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Various Connections and Their Relations

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Abstract

We investigate the properties of Galois, dual Galois, residuated, and dual residuated connections on posets. In particular, we show that their connections are related to relations.

Key Words: Galois, dual Galois, residuated, dual residuated

1. Introduction and Preliminaries

Galois, dual Galois, residuated and dual residuated connections are defined by relationship between posets. Wille [11] introduced the formal concept lattices by allowing some uncertainty in data as examples as Galois, dual Galois, residuated and dual residuated connections. Galois connection analysis is an important mathematical tool for data analysis and knowledge processing [1-5,8,9,11].

In this paper, we investigate the properties of Galois, dual Galois, residuated and dual residuated connections. We find generating functions which induce Galois, dual Galois, residuated and dual residuated connections. In particular, we show that their connections related to relations.

Let X be a set. A relation $e_X \subset X \times X$ is called a partially order set (simply, poset)if it is reflexive, transitive and anti-symmetric. We can define a poset $e_{P(X)} \subset P(X) \times P(X)$ as $(A, B) \in e_{P(X)}$ iff $A \subset B$ for $A, B \in P(X)$. If (X, e_X) is a poset and we define a function $(x, y) \in e_X^{-1}$ iff $(y, x) \in e_X$, then (X, e_X^{-1}) is a poset.

Definition 1.1. [10] Let (X, e_X) and (Y, e_Y) be posets and $f : X \to Y$ and $g : Y \to X$ maps.

(1) (e_X, f, g, e_Y) is called a Galois connection if for all $x \in X, y \in Y$, $(y, f(x)) \in e_Y$ iff $(x, g(y)) \in e_X$.

(2)(e_X, f, g, e_Y) is called a dual Galois connection if for all $x \in X, y \in Y$, $(f(x), y) \in e_Y$ iff $(g(y), x) \in e_Y$.

(3) (e_X, f, g, e_Y) is called a residuated connection if for all $x \in X, y \in Y$, $(f(x), y) \in e_Y$ iff $(x, g(y)) \in e_Y$.

(4) (e_X, f, g, e_Y) is called a dual residuated connection if for all $x \in X, y \in Y$, $(y, f(x)) \in e_Y$ iff $(g(y), x) \in e_Y$.

Remark 1.2. [10] Let (X, e_X) and (Y, e_Y) be a poset and $f : X \to Y$ and $g : Y \to X$ maps.

(1) (e_X, f, g, e_Y) is a Galois (resp. dual Galois) connection iff (e_Y, g, f, e_X) is a Galois (resp. dual Galois) connection.

(2) (e_X, f, g, e_Y) is a Galois (resp. residuated) connection iff $(e_X^{-1}, f, g, e_Y^{-1})$ is a dual (resp. dual residuated) Galois connection.

(3) (e_X, f, g, e_Y) is a residuated (resp. dual residuated) connection iff $(e_Y^{-1}, g, f, e_X^{-1})$ is a residuated (resp. dual residuated) connection.

(4) (e_X, f, g, e_Y) is a Galois (resp. dual Galois) connection iff (e_X, f, g, e_Y^{-1}) is a residuated (resp. dual residuated) connection.

(5) (e_X, f, g, e_Y) is a residuated connection iff (e_Y, g, f, e_X) is a dual residuated connection.

Theorem 1.3. [10] Let (X, e_X) and (Y, e_Y) be a poset and $f : X \to Y$ and $g : Y \to X$ maps.

(1) (e_X, f, g, e_Y) is a Galois connection if f, g are antitone maps and $(y, f(g(y))) \in e_Y$ and $(x, g(f(x))) \in e_X$.

(2) (e_X, f, g, e_Y) is a dual Galois connection if f, g are antitone maps and $(f(g(y)), y) \in e_Y$ and $(g(f(x)), x) \in e_X$.

(3) (e_X, f, g, e_Y) is a residuated connection if f, g are isotone maps and $(f(g(y)), y) \in e_Y$ and $(x, g(f(x))) \in e_X$.

(4) (e_X, f, g, e_Y) is called a dual residuated connection if f, g are isotone maps and $(y, f(g(y))) \in e_Y$ and $(g(f(x)), x) \in e_X$.

(5) (e_X, f, g, e_Y) is a Galois (resp. residuated) connection iff $(e_X^{-1}, f, g, e_Y^{-1})$ is a dual (resp. dual residuated) Galois connection.

2. Various Connection and Their Relations

Theorem 2.1. Let $(P(X), e_{P(X)})$ and $(P(Y), e_{P(Y)})$ be a poset and $F : P(X) \to P(Y)$ and $G : P(Y) \to P(X)$ maps.

(1) $(e_{P(X)}, F, G, e_{P(Y)})$ is a Galois connection iff $F(\bigcup_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i)$.

