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GENERATING NEW FRAMES IN $L^2(\mathbb{R})$ BY CONVOLUTIONS

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ABSTRACT. Let $\mathbf{c} = \{c_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ and $\{f_n\}_{n\in\mathbb{Z}}$ be a frame (Riesz basis, respectively) of $L^2(\mathbb{R})$. We obtain necessary and sufficient conditions on $\mathbf{c}$ under which $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ becomes a frame (Riesz basis, respectively) of $L^2(\mathbb{R})$, where $\lambda > 0$ and $\big(\mathbf{c} *_\lambda f\big)(t) := \sum_{n\in\mathbb{Z}} c_n f(t - n\lambda)$. When $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ becomes a frame of $L^2(\mathbb{R})$, we present its frame operator and the canonical dual frame in a simple form. Some interesting examples are included.

1. PRELIMINARIES

A frame is an overcomplete system in a Hilbert space H that provides basis-like representations of vectors in H. In general, the representations are not unique and this nonuniqueness (or redundancy) gives rise to robust and stable representations, which explains the usefulness of frames in applications. For backgrounds on frames, we refer to the books [3, 8, 10] or the research tutorials [2, 6].

A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in a (separable) Hilbert space H equipped with the inner product $\langle\, ,\rangle$ is

- a Bessel sequence of H (with a bound B) if there is a constant $B > 0$ such that
$$\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \le B\,\|f\|^2\,,\ f \in H;$$
- a frame of H (with bounds (A, B)) if there are constants $B \ge A > 0$ such that
$$A\|f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \le B\,\|f\|^2\,,\ f \in H;$$

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- a Riesz basis of H (with bounds (A, B)) if it is complete in H and there are constants $B \geq A > 0$ such that
$$A\,\|\mathbf{c}\|_2^2 \leq \|\sum_{n\in\mathbb{Z}} c_n f_n\|^2 \leq B\,\|\mathbf{c}\|_2^2\,,\ \mathbf{c} = \{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}),$$
where $\|\mathbf{c}\|_2^2 := \sum_{n\in\mathbb{Z}} |c_n|^2$.

If a frame ceases to be a frame when any one of its members is removed, it is called an *exact* frame.

It follows from the definition that Riesz bases are frames. In fact, a frame of H is exact if and only if it is a Riesz basis of H (see Theorem 6.1.1 in [3]).

For a frame $\{f_n\}_{n\in\mathbb{Z}}$ of H with bounds (A, B), let
$$S(f) = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle f_n,\ f \in H$$
be the *frame operator* of $\{f_n\}_{n\in\mathbb{Z}}$. Then S is a self-adjoint automorphism on H and $\{S^{-1}(f_n)\}_{n\in\mathbb{Z}}$ is also a frame of H with bounds $(\frac{1}{B}, \frac{1}{A})$, called the *canonical dual frame* of $\{f_n\}_{n\in\mathbb{Z}}$. We then have the frame expansion property;
$$f = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle S^{-1}(f_n) = \sum_{n\in\mathbb{Z}} \langle f, S^{-1}(f_n)\rangle f_n,\ f \in H, \tag{1.1}$$
which converges unconditionally in H. By definition, a frame is preserved under any automorphism of H.

Riesz bases of H are characterized as the families $\{Ue_k\}_{k\in\mathbb{Z}}$ where $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of H and U is an automorphism on H (see Theorem 3.6.6 in [3]).

Fourier transform is defined by
$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) := \int_{-\infty}^{\infty} f(t)\, e^{-it\xi}\, dt,\ f(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$
so that $\frac{1}{\sqrt{2\pi}}\mathcal{F}[\cdot]$ extends to be a unitary operator on $L^2(\mathbb{R})$.

