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# Mathematical Thinking through Different Representations and Analogy[1]

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Mathematical thinking is a core element in mathematics education and classroom learning. This paper wish to investigate how primary four (grade 4) students develop their mathematical thinking through working on tasks in multiplication where greatest products of multiplication are required. The tasks include the format of many digit times one digit, 2 digits times 2 digits up to 3 digits times 3 digits. It is found that the process of mathematical thinking of students depends on their own representation in obtaining the product. And the solution is obtained through a pattern/analogy and "pattern plus analogy" process. This specific learning process provides data for understanding structure and mapping in problem solving. The result shows that analogy allows successful extension of solution structure in the tasks.

*Keywords*: mathematical thinking, solution structure
*ZDM Classification:* C32, F32
*MSC2000 Classification:* 97C50

## INTRODUCTION

The aim of the paper is to discuss how a class of grade four students communicate their thinking through working on problems in finding the greatest product in multiplication, and that mathematical thinking is enhanced through analogy and pattern observation. This enables them to use different representations, algebraic and geometric. To achieve this aim, a set of mathematics tasks with progressive mathematisation and structure is designed. Progressive mathematisation means student can "acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections, structure" (Treffers & Beishuizen, 1999).

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According to Bruner (1960), students learn best when students do the categorization according to their own thinking. This is a form of coding system devised by the learners themselves and such process allows them to transfer information more effectively. In recognising the structure, students do best in this investigation. Reid (2002) clarifies pattern of reasoning which are mathematical. He made a distinction between a mathematical reasoning and scientific reasoning, on the basis of the criteria used to accept or reject reasoning in each domain.

Relational complexity may be defined in terms of dimensionality. Halford (1993) shown that there are four levels of dimensionality. English & Halford (1995) discussed structure mapping and suggested that it consists a set of rules of mapping elements from a structure A to another structure B so that the relationships of the objects in the two structures are preserved. English & Halford (1995) defined four levels of structure mapping. The four levels are element mappings, relational mappings, system mappings, and multiple system mappings.

There are several ways for students to deliver their reasoning. The first reasoning process is testing that confirms a conjecture. The second reasoning process is testing that refutes an implicit conjecture. The third reasoning process is barring exceptions to a generalization, in which contradictions are not accepted as counter-examples.

In this study, the sequence starts with pattern observation, conjecturing that the pattern generally, testing the conjecture, generalize the conjecture, and finally use the conjecture for generalization for simple deduction (pattern → conjecture → test → generalize → deduction).

## THE STUDY

To allow thinking to happen, it is important to select appropriate set of tasks and allow students to express their solution through different representations. The using of different question format helps children to relate the process of solution and the original mathematics task. And teaching tasks of mathematical structure is helpful for students in solving the tasks (Cheng, 2008).

It is a series of 4 lessons of multiplication. It is a view that children can be flexible in their multiplication process and invent some of their multiplication (Ambrose, Baek & Carpenter, 2003). The first lesson is an investigation of finding the greatest product of one digit times multiple digits. It is expected that students can explore the problem, obtains their solutions, and can explain their answer. The second to fourth lesson is to apply the result that they got in the first lesson and use it to extend to new type of tasks, finding the greatest product of multiplication of 2 digits with 2 digits and in other format.

Students need deliver their reasoning through representations that is induced by teacher or developed by students themselves.

The following is the sequence of tasks:

*Find the greatest product in the following format.*

(A) $\begin{array}{r} \square\,\square\,\square \\ \times \quad\quad \square \\ \hline \end{array}$ (B) $\begin{array}{r} \square\,\square \\ \times \quad \square\,\square \\ \hline \end{array}$ (C) $\begin{array}{r} \square\,\square\,\square \\ \times \quad \square\,\square \\ \hline \end{array}$ (D) $\begin{array}{r} \square\,\square\,\square \\ \times\, \square\,\square\,\square \\ \hline \end{array}$

The beginning questions use some simple number so that students can obtain answers easily. It encourages students to use generic skills such as listing and observing pattern. Students are required to explain their answer based on pattern and observation and through different representations.