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(2) $(e_{P(X)}, F, G, e_{P(Y)})$ is a residuated connection iff $F(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i)$.

(3) $(e_{P(X)}, F, G, e_{P(Y)})$ is a dual Galois connection iff $F(\bigcap_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i)$.

(4) $(e_{P(X)}, F, G, e_{P(Y)})$ is a dual residuated connection iff $F(\bigcap_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i)$.

Proof. (1) $(\Rightarrow)$ Since $(B, F(\bigcup_{i\in\Gamma} A_i)) \in e_{P(Y)}$ iff $(\bigcup_{i\in\Gamma} A_i, G(B)) \in e_{P(X)}$ iff $(\forall i \in \Gamma)(A_i, G(B)) \in e_{P(X)}$ iff $(\forall i \in \Gamma)(B, F(A_i)) \in e_{P(Y)}$ iff $(B, \bigcap_{i\in\Gamma} F(A_i)) \in e_{P(Y)}$, put $B = \{y\}$ for each $y \in Y$, $y \in F(\bigcup_{i\in\Gamma} A_i)$ iff $(\{y\}, F(\bigcup_{i\in\Gamma} A_i)) \in e_{P(Y)}$ iff $(\{y\}, \bigcap_{i\in\Gamma} F(A_i)) \in e_{P(Y)}$ iff $y \in \bigcap_{i\in\Gamma} F(A_i)$. Hence $F(\bigcup_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i)$.

$(\Leftarrow)$ Define $G(B) = \bigcup\{C \mid B \subset F(C)\}$. If $B \subset F(C)$, then $C \subset G(B)$. If $C \subset G(B)$, then $F(C) \supset F(G(B)) \supset B$.

(2)$(\Rightarrow)$ Since $(F(\bigcup_{i\in\Gamma} A_i), B) \in e_{P(Y)}$ iff $(\bigcup_{i\in\Gamma} A_i, G(B)) \in e_{P(X)}$ iff $(\forall i \in \Gamma)(A_i, G(B)) \in e_{P(X)}$ iff $(\forall i \in \Gamma)(F(A_i), B) \in e_{P(Y)}$ iff $(\bigcup_{i\in\Gamma} F(A_i), B) \in e_{P(Y)}$, then $F(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i)$.

$(\Leftarrow)$ Define $G(B) = \bigcup\{C \mid F(C) \subset B\}$. If $F(C) \subset B$, then $C \subset G(B)$. If $C \subset G(B)$, then $F(C) \subset F(G(B)) \subset B$.

(3)$(\Rightarrow)$ Since $(F(\bigcap_{i\in\Gamma} A_i), B) \in e_{P(Y)}$ iff $(G(B), \bigcap_{i\in\Gamma} A_i) \in e_{P(X)}$ iff $(\forall i \in \Gamma)(G(B), A_i) \in e_{P(X)}$ iff $(\forall i \in \Gamma)(F(A_i), B) \in e_{P(X)}$ iff $(\bigcup_{i\in\Gamma} F(A_i), B) \in e_{P(X)}$, then $F(\bigcap_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i)$.

$(\Leftarrow)$ Define $G(B) = \bigcap\{C \mid F(C) \subset B\}$. If $F(C) \subset B$, then $G(B) \subset C$. If $G(B) \subset C$, then $F(C) \subset F(G(B)) \subset B$.

(4)$(\Rightarrow)$ Since $(B, F(\bigcap_{i\in\Gamma} A_i)) \in e_{P(Y)}$ iff $(G(B), \bigcap_{i\in\Gamma} A_i) \in e_{P(X)}$ iff $(\forall i \in \Gamma)(G(B), A_i) \in e_{P(X)}$ iff $(\forall i \in \Gamma)(B, F(A_i)) \in e_{P(Y)}$ iff $(B, \bigcap_{i\in\Gamma} F(A_i)) \in e_{P(Y)}$, then $F(\bigcap_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i)$.

$(\Leftarrow)$ Define $G(B) = \bigcap\{C \mid B \subset F(C)\}$. If $B \subset F(C)$, then $G(B) \subset C$. If $G(B) \subset C$, then $F(C) \supset F(G(B)) \supset B$.

□

Theorem 2.2. Let $(P(X), e_{P(X)})$ and $(P(Y), e_{P(Y)})$ be a poset and $F : P(X) \to P(Y)$ and $G : P(Y) \to P(X)$ maps.