For any $\lambda > 0$, define the scaling operator $\delta_\lambda : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $(\delta_\lambda f)(t) := f(t/\lambda)$. Then
$$\widehat{\big(\delta_\lambda f\big)}(\xi) = \lambda \widehat{f}(\lambda\,\xi). \tag{1.2}$$

For any $\lambda > 0$, $\mathbf{c} = \{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and $f \in L^2(\mathbb{R})$, let $\bar{\mathbf{c}} := \{\overline{c_n}\}_{n\in\mathbb{Z}}$, $\widetilde{\mathbf{c}} := \{\overline{c_{-n}}\}_{n\in\mathbb{Z}}$ and $\big(\mathbf{c} *_\lambda f\big)(t) := \sum_{n\in\mathbb{Z}} c_n f(t - n\lambda)$ be the generalized discrete-continuous convolution product. Then $\big(\mathbf{c} *_\lambda f\big)(t) \in L^1_{loc}(\mathbb{R})$ (see Theorem 2 in [11]).

We also let
$$\widehat{\mathbf{c}}(\xi) := \sum_{n\in\mathbb{Z}} c_n e^{-in\xi}.$$
Then $\widehat{\mathbf{c}}(\xi) \in L^2[0, 2\pi]$ is 2π-periodic and $\widehat{\widetilde{\mathbf{c}}}(\xi) = \overline{\widehat{\mathbf{c}}(\xi)}$. Moreover, if $\mathbf{c} \in \ell^1(\mathbb{Z})$, then $\widehat{\mathbf{c}}(\xi) \in C[0, 2\pi]$.

2. GENERATING NEW FRAMES IN $L^2(\mathbb{R})$ BY CONVOLUTIONS

Lemma 2.1. *Let $f \in L^2(\mathbb{R})$ and $\mathbf{c} \in \ell^2(\mathbb{Z})$ be such that $\big(\mathbf{c} *_\lambda f\big)(t)$ converges in $L^2(\mathbb{R})$.*

(a) *If $\{f(t-n\lambda)\}_{n\in\mathbb{Z}}$ is a Bessel sequence of $L^2(\mathbb{R})$, then $G_{f,\lambda}(\xi) := \sum\limits_{n\in\mathbb{Z}} |\widehat{f}(\xi+\frac{2\pi}{\lambda}n)|^2 \in L^1[0,\frac{2\pi}{\lambda}] \bigcap L^\infty[0,\frac{2\pi}{\lambda}]$.*

(b) *(cf. Lemma 2.2 in [7]) If either $\mathbf{c} \in \ell^1(\mathbb{Z})$ or $\{f(t-n\lambda)\}_{n\in\mathbb{Z}}$ is a Bessel sequence of $L^2(\mathbb{R})$, then $\widehat{\mathbf{c} *_\lambda f}(\xi) = \widehat{\mathbf{c}}(\lambda\xi)\widehat{f}(\xi)$ in $L^2(\mathbb{R})$.*

Proof. See Theorem 7.2.3 in [3] for (a). In order to prove (b), let $\mathbf{c}_N := \{c_{N,n}\}_{n\in\mathbb{Z}}, N \geq 1$ be the sequence defined by $c_{N,n} = c_n$ if $|n| \leq N$ and 0 otherwise. Then $\widehat{\mathbf{c}_N}(\lambda\xi)\widehat{f}(\xi) = \sum_{|n|\leq N} c_n e^{-in\lambda\xi}\widehat{f}(\xi) \to \widehat{\mathbf{c} *_\lambda f}(\xi)$ in $L^2(\mathbb{R})$ as $N \to \infty$ since $\big(\mathbf{c} *_\lambda f\big)(t) = \sum_{n\in\mathbb{Z}} c_n f(t - n\lambda)$ converges in $L^2(\mathbb{R})$. Therefore it is enough to show that $\widehat{\mathbf{c}_N}(\lambda\xi)\widehat{f}(\xi) \to \widehat{\mathbf{c}}(\lambda\xi)\widehat{f}(\xi)$ in $L^2(\mathbb{R})$ as $N \to \infty$. To see this, note that

$$\begin{aligned}\|\widehat{\mathbf{c}}(\lambda\xi)\widehat{f}(\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)\widehat{f}(\xi)\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |\widehat{\mathbf{c}}(\lambda\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)|^2 |\widehat{f}(\xi)|^2 d\xi \\ &= \int_0^{2\pi/\lambda} |\widehat{\mathbf{c}}(\lambda\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)|^2 G_{f,\lambda}(\xi) d\xi.\end{aligned}$$