**Task A.** finding the greatest product of multiple digits times one digit

| Question A1: | |
|---|---|
| Use the numbers 1, 2, and 3 to fill in the boxes to make the greatest product. | $\begin{array}{r} \square\,\square \\ \times \quad \square \\ \hline \end{array}$ |

The first question (A1) requires students to find the greatest product of 2 digits time 1 digit, using the number 1, 2 and 3. Students are asked to guess the arrangement of the number 1, 2, and 3 which produce the greatest product. Many students can see the correct answer right the way. Then they are asked to explain their answer by listing all the possible products.

$$\begin{array}{r} 1\;2 \\ \times \quad 3 \\ \hline 3\;6 \end{array} \qquad \begin{array}{r} 1\;3 \\ \times \quad 2 \\ \hline 2\;6 \end{array} \qquad \begin{array}{r} 2\;1 \\ \times \quad 3 \\ \hline 6\;3 \end{array} \qquad \begin{array}{r} 2\;3 \\ \times \quad 1 \\ \hline 2\;3 \end{array} \qquad \begin{array}{r} 3\;2 \\ \times \quad 1 \\ \hline 3\;2 \end{array} \qquad \begin{array}{r} 3\;1 \\ \times \quad 2 \\ \hline 6\;2 \end{array}$$

Then students were asked to explain why this product is the greatest and work on another question with different digits. For example, students need to find the greatest product by arranging the numbers 2, 6, 8. Nearly all students concluded that large numbers should be placed at the position P and Q.

| Question A2: | |
|---|---|
| Use the numbers 2, 6, and 8 to fill in the boxes to make the greatest product. | $\begin{array}{r} \square\,\square \\ \times \quad \square \\ \hline \end{array}$ |

$$\begin{array}{r} 6\ 2 \\ \times \quad 8 \\ \hline 4\ 9\ 6 \end{array} \qquad \begin{array}{r} 8\ 2 \\ \times \quad 6 \\ \hline 4\ 9\ 2 \end{array} \qquad \begin{array}{r} P\ \square \\ \times \quad Q \\ \hline \end{array}$$

(P and Q are crucial position)

Though students' could easily noted that the arrangement of 62×8 gives a greater product than 82×6, but explaining why the two products differ by 4 is not easy.

**Investigation of the difference of products**

When invited to use what they knows for comparison, the following vertical representation in multiplication are shown by students.

$$\begin{array}{r} 6\ 2 \\ \times \quad 8 \\ \hline 4\ 8\ 0 \\ 1\ 6 \\ \hline \end{array} \qquad \begin{array}{r} 8\ 2 \\ \times \quad 6 \\ \hline 4\ 8\ 0 \\ 1\ 2 \\ \hline \end{array}$$

Then, a second approach to compare the two is by listing the sum as follows:

| | |
|---|---|
| 60 | 2 |
| 60 | 2 |
| 60 | 2 |
| 60 | 2 |
| 60 | 2 |
| 60 | 2 |
| 60 | 2 |
| 60 | 2 |

480 + 16

| | |
|---|---|
| 80 | 2 |
| 80 | 2 |
| 80 | 2 |
| 80 | 2 |
| 80 | 2 |
| 80 | 2 |

480 + 12

From these representations, it could be seen that the different arise from 16 and 12, which is 4. And 16 is 2×8, 12 is 2×6.

The above representations lead to another search of representation of solution. This time based on the distributive law of multiplication, where the difference 4 is easily explained.

$62 \times 8 = (60+2) \times 8 = 480 + 16$
$82 \times 6 = (80+2) \times 6 = 480 + 12$

The above three representations are all algebraic, with the second one some potential to be pictorial. This helps students to understand why such arrangement gives greater product. Then students are asked to work on tasks with extended structure.

***Extension 1 (Using Pattern Observation for extend structure)***

In this extension, students are asked to solve the task of 3 digits times 1 digit.

| Question A2: | |
|---|---|
| Use the numbers 1, 2, 3, and 4 to fill in the boxes to make the greatest product. | □ □ □ <br> × □ |

Many students predicted that the greatest product is either 321×4 or 421×3, through pattern observation. And with a short moment, many concluded that 321 × 4 is greater than 421×3.