(1) $(e_{P(X)}, F, G, e_{P(Y)})$ is a Galois connection iff there exists $R \subset X \times Y$ such that

$$F(A) = \{y \in Y \mid (\forall x \in X)(x \in A \to (x,y) \in R)\}$$

$$G(B) = \{x \in X \mid (\forall y \in Y)(y \in B \to (x,y) \in R)\}.$$

(2) $(e_{P(X)}, F, G, e_{P(Y)})$ is a residuated connection iff there exists $R \subset X \times Y$ such that

$$F(A) = \{y \in Y \mid (\exists x \in X)(x \in A \;\&\; (x,y) \in R)\}$$

$$G(B) = \{x \in X \mid (\forall y \in Y)((x,y) \in R \to y \in B)\}$$

(3) $(e_{P(X)}, F, G, e_{P(Y)})$ is a dual Galois connection iff there exists $R \subset X \times Y$ such that

$$F(A) = \{y \in Y \mid (\exists x \in X)(x \in A^c \;\&\; (x,y) \in R)\}$$

$$G(B) = \{x \in X \mid (\exists y \in Y)(y \in B^c \;\&\; (x,y) \in R)\}$$

(4) $(e_{P(X)}, F, G, e_{P(Y)})$ is a dual residuated connection iff there exists $R \subset X \times Y$ such that

$$F(A) = \{y \in Y \mid (\forall x \in X)((x,y) \in R \to x \in A)\}$$

$$G(B) = \{x \in X \mid (\exists y \in Y)(y \in B \;\&\; (x,y) \in R)\}$$

Proof. (1)$(\Rightarrow)$

$$\begin{aligned} &y \in F(\{x\}) \text{ iff } (\{y\}, F(\{x\})) \in e_{P(Y)} \\ &\text{iff } (\{x\}, G(\{y\})) \in e_{P(X)} \text{ iff } x \in G(\{y\}). \end{aligned}$$

Put $(x,y) \in R$ iff $y \in F(\{x\})$ iff $x \in G(\{y\})$. Then

$$\begin{aligned} &y \in F(A) \\ &\text{iff } (\{y\}, F(A)) \in e_{P(Y)} \text{ iff } (A, G(\{y\})) \in e_{P(X)} \\ &\text{iff } (\forall x \in X)\Big(x \in A \to x \in G(\{y\})\Big) \\ &\text{iff } (\forall x \in X)\Big(x \in A \to (x,y) \in R\Big). \end{aligned}$$

$$\begin{aligned} &x \in G(B) \\ &\text{iff } (\{x\}, G(B)) \in e_{P(X)} \text{ iff } (B, F(\{x\})) \in e_{P(Y)} \\ &\text{iff } (\forall y \in Y)\Big(y \in B \to y \in F(\{x\})\Big) \\ &\text{iff } (\forall y \in Y)\Big(y \in B \to (x,y) \in R\Big). \end{aligned}$$

$(\Leftarrow)$

$$\begin{aligned} &(B, F(A)) \in e_{P(Y)} \\ &\text{iff } (\forall y \in Y)(y \in B \to y \in F(A)) \\ &\text{iff } (\forall y \in Y)(y \in B \to (\forall x \in X)(x \in A \to (x,y) \in R)) \\ &\text{iff } (\forall x \in X)(x \in A \to (\forall y \in Y)(y \in B \to (x,y) \in R)) \\ &\text{iff } (\forall x \in X)(x \in A \to x \in G(B)) \\ &\text{iff } (A, G(B)) \in e_{P(Y)}. \end{aligned}$$

(2)$(\Rightarrow)$

$$\begin{aligned} &y \in F(\{x\})^c \text{ iff } (F(\{x\}), \{y\}^c) \in e_{P(Y)} \\ &\text{iff } (\{x\}, G(\{y\}^c)) \in e_{P(X)} \text{ iff } x \in G(\{y\}^c), \end{aligned}$$

Put $(x,y) \in R$ iff $y \in F(\{x\})$. By Theorem 2.1 (2), since $F(\cup_{i\in\Gamma} A_i) = \cup_{i\in\Gamma} F(A_i)$, then

$$\begin{aligned} y \in F(A) \quad &\text{iff } y \in F(\textstyle\bigcup_{x\in A}\{x\}) \\ &\text{iff } y \in \textstyle\bigcup_{x\in A} F(\{x\}) \\ &\text{iff } (\exists x \in X)(x \in A \;\&\; y \in F(\{x\})) \\ &\text{iff } (\exists x \in X)(x \in A \;\&\; (x,y) \in R). \end{aligned}$$

$x \in G(B)$
iff $(\{x\}, G(B)) \in e_{P(X)}$ iff $(F(\{x\}), B) \in e_{P(Y)}$
iff $(\forall y \in Y)(y \in F(\{x\}) \to y \in B)$
iff $(\forall y \in Y)((x, y) \in R \to y \in B)$

$(\Leftarrow)$

$(F(A), B) \in e_{P(Y)}$
iff $(\forall y \in Y)(y \in F(A) \to y \in B)$
iff $(\forall y \in Y)((\exists x \in X)(x \in A \,\&\, y \in F(\{x\}) \to y \in B)$
iff $(\forall y \in Y)(\forall x \in X)(x \in A \to ((x, y) \in R \to y \in B)$
iff $(\forall x \in X)(x \in A \to (\forall y \in Y)((x, y) \in R \to y \in B)$
iff $(\forall x \in X)(x \in A \to x \in G(B))$
iff $(A, G(B)) \in e_{P(X)}$.