For the last integral, it is easy to see that if $\mathbf{c} \in \ell^1(\mathbb{Z})$, then

$$\int_0^{2\pi/\lambda} |\widehat{\mathbf{c}}(\lambda\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)|^2 G_{f,\lambda}(\xi) d\xi \leq \|\widehat{\mathbf{c}}(\lambda\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)\|_{L^\infty[0,\frac{2\pi}{\lambda}]} \int_0^{2\pi/\lambda} G_{f,\lambda}(\xi) d\xi$$

and if $\{f(t-n\lambda)\}_{n\in\mathbb{Z}}$ is a Bessel sequence of $L^2(\mathbb{R})$, then

$$\int_0^{2\pi/\lambda} |\widehat{\mathbf{c}}(\lambda\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)|^2 G_{f,\lambda}(\xi) d\xi \leq \|G_{f,\lambda}\|_{L^\infty[0,\frac{2\pi}{\lambda}]} \int_0^{2\pi/\lambda} |\widehat{\mathbf{c}}(\lambda\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)|^2 d\xi.$$

If $\mathbf{c} \in \ell^1(\mathbb{Z})$, then $\widehat{\mathbf{c}}(\lambda\xi) \in C[0, 2\pi/\lambda]$ so that $\|\widehat{\mathbf{c}}(\lambda\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)\|_{L^\infty[0,\frac{2\pi}{\lambda}]} \to 0$ as $N \to \infty$. If $\{f(t-n\lambda)\}_{n\in\mathbb{Z}}$ is a Bessel sequence of $L^2(\mathbb{R})$, then $\|G_{f,\lambda}\|_{L^\infty[0,\frac{2\pi}{\lambda}]} < \infty$ by (a) and $\widehat{\mathbf{c}_N}(\lambda\xi) \to \widehat{\mathbf{c}}(\lambda\xi)$ in $L^2[0, 2\pi/\lambda]$ as $N \to \infty$. In both cases, $\|\widehat{\mathbf{c}}(\lambda\xi)\widehat{f}(\xi) - \widehat{\mathbf{c}_N}(\lambda\xi)\widehat{f}(\xi)\|^2_{L^2(\mathbb{R})} \to 0$ as $N \to \infty$ so that $\widehat{\mathbf{c}_N}(\lambda\xi)\widehat{f}(\xi) \to \widehat{\mathbf{c}}(\lambda\xi)\widehat{f}(\xi)$ in $L^2(\mathbb{R})$ as $N \to \infty$. □

Now, we prove a statement which is similar to Proposition 6 in [1].

Lemma 2.2. *Let $\mathbf{c} \in \ell^1(\mathbb{Z})$ and $\Phi_{\mathbf{c},\lambda}(f) := \mathbf{c} *_\lambda f$. Then $\Phi_{\mathbf{c},\lambda}$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ with its adjoint $\Phi^*_{\mathbf{c},\lambda} = \Phi_{\widetilde{\mathbf{c}},\lambda}$ and $\Phi_{\mathbf{c},\lambda}$ is an automorphism of $L^2(\mathbb{R})$ if and only if $\min\limits_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$.*