The progress of their thinking is based on pattern observation. With the analogy of the earlier problem, there is a dominant effect of the largest number in certain "critical" position, which is Q, as the number at position Q will multiple all other numbers, hence the largest number at Q is a reasonable guess. Such analogy is based on deduction of intuition. The intuition is based on pattern observation. In order to look deeper into the tasks, students are asked to find ways to compare products to justify their answer.

```
  □ □ □
×     Q        (number at Q position is crucial)
```

First representation for multiplication by students

Most multiplication process in school is done in vertical format. However, students are encouraged to perform multiplication in the following vertical format so that they can compare different layers of number. From the partition of layer, students can compare the sum of product layer by layer and conclude that 321×4 is in fact the largest.

```
      3 2 1                        4 2 1
×         4                  ×         3
  ---------                    ---------
    1 2 0 0       (equal)        1 2 0 0
        8 0       (-2 0)             6 0
          4       (-1)                 3
  ---------                    ---------
    1 2 8 4                      1 2 6 3
```

(321×4 = 1200 + 80 + 4) (421×3 = 1200 + 60 + 3)

Second representation for multiplication by students

The above format can be represented in a row operation. Using sum of rows to explain the differences of the two products of 321×4 and 421×3.

| | | |
|---|---|---|
| 300 | 20 | 1 |
| 300 | 20 | 1 |
| 300 | 20 | 1 |
| 300 | 20 | 1 |

1200 + 80 + 4

| | | |
|---|---|---|
| 400 | 20 | 1 |
| 400 | 20 | 1 |
| 400 | 20 | 1 |

1200 + 60 + 3

Third representation for multiplication by students

Some students use the area concepts to explain why the arrangement gives greatest product. This is a pictorial representation of the above calculation. In fact the sum of the three numbers "1200 + 60 + 3" and "1200 + 80 + 4" can be compared.

| | 300 | 20 | 1 |
|---|---|---|---|
| 4 | 1200 | 80 | 4 |

| | 400 | 20 | 1 |
|---|---|---|---|
| 3 | 1200 | 60 | 3 |

With the three representations, students investigate further on.

| Question A3: | |
|---|---|
| Using the numbers 1, 2, 3, 5, 6 to fill in the boxes to make the greatest product. | □ □ □ □ × □      P 3 2 1 × Q |

Almost all of the students can transfer their answer of the previous question to this question, that the two greatest numbers should be put in at the position P and Q. And among the two arrangement 5321×6 and 6321×5, the greatest product is 5321×6.

Another logical argument provided by students is deduction. As in previous work, the only two possible products are 5321×6 and 6321×5. Some students noticed that there is a common part 321 in the two products, so they just compare the products 321×6 and 321×5. This is the same as using distributive law to compare (5000 + 321)×6 and (6000 + 321)×5. As "5000× 6 = 6000× 5", students knew that they only need to compare the part "321× 6 and 321× 5", and concluded that "5321×6" is the greatest.

***Extension 2 (Using Analogy)***

In the second lesson, a new version of the task is introduced.

| Question B1: | |
|---|---|
| Using the numbers 1, 2, 3, 4 to fill in the boxes to make the greatest product. | $\begin{array}{r} \square\,\square \\ \times\ \square\,\square \\ \hline \end{array}$ |

Based on the analogy of the results in the first lesson, the number 4 and 3 will occupy the position of P and Q. That left with two possibilities, 41×32 and 42×31. As 42×31 = 1312 and 41×32 = 1302, they concluded that 41×32 is the greatest product.

$$\begin{array}{r} P\ \square \\ \times\quad Q\ \square \\ \hline \end{array}$$

The next question is as follow:

| Question B2 and answer: | |
|---|---|
| Using the numbers 1, 3, 5, 7 to fill in the boxes to make the greatest product. | $\begin{array}{r} 7\ 1 \\ \times\ 5\ 3 \\ \hline \end{array}$ |

Almost all students could answer the question by pattern observation and analogy. They fill in the four numbers inn the boxes with respective order as in the earlier task. Students also try to explain why the difference of the two product 71×53 = 3763 and 73×51 = 3723 is 40 through the following vertical format.

One of the explanations is to obtain the product line by line, which is as follow.

$$\begin{array}{r} 7\ 1 \\ \times\quad 5\ 3 \\ \hline 3\ 5\ 0\ 0 \\ 5\ 0 \\ 2\ 1\ 0 \\ 3 \\ \hline 3\ 7\ 6\ 3 \end{array} \quad \begin{array}{l} \\ \\ \text{(equal)} \\ (+\ 100) \\ (-140) \\ \text{(equal)} \\ \\ \end{array} \quad \begin{array}{r} 7\ 3 \\ \times\quad 5\ 1 \\ \hline 3\ 5\ 0\ 0 \\ 1\ 5\ 0 \\ 7\ 0 \\ 3 \\ \hline 3\ 7\ 2\ 3 \end{array}$$