(3)$(\Rightarrow)$

$y \in F(\{x\}^c)^c$ iff $(F(\{x\}^c), \{y\}^c) \in e_{P(Y)}$
iff $(G(\{y\}^c), \{x\}^c) \in e_{P(X)}$ iff $x \in G(\{y\}^c)^c$.

Put $(x, y) \in R$ iff $y \in F(\{x\}^c)$. By Theorem 2.1 (3), since $F(\cap_{i\in\Gamma} A_i) = \cup_{i\in\Gamma} F(A_i)$, then

$y \in F(A)$
iff $y \in F(\bigcap_{x\in A^c} \{x\}^c)$ iff $y \in \bigcup_{x\in A^c} F(\{x\}^c)$
iff $(\exists x \in X)(x \in A^c \,\&\, y \in F(\{x\}^c))$
iff $(\exists x \in X)(x \in A^c \,\&\, (x, y) \in R)$.

$x \in G(B)^c$
iff $(G(B), \{x\}^c) \in e_{P(X)}$ iff $(F(\{x\}^c), B) \in e_{P(Y)}$
iff $(B^c, F(\{x\}^c)^c) \in e_{P(Y)}$
iff $(\forall y \in Y)(y \in B^c \to y \in F(\{x\}^c)^c)$.

$x \in G(B)$ iff $(\exists y \in Y)(y \in B^c \,\&\, (x, y) \in R)$.

$(\Leftarrow)$

$(F(A), B) \in e_{P(Y)}$
iff $(\forall y \in Y)(y \in F(A) \to y \in B)$
iff $(\forall y \in Y)((\exists z \in X)(z \in A^c \,\&\, (y, z) \in R) \to y \in B)$
iff $(\forall y \in Y)(\forall z \in X)((z \in A^c \to ((y, z) \in R \to y \in B)$
iff $(\forall z \in X)((z \in A^c \to (\forall y \in Y)((y, z) \in R \to y \in B)$
iff $(\forall z \in X)((\exists y \in Y)((y, z) \in R \,\&\, y \in B^c) \to z \in A)$
iff $(G(B), A) \in e_{P(X)}$.

(4) Since

$y \in F(\{x\}^c)$ iff $(\{y\}, F(\{x\}^c)) \in e_{P(Y)}$
iff $(G(\{y\}), \{x\}^c) \in e_{P(X)}$ iff $x \in G(\{y\})^c$.

Put $(x, y) \in R$ iff $y \in F(\{x\}^c)^c$. By Theorem 2.1 (4), since $F(\cap_{i\in\Gamma} A_i) = \cap_{i\in\Gamma} F(A_i)$, then

$y \in F(A)$
iff $y \in F(\bigcap_{x\in A^c} \{x\}^c)$ iff $y \in \bigcap_{x\in A^c} F(\{x\}^c)$
iff $(\forall x \in X)(x \in A^c \to y \in F(\{x\}^c))$
iff $(\forall x \in X)(y \in F(\{x\}^c)^c \to x \in A)$
iff $(\forall x \in X)((x, y) \in R \to x \in A)$.

$x \in G(B)^c$
iff $(G(B), \{x\}^c) \in e_{P(X)}$ iff $(B, F(\{x\}^c)) \in e_{P(Y)}$
iff $(\forall y \in Y)(y \in B \to y \in F(\{x\}^c))$.

$x \in G(B)$ iff $(\exists y \in Y)(y \in B \,\&\, (x, y) \in R)$.

$(\Leftarrow)$

$(B, F(A)) \in e_{P(Y)}$
iff $(\forall y \in Y)(y \in B \to y \in F(A))$
iff $(\forall y \in Y)(y \in B \to (\forall z \in X)((z, y) \in R \to z \in A))$
iff $(\forall y \in Y)(\forall z \in X)((y \in B \,\&\, (z, y) \in R) \to z \in A))$
iff $(\forall z \in X)((\exists y \in Y)(y \in B \,\&\, (z, y) \in R) \to z \in A))$
iff $(G(B), A) \in e_{P(X)}$.