Proof. Let $F_N(t) := \sum_{|n|\le N} c_n f(t-n\lambda), N \ge 1$. Then for any $N > M \ge 1$,

$$\begin{aligned} \|F_N - F_M\|_{L^2(\mathbb{R})} &= \|\sum_{M<|n|\le N} c_n f(t-n\lambda)\|_{L^2(\mathbb{R})} \\ &\le \sum_{M<|n|\le N} |c_n| \cdot \|f(t-n\lambda)\|_{L^2(\mathbb{R})} \\ &= \|f\|_{L^2(\mathbb{R})} \cdot \sum_{M<|n|\le N} |c_n| \to 0 \text{ as } M\to\infty \end{aligned}$$

since $\mathbf{c} \in \ell^1(\mathbb{Z})$. Thus, $\{F_N\}_{N=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R})$ and converges to $\big(\mathbf{c} *_\lambda f\big)(t)$ in $L^2(\mathbb{R})$ so that $\Phi_{\mathbf{c},\lambda}$ is well-defined.

Since $\|\Phi_{\mathbf{c},\lambda}(f)\|_{L^2(\mathbb{R})} = \|\sum_{n\in\mathbb{Z}} c_n f(t-n\lambda)\|_{L^2(\mathbb{R})} \le \|\mathbf{c}\|_1 \|f\|_{L^2(\mathbb{R})}$ where $\|\mathbf{c}\|_1 := \sum_{n\in\mathbb{Z}} |c_n|$, $\Phi_{\mathbf{c},\lambda}$ is bounded and $\|\Phi_{\mathbf{c},\lambda}\| \le \|\mathbf{c}\|_1$. For any $a \in \mathbb{R}$, let $T_a f(t) := f(t-a)$. Then trivially T_a is an automorphism of $L^2(\mathbb{R})$ and $T_a^* = T_{-a}$. Since $\Phi_{\mathbf{c},\lambda} = \sum_{n\in\mathbb{Z}} c_n T_{n\lambda}$,

$$\Phi_{\mathbf{c},\lambda}^* = \sum_{n\in\mathbb{Z}} \bar{c}_n T_{n\lambda}^* = \sum_{n\in\mathbb{Z}} \bar{c}_n T_{-n\lambda} = \Phi_{\tilde{\mathbf{c}},\lambda}.$$

Note that $\widehat{\mathbf{c}}(\xi) = \widehat{\mathbf{c}}(\xi+2\pi) = \sum_{n\in\mathbb{Z}} c_n e^{-in\xi} \in C[0,2\pi]$ and $\widehat{\mathbf{c} *_\lambda f_n} = \big(\delta_{\frac{1}{\lambda}}\widehat{\mathbf{c}}\big)\widehat{f}$ in $L^2(\mathbb{R})$ by Lemma 2.1(b).

Assume that $\Phi_{\mathbf{c},\lambda}$ is an automorphism of $L^2(\mathbb{R})$ so that there exist constants $M \ge m > 0$ such that $m\,\|f\|_{L^2(\mathbb{R})} \le \|\mathbf{c} *_\lambda f\|_{L^2(\mathbb{R})} \le M\,\|f\|_{L^2(\mathbb{R})}$ for any $f \in L^2(\mathbb{R})$. Then

$$m^2 \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi \le \int_{\mathbb{R}} |\widehat{\mathbf{c}}(\lambda\,\xi)|^2 |\widehat{f}(\xi)|^2 d\xi \le M^2 \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi,\ f \in L^2(\mathbb{R})$$

so that $m \le |\widehat{\mathbf{c}}(\xi)| \le M$ on $\mathbb{R}$.

Conversely, assume $m := \min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$ and let $M := \max_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)|$. Then

$$m\,\|f\|_{L^2(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|\delta_{\frac{1}{\lambda}}\widehat{\mathbf{c}}\widehat{f}\|_{L^2(\mathbb{R})} = \|\mathbf{c} *_\lambda f\|_{L^2(\mathbb{R})} \le M\,\|f\|_{L^2(\mathbb{R})}\,,\ f \in L^2(\mathbb{R}) \tag{2.1}$$

