc from University of Madras, in 1978 and Ph.D Degree in Control Theory from University of Madras, in 1985. He is currently a professor at Department of Mathematics, Bharathiar University, Coimbatore, since 1994. He is a member of American Mathematical Society and reviewer for Mathematical Reviews. He is the author or co author of more than 300 scholarly publications. His research interest includes control theory, abstract differential equations, integral equations, partial differential equations and fractional differential equations.

Department of Mathematics, Bharathiar University, Coimbatore, India.
e-mail: kb.maths.bu@gmail.com