□

Example 2.3. Let $(X = \{a, b, c\}, e_{P(X)})$ and $(Y = \{x, y, z\}, e_{P(Y)})$ be posets with relation

$$R = \{(a, x), (a, z), (b, x), (b, y), (c, z)\}.$$

(1) From Theorem 2.3 (1), $(e_{P(X)}, F, G, e_{P(Y)})$ is a Galois connection with

$F(\emptyset) = Y, F(\{a\}) = \{x, z\}, F(\{b\}) = \{x, y\}$,
$F(\{c\}) = \{z\}, F(\{a, b\}) = \{x\}, F(\{a, c\}) = \{z\}$,
$F(\{b, c\}) = F(X) = \emptyset$,
$G(\emptyset) = X, G(\{x\}) = \{a, b\}, G(\{y\}) = \{b\}$,
$G(\{z\}) = \{a, c\}, G(\{x, y\}) = \{b\}, G(\{z, x\}) = \{a\}$,
$G(\{y, z\}) = G(Y) = \emptyset$.

(2) From Theorem 2.3 (2), $(e_{P(X)}, F, G, e_{P(Y)})$ is a residuated connection with

$F(\emptyset) = \emptyset, F(\{a\}) = \{x, z\}, F(\{b\}) = \{x, y\}$,
$F(\{c\}) = \{z\}, F(\{a, c\}) = \{x, z\}$,
$F(\{a, b\}) = F(\{b, c\}) = F(X) = Y$,
$G(\emptyset) = G(\{x\}) = G(\{y\}) = \emptyset$,
$G(\{z\}) = \{c\}, G(\{x, y\}) = \{b\}, G(\{z, x\}) = \{a, c\}$,
$G(\{y, z\}) = \{b, c\}, G(Y) = X$.

(3) From Theorem 2.3 (3), $(e_{P(X)}, F, G, e_{P(Y)})$ is a dual Galois connection with

$F(\emptyset) = Y, F(\{a\}) = F(\{c\}) = Y$,
$F(\{b\}) = \{x, z\}, F(\{a, c\}) = \{x, y\}, F(\{a, b\}) = \{z\}$,
$F(\{b, c\}) = \{x, z\}, F(X) = \emptyset$,
$G(\emptyset) = G(\{x\}) = G(\{y\}) = X, G(\{z\}) = \{a, b\}$,
$G(Y) = \emptyset, G(\{x, y\}) = \{a, c\}$,
$G(\{z, x\}) = \{b\}, G(\{y, z\}) = \{a, b\}$.

(4) From Theorem 2.3 (4), $(e_{P(X)}, F, G, e_{P(Y)})$ is a dual residuated connection with

$F(\emptyset) = \emptyset, F(\{a\}) = F(\{c\}) = \emptyset$,
$F(\{b\}) = \{y\}, F(X) = \emptyset, F(\{a, c\}) = \{z\}$,
$F(\{a, b\}) = \{x, y\}, F(\{b, c\}) = \{y\}$,
$G(\emptyset) = \emptyset, G(\{x\}) = \{a, b\}, G(\{y\}) = \{b\}$,
$G(\{z\}) = \{a, c\}, G(\{x, y\}) = \{a, b\}$,
$G(\{z, x\}) = G(\{y, z\}) = G(Y) = X$.

From the following theorem, we find generating functions which induce Galois, dual Galois, residuated and dual residuated connections.

Theorem 2.4. Let $(P(X), e_{P(X)})$ and $(P(Y), e_{P(Y)})$ be a poset and $F : P(X) \to P(Y)$ and $G : P(Y) \to P(X)$ maps.

(1) $(e_{P(X)}, F, G, e_{P(Y)})$ is a Galois connection iff there exists $F : P(X) \to P(Y)$ with $F(\{x\}) = \gamma_x$ such that

$$F(A) = \{y \in Y \mid (\forall x \in X)(x \in A \to y \in \gamma_x)\}.$$

(2) $(e_{P(X)}, F, G, e_{P(Y)})$ is a residuated connection iff there exists $F : P(X) \to P(Y)$ with $F(\{x\}) = \gamma_x$ such that

$$F(A) = \{y \in Y \mid (\exists x \in X)(x \in A \;\&\; y \in \gamma_x)\}.$$

(3) $(e_{P(X)}, F, G, e_{P(Y)})$ is a dual Galois connection iff there exists $F : P(X) \to P(Y)$ with $F(\{x\}^c) = \gamma_x$ such that

$$F(A) = \{y \in Y \mid (\exists z \in X)(z \in A^c \;\&\; y \in \gamma_x)\}$$

(4) $(e_{P(X)}, F, G, e_{P(Y)})$ is a dual residuated connection iff there exists $F : P(X) \to P(Y)$ with $F(\{x\}^c)^c = \gamma_x$ such that

$$F(A) = \{y \in Y \mid (\forall z \in X)(y \in \gamma_x \to z \in A)\}.$$

Proof. (1) $(\Rightarrow)$ Since $(\{y\}, F(\{x\})) \in e_{P(Y)}$ iff $(\{x\}, G(\{y\})) \in e_{P(Y)}$, $y \in F(\{x\})$ iff $x \in G(\{y\})$. Thus

$$\begin{aligned} y \in F(A) \quad & \text{iff } (\{y\}, F(A)) \in e_{P(Y)} \\ & \text{iff } (A, G(\{y\}) \in e_{P(X)} \\ & \text{iff } \vdash (\forall x \in X)((x \in A) \to x \in G(\{y\}))\} \\ & \text{iff } \vdash (\forall x \in X)((x \in A) \to y \in F(\{x\}))\} \end{aligned}$$