so that $\Phi_{\mathbf{c},\lambda}$ is bounded above and below on $L^2(\mathbb{R})$. For any g $\in L^2(\mathbb{R})$, $\frac{1}{\widehat{\mathbf{c}}(\lambda\,\xi)}\widehat{g}(\xi) \in L^2(\mathbb{R})$ since $\frac{1}{\widehat{\mathbf{c}}(\lambda\,\xi)} \in L^\infty(\mathbb{R})$. Then $f := \mathcal{F}^{-1}\{\frac{1}{\widehat{\mathbf{c}}(\lambda\,\xi)}\widehat{g}(\xi)\} \in L^2(\mathbb{R})$ is such that $\Phi_{\mathbf{c},\lambda}(f) = g$. Thus, $\Phi_{\mathbf{c},\lambda}$ becomes an automorphism of $L^2(\mathbb{R})$. □

Using the fact that frames and Riesz bases are preserved under automorphisms, we directly obtain the following as a consequence of Lemma 2.2.

Theorem 2.3. *Let* $\mathbf{c} \in \ell^1(\mathbb{Z})$ *be such that* $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$*. Then* $\{f_n\}_{n\in\mathbb{Z}}$ *is a frame (Riesz basis, respectively) of* $L^2(\mathbb{R})$ *if and only if* $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ *is a frame (Riesz basis, respectively) of* $L^2(\mathbb{R})$.

In fact, the condition $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$ for $\mathbf{c} \in \ell^1(\mathbb{Z})$ is exactly what we need for $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ to be a frame of $L^2(\mathbb{R})$ assuming that $\{f_n\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$.

Theorem 2.4. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a frame of $L^2(\mathbb{R})$ with frame bounds (A, B) and $\mathbf{c} \in \ell^1(\mathbb{Z})$. Then $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$ if and only if $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$. Moreover, in this case, (m^2A, M^2B) are frame bounds for $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ where $m := \min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)|$ and $M := \max_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)|$.*

Proof. Note that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\langle g, \mathbf{c} *_\lambda f_n\rangle|^2 &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \left|\langle \widehat{g}, \big(\delta_{\frac{1}{\lambda}}\widehat{\mathbf{c}}\big)\widehat{f_n}\rangle\right|^2 = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \left|\langle \overline{\big(\delta_{\frac{1}{\lambda}}\widehat{\mathbf{c}}\big)}\widehat{g}, \widehat{f_n}\rangle\right|^2 \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \left|\langle \big(\delta_{\frac{1}{\lambda}}\widehat{\widetilde{\mathbf{c}}}\big)\widehat{g}, \widehat{f_n}\rangle\right|^2 = \sum_{n\in\mathbb{Z}} |\langle \widetilde{\mathbf{c}} *_\lambda g, f_n\rangle|^2, \ g \in L^2(\mathbb{R}).\end{aligned}$$

Assume $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$. Then $\Phi_{\mathbf{c},\lambda}$ is an automorphism of $L^2(\mathbb{R})$ by Lemma 2.2 so that $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ is also a frame of $L^2(\mathbb{R})$. Moreover, $m\, \|g\|_{L^2(\mathbb{R})} \le \|\widetilde{\mathbf{c}} *_\lambda g\|_{L^2(\mathbb{R})} \le M\, \|g\|_{L^2(\mathbb{R})}$ for any $g \in L^2(\mathbb{R})$ by (2.1) so that (m^2A, M^2B) are frame bounds for $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$.