There are other explanations as well. Some students transform the algebraic representation to geometric representation. This is the influence of the pictorial representations from earlier tasks. These numbers are compared with similar entries as in vertical multiplication.

| | 70 | 1 |
|---|---|---|
| 50 | 3500 | 50 |
| 3 | 210 | 3 |

| | 70 | 3 |
|---|---|---|
| 50 | 3500 | 150 |
| 1 | 70 | 3 |

**Correct answer with insufficient reasoning**

Correct answer may not mean good reasoning. One response is that the numbers should be "equally" divided into two groups. Equally means a small number should pair with a larger number. And in this case, the four numbers are 1, 3, 5 and 7, and hence number 1 pair with 7 and number 3 pair with 5. The result arrangement is 71×53. Though this compensation view resulted in correct answer, it could not be further used in more complicated structure.

Another example is the using of cross multiplication. Students knew that there are four products and two of the 4 products are equal. They are 70×50 and 3×1. Hence they developed the following comparison, comparing (7×1 + 5×3) and (7×3 + 5×1). The former is 22 and the later 26, which implies that the later one gives the greatest product.


Students are using more diagram approach in explaining why the product they got is the greatest. There is a big decrease in the number of students using row format, as it is clumsier and easily replaced by diagram.

The following table summaries the changes.

**Table 1.** Summary of the changes

| Representations used | 1 digit× multiple digits | 2 digits × 2 digits |
|---|---|---|
| Vertical format | 14 | 5 |
| Row format | 10 | 3 |
| Diagram approach | 8 | 24 |
| Total number of students | 32 | 32 |

***Extension 3 (Solving tasks with Pattern Observation and Analogy)***

The third lesson is the solving of an extended task, 3 digits times 2 digits and 3 digits times 3 digits.

| Question C1: | |
|---|---|
| Use the numbers 1, 2, 3, 4, and 5 to fill in the boxes to make the greatest product. | $\begin{array}{r}\square\ \square\ \square\\ \times\quad \square\ \square\\ \hline\end{array}$ $\quad$ $\begin{array}{r}P\ \square\ \square\\ \times\quad Q\ \square\\ \hline\end{array}$ |

Using analogy, students knew that the two numbers "4 and 5" should be placed at the position P or Q. There remains the questions of whether P > Q, or Q > P.

By analogy of the task in the first lesson (multiple digits × 1 digit), it seems that the largest number be placed at Q. This is an assumption, the rest is deduction.

One of the responses was to assume that the numbers used are "1, 1, 3, 4, and 5". Since there are two smallest number and they are "1", they will form the unit digit of the two numbers. The question reduces to finding the greatest product in the form of $\square\square\boxed{1}\times\square\boxed{1}$ . As the answer of using 3, 4, and 5 to form the largest product $\square\square\times\square$is known. Students deduce that the greatest product must be $\boxed{43}\,\boxed{1}\times\boxed{5}\,\boxed{1}$ .

Back to the original question of using "1, 2, 3, 4 and 5", the greatest product must be 432×51 or 431×52. The process of thinking is as follow.

$$\begin{array}{r}4\ 3\,\vdots\,\square\\ \times\quad 5\,\vdots\,\square\\ \hline\end{array} \Rightarrow \begin{array}{r}4\ 3\,\vdots\,1\\ \times\quad 5\,\vdots\,1\\ \hline\end{array} \Rightarrow \text{(i)}\begin{array}{r}4\ 3\,\vdots\,2\\ \times\quad 5\,\vdots\,1\\ \hline\end{array} \text{ or } \text{(ii)}\begin{array}{r}4\ 3\,\vdots\,1\\ \times\quad 5\,\vdots\,2\\ \hline\end{array}$$

Product (i) will give a larger value than (ii) as $\boxed{431}\times 2$ is larger than $\boxed{432}\times 1$.
This was checked with the following calculations.

$$\text{(i)}\begin{array}{r}4\ 3\ 1\\ \times\quad 5\ 2\\ \hline 2\ 2\ 4\ 1\ 2\end{array} \quad \text{(ii)}\begin{array}{r}4\ 3\ 2\\ \times\quad 5\ 1\\ \hline 2\ 2\ 0\ 3\ 2\end{array} \quad \text{(iii)}\begin{array}{r}5\ 3\ 2\\ \times\quad 4\ 1\\ \hline 2\ 1\ 8\ 1\ 2\end{array} \quad \text{(iv)}\begin{array}{r}5\ 3\ 1\\ \times\quad 4\ 2\\ \hline 2\ 2\ 3\ 0\ 2\end{array}$$