$(\Leftarrow)$ Since $F(\bigcup_{i\in\Gamma} A_i) = \{y \in Y \mid (\forall x \in X)(x \in \bigcup_{i\in\Gamma} A_i \to y \in \gamma_x)\} = \{y \in Y \mid (\forall i \in \Gamma)(\forall x \in X)(x \in A_i \to y \in \gamma_x)\}$, then $F(\bigcup_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i)$. We define

$$\begin{aligned} G(B) \; &= \bigcup\{C \Vdash B \to F(C)\} \\ &= \bigcup\{C \Vdash (\forall y \in Y)(y \in B \to (\forall x \in X) \\ &\quad (x \in C \to y \in F(\{x\}))\} \\ &= \bigcup\{C \Vdash (\forall y \in Y)(\forall x \in X)(x \in C \to \\ &\quad (y \in B \to y \in F(\{x\}))\} \\ &= \bigcup\{C \Vdash (\forall x \in X)(x \in C \to (\forall y \in Y) \\ &\quad (y \in B \to y \in F(\{x\}))\} \\ &= \{x \in X \mid (\forall y \in Y)(y \in B \to y \in F(\{x\}))\}. \end{aligned}$$

$$\begin{aligned} &(B, F(A)) \in e_{P(Y)} \\ &\text{iff } (\forall y \in Y)(y \in B \to y \in F(A)) \\ &\text{iff } (\forall y \in Y)(y \in B \to (\forall x \in X)(x \in A \to y \in F(\{x\})) \\ &\text{iff } (\forall x \in X)(x \in A \to (\forall y \in Y)(y \in B \to y \in F(\{x\})) \\ &\text{iff } (\forall x \in X)(x \in A \to x \in G(B)) \\ &\text{iff } (A, G(B)) \in e_{P(Y)}. \end{aligned}$$

(2)$(\Rightarrow)$ Since $(F(\{x\}), \{y\}^c) \in e_{P(Y)}$ iff $(\{x\}, G(\{y\}^c)) \in e_{P(Y)}$, then $y \in (F(\{x\})^c$ iff $x \in G(\{y\}^c))$. Thus,

$$\begin{aligned} y \in F(A)^c \quad & \text{iff } (F(A), \{y\}^c) \in e_{P(Y)} \\ & \text{iff } (A, G(\{y\}^c)) \in e_{P(X)} \\ & \text{iff } \vdash (\forall x \in X)((x \in A) \to x \in G(\{y\}^c)) \\ & \text{iff } \vdash (\forall x \in X)((x \in A) \to y \in F(\{x\})^c) \end{aligned}$$

Thus, $y \in F(A)$ iff $(\exists x \in X)(x \in A \;\&\; y \in F(\{x\}))$.

$(\Leftarrow)$ Since $F(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i)$,

$$\begin{aligned} &G(B) \\ &= \bigcup\{C \Vdash F(C) \to B\} \\ &= \bigcup\{C \Vdash (\forall y \in Y)(y \in F(C) \to y \in B)\} \\ &= \bigcup\{C \Vdash (\forall y \in Y)((\exists x \in X)(x \in C \;\&\; y \in F(\{x\}) \\ &\quad \to y \in B)\} \\ &= \bigcup\{C \Vdash (\forall y \in Y)(\forall x \in X)(x \in C \to (y \in F(\{x\}) \\ &\quad \to y \in B)\} \\ &= \bigcup\{C \Vdash (\forall x \in X)(x \in C \to (\forall y \in Y)(y \in F(\{x\}) \\ &\quad \to y \in B)\} \\ &= \{x \in X \mid (\forall y \in Y)(y \in F(\{x\}) \to y \in B)\} \end{aligned}$$

$$\begin{aligned} &(F(A), B) \in e_{P(Y)} \\ &\text{iff } (\forall y \in Y)(y \in F(A) \to y \in B) \\ &\text{iff } (\forall y \in Y)((\exists x \in X)(x \in A \;\&\; y \in F(\{x\}) \to y \in B) \\ &\text{iff } (\forall y \in Y)(\forall x \in X)(x \in A \to (y \in F(\{x\}) \to y \in B) \\ &\text{iff } (\forall x \in X)(x \in A \to (\forall y \in Y)(y \in F(\{x\}) \to y \in B) \\ &\text{iff } (\forall x \in X)(x \in A \to x \in G(B)) \\ &\text{iff } (A, G(B)) \in e_{P(X)}. \end{aligned}$$