Conversely, assume that $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$ with bounds (A', B'). Then

$$A'\, \|g\|^2_{L^2(\mathbb{R})} \le \sum_{n\in\mathbb{Z}} |\langle g, \mathbf{c} *_\lambda f_n\rangle|^2 \le B'\, \|g\|^2_{L^2(\mathbb{R})}, \ g \in L^2(\mathbb{R}).$$

On the other hand,

$$A\, \|\widetilde{\mathbf{c}} *_\lambda g\|^2_{L^2(\mathbb{R})} \le \sum_{n\in\mathbb{Z}} |\langle \widetilde{\mathbf{c}} *_\lambda g, f_n\rangle|^2 \le B\, \|\widetilde{\mathbf{c}} *_\lambda g\|^2_{L^2(\mathbb{R})}, \ g \in L^2(\mathbb{R}).$$

Hence, we get $A\, \|\widetilde{\mathbf{c}} *_\lambda g\|^2_{L^2(\mathbb{R})} \le B'\, \|g\|^2_{L^2(\mathbb{R})}$ and $A'\, \|g\|^2_{L^2(\mathbb{R})} \le B\, \|\widetilde{\mathbf{c}} *_\lambda g\|^2_{L^2(\mathbb{R})}$ for any $g \in L^2(\mathbb{R})$. Combining these inequalities, we obtain

$$\frac{A'}{B}\, \|g\|^2_{L^2(\mathbb{R})} \le \|\widetilde{\mathbf{c}} *_\lambda g\|^2_{L^2(\mathbb{R})} \le \frac{B'}{A}\, \|g\|^2_{L^2(\mathbb{R})}, \ g \in L^2(\mathbb{R}),$$

which is equivalent to

$$\frac{A'}{B}\int_{\mathbb{R}} |\widehat{g}(\xi)|^2\, d\xi \le \int_{\mathbb{R}} |\widehat{\widetilde{\mathbf{c}}}(\lambda\,\xi)|^2 |\widehat{g}(\xi)|^2 d\xi \le \frac{B'}{A}\int_{\mathbb{R}} |\widehat{g}(\xi)|^2\, d\xi, \ g \in L^2(\mathbb{R}).$$

Since $\widehat{\widetilde{\mathbf{c}}}(\xi) = \overline{\widehat{\mathbf{c}}(\xi)}$, the last inequality is again equivalent to $\frac{A'}{B} \le |\widehat{c}(\xi)|^2 \le \frac{B'}{A}$ on $\mathbb{R}$, which completes the proof. $\square$

Example 2.5. Let $\{f_n\}_{n\in\mathbb{Z}}$ be a frame of $L^2(\mathbb{R})$ and $a, b \in \mathbb{C}$. Then $\{af_n(t) + bf_n(t-1)\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$ if and only if $|a| \ne |b|$. To see this, let $\{c_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers with $c_0 = a$, $c_1 = b$ and $c_n = 0$, otherwise. It is easy to check that $\widehat{\mathbf{c}}(\xi) = a + b\, e^{-i\xi}$ and so $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$ if and only if $|a| \ne |b|$.

In general, it's not easy to check $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$ for an arbitrary $\mathbf{c} \in \ell^1(\mathbb{Z})$. However, we have:

Lemma 2.6. *Let* $\mathbf{c} \in \ell^1(\mathbb{Z})$ *be such that* $2 \max_{n\in\mathbb{Z}} |c_n| > \sum_{n\in\mathbb{Z}} |c_n|$. *Then* $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$.

Proof. Since $\mathbf{c} \in \ell^1(\mathbb{Z})$, we have $\widehat{\mathbf{c}}(\xi) \in C[0, 2\pi]$. Let $|c_j| = \max_{n\in\mathbb{Z}} |c_n|$. Then $|\widehat{\mathbf{c}}(\xi)| = |\sum_{n\in\mathbb{Z}} c_n e^{-in\xi}| \geq |c_j| - |\sum_{n\neq j} c_n e^{-in\xi}| \geq |c_j| - \sum_{n\neq j} |c_n| > 0$ so that $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$. □

In other words, if one of the coefficients in $\{c_n\}_{n\in\mathbb{Z}}$ "dominates" the others, then the condition $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$ is satisfied, which plays a crucial role in our setting. This sufficient condition for $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$ leads directly to the following corollary.