The reasoning is then transferred to the following task.

| Question D1: | |
|---|---|
| Using the numbers 1, 3, 5, 7, 8, 9 to fill in the boxes to make the greatest product. | $\begin{array}{r}\square\ \square\ \square\\ \times\ \square\ \square\ \square\\ \hline\end{array}$ $\quad$ $\begin{array}{r}P\ \square\ \square\\ \times\ Q\ \square\ \square\\ \hline\end{array}$ |

Students are asked to refer to their previous work of Question B2 for insight in solving this task.

| Previous Question (B2): | New Question D1 (Using Analogy): |
|---|---|
| The greatest product for using the numbers 1, 3, 5, 7 is<br>  7 1<br>× 5 3 | Using the numbers 1, 3, 5, 7, 8, 9 to fill in the boxes to make the greatest product.<br>  □ □ □<br>× □ □ □ |

Here the greatest product formed by "1, 3, 5, 7, 8, and 9" in the format of □□□×□□□ is by extending the results of □□×□□, with the addition of two more numbers "8 and 9". Many students predicted that the greatest product is 830223 and confirmed by representation used below.

$$\begin{array}{r} 7\ 1 \\ \times\ 5\ 3 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 9\ 5\ 1 \\ \times\ 8\ 7\ 3 \\ \hline 8\ 3\ 0\ 2\ 2\ 3 \end{array}$$

The following table gives the information on the approaches taken by students in tackling the problem. As times go by, more students are using both pattern and analogy to solve the tasks. It also shows that using both approaches could achieve in faster results.

**Table 2.** Information on the approaches by students in tackling the problem

| Approaches in explanation | 3 digits×2 digits | 3 digits×3 digits |
|---|---|---|
| Pattern Observation only | 13 | 7 |
| Analogy only | 6 | 8 |
| Pattern Observation and Analogy | 13 | 17 |
| Total number of students | 32 | 32 |

***Extension 4 (Individual investigation and Extensions)***

In the forth lesson, students are asked to extend their investigation, to find the greatest product of three numbers with their own arrangement of 5 or 6 integers. The following two extensions are the most common extensions given by students.

| Students extension 1: | Students extension 2: |
|---|---|
| Use the numbers 1, 2, 3, 4, and 5 to fill in the boxes to make the greatest product.<br>□□ 4 1<br>□□ 3 2<br>× □ × 5<br>6 5 6 0 | Use the numbers 1, 2, 3, 4, 5, and 6 to fill in the boxes to make the greatest product.<br>□□ 6 1<br>□□ 5 2<br>× □□ × 4 3<br>13 63 96 |

For students extension 1, based on the analogy of the results of □□×□ and □□×□□, nearly all students can obtained their answer. And for students' extension 2, the analogy they used is the repeated usage of the pattern □□×□□. In these extensions, only algebraic representation is used by students. The ways students induced the answer base on analogy in line with other study on induction and analogy (English, 1997), and individual learners have their specific process of mathematical thinking, sometimes through their own invention of representations.

**Table 3.** The stages of structural correspondence

| Stages | Remark |
|---|---|
| 1.<br>Formulation of schema, connection and compression of cognitive units | Task: find the greatest product of multiple digits × 1 digit.<br>□□<br>× □ |
| 2.<br>Structure mapping by pattern observation and recognition. | Task: find the greatest product of multiple digits × 1 digit.<br>□□□□<br>× □ |
| 3.<br>Extended Structure Mapping through analogy | Task: find the greatest product of 2 digits × 2 digits.<br>□□<br>× □□ |
| 4.<br>Extension and Generalization through Pattern and Analogy | Task: find the greatest product of 3 digits × 2 digits and of 3 digits × 3 digits.<br>□□□ □□□<br>□□ × □□□ |

## CONCLUSION

One of the definitions of reasoning is the transfer of structural information from a base system to the target system (English, 1997). This was also shared by Gentner (1983) and Vosniadou (1989). In this study, the specific task of finding greatest product shows that pattern observation (the crucial position of the largest integer) and analogy (examining the structure of "2 digits × 2 digits" from "1 digit × multiple digits") are used together by students in solving more irregular problem. It suggests that different cognitive units could be related and compressed into a schema through structural correspondence, and such schema is meaningful when related to mathematical structure. This could also be described by the stages of structural correspondence in Cheng (2010) as in Table 3.

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