(3)$(\Rightarrow)$ Since $(F(\{x\}^c), \{y\}^c) \in e_{P(Y)}$ iff $(G(\{y\}^c), \{x\}^c) \in e_{P(X)}$, $y \in F(\{x\}^c)^c$ iff $x \in G(\{y\}^c)^c)$. Thus,

$$\begin{aligned} y \in F(A)^c \quad & \text{iff } (F(A), \{y\}^c) \in e_{P(Y)} \\ & \text{iff } (G(\{y\}^c), A) \in e_{P(X)} \\ & \text{iff } \vdash (\forall x \in X)(x \in G((\{y\}^c) \to x \in A) \\ & \text{iff } \vdash (\forall x \in X)(x \in A^c \to x \in G(\{y\}^c)^c) \\ & \text{iff } \vdash (\forall x \in X)(x \in A^c \to y \in F(\{x\}^c)^c) \end{aligned}$$

Hence $y \in F(A)$ iff $\vdash (\exists x \in X)(x \in A^c \;\&\; y \in F(\{x\}^c))$.

$(\Leftarrow)$ Since Since $F(\bigcap_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i)$, we define

$$\begin{aligned} &G(B) \\ &= \bigcap\{C \Vdash F(C) \to B\} \\ &= \bigcap\{C \Vdash (\forall y \in Y)(y \in F(C) \to y \in B)\} \\ &= \bigcap\{C \Vdash (\forall y \in Y)((\exists z \in X)(z \in C^c \;\&\; y \in F(\{z\}^c) \\ &\quad \to y \in B)\} \\ &= \bigcap\{C \Vdash (\forall y \in Y)(\forall z \in X)(z \in C^c \to (y \in F(\{z\}^c)) \\ &\quad \to y \in B)\} \\ &= \bigcap\{C \Vdash (\forall z \in X)(z \in C^c \to (\forall y \in Y)(y \in F(\{z\}^c) \\ &\quad \to y \in B)\} \\ &= \bigcap\{C \Vdash (\forall z \in X)((\exists y \in Y)(y \in F(\{z\}^c) \;\&\; y \in B^c) \\ &\quad \to z \in C)\} \\ &= \{x \in X \mid (\exists y \in Y)(y \in F(\{z\}^c) \;\&\; y \in B^c)\} \end{aligned}$$

$(F(A),B)\in e_{P(Y)}$
iff $(\forall y\in Y)(y\in F(A)\to y\in B)$
iff $(\forall y\in Y)((\exists z\in X)(z\in A^c \,\&\, y\in F(\{z\}^c)$
$\to y\in B)$
iff $(\forall y\in Y)(\forall z\in X)((z\in A^c\to (y\in F(\{z\}^c)$
$\to y\in B)$
iff $(\forall z\in X)((z\in A^c\to (\forall y\in Y)(y\in F(\{z\}^c)$
$\to y\in B)$
iff $(\forall z\in X)((\exists y\in Y)(y\in F(\{z\}^c)\,\&\, y\in B^c)$
$\to z\in A)$
iff $(G(B),A)\in e_{P(X)}$.

(4)($\Rightarrow$) Since $(\{y\},F(\{x\}^c))\in e_{P(Y)}$ iff $(G(\{y\}),\{x\}^c)\in e_{P(X)}$, $y\in F(\{x\}^c)$ iff $x\in G(\{y\})^c)$. Thus,

$$\begin{aligned} y\in F(A) \quad &\text{iff } (\{y\},F(A))\in e_{P(Y)}\\ &\text{iff } (G(\{y\}),A)\in e_{P(X)}\\ &\text{iff } (\forall x\in X)(x\in G(\{y\})\to (x\in A))\\ &\text{iff } (\forall x\in X)(x\in A^c\to x\in G(\{y\})^c)\\ &\text{iff } (\forall x\in X)(x\in A^c\to y\in F(\{x\}^c))\\ &\text{iff } (\forall z\in X)(y\in F(\{x\}^c)^c\to z\in A)\end{aligned}$$

($\Leftarrow$) Since $F(\bigcap_{i\in\Gamma}A_i)=\bigcap_{i\in\Gamma}F(A_i)$, we define

$G(B)$
$=\bigcap\{C\mid\vdash B\to F(C)\}$
$=\bigcap\{C\mid (\forall y\in Y)(y\in B\to (\forall z\in X)(y\in F(\{z\}^c)^c$
$\to z\in C))\}$
$=\bigcap\{C\mid (\forall y\in Y)(\forall z\in X)(y\in B\,\&\, y\in F(\{z\}^c)^c)$
$\to z\in C)\}$
$=\bigcap\{C\mid (\forall z\in X)((\exists y\in Y)(y\in B\,\&\, y\in F(\{z\}^c)^c)$
$\to z\in C)\}$
$=\{x\in X\mid (\exists y\in Y)(y\in B\,\&\, y\in F(\{x\}^c)^c)\}$.