Corollary 2.7. *Let* $\mathbf{c} \in \ell^1(\mathbb{Z})$ *be such that* $2 \max_{n\in\mathbb{Z}} |c_n| > \sum_{n\in\mathbb{Z}} |c_n|$. *Then* $\{f_n\}_{n\in\mathbb{Z}}$ *is a frame (Riesz basis, respectively) of* $L^2(\mathbb{R})$ *if and only if* $\{\mathbf{c} *_\lambda f_n\}_{n\in\mathbb{Z}}$ *is a frame (Riesz basis, respectively) of* $L^2(\mathbb{R})$.

The following example provides another point of view on the perturbation of frames (see [4]).

Example 2.8. Let $\{f_n\}_{n\in\mathbb{Z}}$ be a frame of $L^2(\mathbb{R})$ and $\epsilon_1, \epsilon_2 \in \mathbb{C}$ be complex numbers with $|\epsilon_1| + |\epsilon_2| < \frac{1}{2}$. Then for any $\delta > 0$, $\{\epsilon_1 f_n(t-\delta) + (1-\epsilon_1-\epsilon_2) f_n(t) + \epsilon_2 f_n(t+\delta)\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$ by Corollary 2.7. This frame can be understood as a perturbation of the original frame. Moreover,

(i) if $\{f_n\}_{n\in\mathbb{Z}}$ is exact, then $\{\epsilon_1 f_n(t-\delta) + (1-\epsilon_1-\epsilon_2) f_n(t) + \epsilon_2 f_n(t+\delta)\}_{n\in\mathbb{Z}}$ is also exact;

(ii) if $f_n(t) = f(t-n)$, $n \in \mathbb{Z}$, then for any $\delta > 0$, $\{\epsilon_1 f(t-n-\delta) + (1-\epsilon_1-\epsilon_2) f(t-n) + \epsilon_2 f(t-n+\delta)\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$.

Let $V : H \to L^2(\mathbb{R})$ and $W : L^2(\mathbb{R}) \to H$ be bounded linear operators where H is a Hilbert space. For $\mathbf{c} \in \ell^1(\mathbb{Z})$, we define $\mathbf{c} *_\lambda V : H \to L^2(\mathbb{R})$ and $W *_\lambda \mathbf{c} : L^2(\mathbb{R}) \to H$ by

$$\mathbf{c} *_\lambda V := \sum_{n\in\mathbb{Z}} c_n T_{n\lambda} V \quad \text{and} \quad W *_\lambda \mathbf{c} := \sum_{n\in\mathbb{Z}} c_n W T_{-n\lambda}.$$

Note that

$$\big(\mathbf{c} *_\lambda V\big)(f) = \big(\Phi_{\mathbf{c},\lambda} \circ V\big)(f), \; f \in H. \tag{2.2}$$

On the other hand, for any $g \in L^2(\mathbb{R})$, let $g_N(t) := \sum_{|n|\le N} c_n W T_{-n\lambda}\, g(t)$, $N \ge 1$. Then for any $N > M \ge 1$,

$$\begin{aligned}\|g_N - g_M\|_H &= \Big\| \sum_{M<|n|\le N} c_n W\big(g(\cdot + n\lambda)\big)\Big\|_H \\ &\le \sum_{M<|n|\le N} |c_n|\, \|W\|\, \|g(\cdot+n\lambda)\|_{L^2(\mathbb{R})} \\ &= \|W\|\, \|g\|_{L^2(\mathbb{R})} \cdot \sum_{M<|n|\le N} |c_n| \to 0 \text{ as } M \to \infty\end{aligned}$$

since $\mathbf{c} \in \ell^1(\mathbb{Z})$. Hence $\sum_{n\in\mathbb{Z}} c_n W T_{-n\lambda}\, g$ converges in H so that $W *_\lambda c$ is well-defined.