$(B,F(A))\in e_{P(Y)}$
iff $(\forall y\in Y)(y\in B\to y\in F(A))$
iff $(\forall y\in Y)(y\in B\to (\forall z\in X)(y\in F(\{z\}^c)^c$
$\to z\in A))$
iff $(\forall y\in Y)(\forall z\in X)((y\in B\,\&\, y\in F(\{z\}^c)^c)$
$\to z\in A))$
iff $(\forall z\in X)((\exists y\in Y)(y\in B\,\&\, y\in F(\{z\}^c)^c)$
$\to z\in A))$
iff $(G(B),A)\in e_{P(X)}$.

□

Example 2.5. Let $(X=\{a,b,c\},e_{P(X)})$ and $(Y=\{x,y,z\},e_{P(Y)})$ be posets. Define $F_i: P(X)\to P(Y)$ for $i=1,2$ with

$$F_1(\{a\})=\{x\},F_1(\{b\})=\{x,y\},F_1(\{c\})=\{y,z\},$$

$$F_2(\{a,b\})=\{x\},F_2(\{a,c\})=\{x,y\},F_2(\{b,c\})=\{y,z\}.$$

(1) From Theorem 2.4 (1), $(e_{P(X)},F_1,G_1,e_{P(Y)})$ is a Galois connection with

$F_1(\emptyset)=Y,F_1(\{a\})=\{x\},F_1(\{b\})=\{x,y\},$
$F_1(\{c\})=\{y,z\},F_1(\{a,b\})=\{x\},F_1(\{b,c\})=\{y\},$
$F_1(\{a,c\})=F(X)=\emptyset$
$G_1(\emptyset)=X,G_1(\{x\})=\{a,b\},G_1(\{y\})=\{b,c\},$
$G_1(\{z\})=\{c\},G_1(\{x,y\})=\{b\},G_1(\{y,z\})=\{c\},$
$G_1(\{z,x\})=G_1(Y)=\emptyset.$

(2) From Theorem 2.4 (2), $(e_{P(X)},F_1,G_1,e_{P(Y)})$ is a residuated connection with

$F_1(\emptyset)=\emptyset,F_1(\{a\})=\{x\},F_1(\{b\})=\{x,y\},$
$F_1(\{c\})=\{y,z\},F_1(\{a,b\})=\{x,y\},$
$F_1(\{a,c\})=F_1(\{b,c\})=F_1(X)=Y,$
$G_1(\emptyset)=G_1(\{y\})=G_1(\{z\})=\emptyset,$
$G_1(\{x\})=\{a\},G_1(\{x,y\})=\{a,b\},G_1(\{z,x\})=\{a\},$
$G_1(\{y,z\})=\{c\},G_1(Y)=X.$

(3) From Theorem 2.4 (3), $(e_{P(X)},F_2,G_2,e_{P(Y)})$ is a dual Galois connection with

$F_2(\emptyset)=Y,F_2(\{b\})=F_2(\{c\})=Y,$
$F_2(\{a\})=\{x,y\},F_2(\{a,c\})=\{x,y\},F_2(\{a,b\})=\{x\},$
$F_2(\{b,c\})=\{y,z\},F_2(X)=\emptyset$
$G_2(\emptyset)=G_2(\{y\})=G_2(\{z\})=X,$
$G_2(\{x\})=\{a,b\},G_2(\{x,y\})=\{a\},G_2(\{z,x\})=\{a,b\},$
$G_2(\{y,z\})=\{b,c\},G_2(Y)=\emptyset.$

(4) From Theorem 2.4 (4), $(e_{P(X)},F,G,e_{P(Y)})$ is a dual residuated connection with

$F_2(\emptyset)=\emptyset,F_2(\{b\})=\emptyset,F_2(\{a\})=\{x\},$
$F_2(\{c\})=\{y\},F_2(\{a,c\})=\{x,y\},F_2(\{a,b\})=\{x\},$
$F_2(\{b,c\})=\{y,z\},F_2(X)=Y,$
$G_2(\emptyset)=\emptyset,G_2(\{x\})=\{a\},G_2(\{y\})=\{c\},$
$G_2(\{z\})=\{b,c\},G_2(\{x,y\})=\{a,c\},G_2(\{y,z\})=\{b,c\},$
$G_2(\{z,x\})=X,G_2(Y)=X.$

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