We can see that $\mathbf{c} *_\lambda V$ and $W *_\lambda \mathbf{c}$ are bounded operators with $\|\mathbf{c} *_\lambda V\| \le \|\mathbf{c}\|_1 \|V\|$ and $\|W *_\lambda \mathbf{c}\| \le \|\mathbf{c}\|_1 \|W\|$ where $\|\cdot\|$ is the corresponding operator norm. Moreover,

$$\big(\mathbf{c} *_\lambda V\big)^* = \big(\sum_{n\in\mathbb{Z}} c_n T_{n\lambda} V\big)^* = \sum_{n\in\mathbb{Z}} \bar{c_n} V^* T_{-n\lambda} = V^* *_\lambda \bar{\mathbf{c}}$$

and

$$\big(W *_\lambda \mathbf{c}\big)^* = \big(\sum_{n\in\mathbb{Z}} c_n W T_{-n\lambda}\big)^* = \sum_{n\in\mathbb{Z}} \bar{c_n} T_{n\lambda} W^* = \bar{\mathbf{c}} *_\lambda W^*.$$

From these observations, we have the following.

Proposition 2.9. *Let $V : H \to L^2(\mathbb{R})$, $W : L^2(\mathbb{R}) \to H$ and $\Gamma : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be bounded linear operators where H is a Hilbert space. If $\mathbf{c}$ and $\mathbf{d} \in \ell^1(\mathbb{Z})$, then $\mathbf{c} *_\lambda V$, $W *_\lambda \mathbf{c}$, $(\mathbf{c} *_\lambda \Gamma) *_\lambda \mathbf{d}$ and $\mathbf{c} *_\lambda (\Gamma *_\lambda \mathbf{d})$ are bounded operators with the following properties.*

(i) $(\mathbf{c} *_\lambda V)^* = V^* *_\lambda \bar{\mathbf{c}}$ *and* $(W *_\lambda \mathbf{c})^* = \bar{\mathbf{c}} *_\lambda W^*$.
(ii) $\|\mathbf{c} *_\lambda V\| \le \|\mathbf{c}\|_1 \|V\|$ *and* $\|W *_\lambda \mathbf{c}\| \le \|\mathbf{c}\|_1 \|W\|$.
(iii) $(\mathbf{c} *_\lambda \Gamma) *_\lambda \mathbf{d} = \mathbf{c} *_\lambda (\Gamma *_\lambda \mathbf{d})$.

Moreover, assume $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$. Then

(iv) *V (resp. W) is isomorphic if and only if $\mathbf{c} *_\lambda V$ (resp. $W *_\lambda \mathbf{c}$) is isomorphic.*
(v) *Γ is positive (invertible, self-adjoint, resp.) if and only if $\mathbf{c} *_\lambda \Gamma *_\lambda \bar{\mathbf{c}}$ is positive (invertible, self-adjoint, resp.).*

Proof. (i) is already proved and (ii) and (iii) are trivial. Note that

$$W *_\lambda \mathbf{c} = (\bar{\mathbf{c}} *_\lambda W^*)^* = (\Phi_{\bar{\mathbf{c}},\lambda} \circ W^*)^* = W \circ \Phi^*_{\bar{\mathbf{c}},\lambda}$$

by (2.2) so that by Lemma 2.2,

$$W *_\lambda \mathbf{c} = W \circ \Phi_{\check{\mathbf{c}},\lambda} \tag{2.3}$$

where $\check{\mathbf{c}} := \{c_{-n}\}_{n\in\mathbb{Z}}$. By combining (2.2) and (2.3) with Lemma 2.2, (iv) and (v) follow. □

Assume that $\{f_n\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$ with bounds (A,B) and $\mathbf{c}\in\ell^1(\mathbb{Z})$ with $\min_{\mathbb{R}}|\widehat{\mathbf{c}}(\xi)|>0$ so that $\{\mathbf{c}*_\lambda f_n\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$. We denote the *synthesis operators* of $\{f_n\}_{n\in\mathbb{Z}}$ and $\{\mathbf{c}*_\lambda f_n\}_{n\in\mathbb{Z}}$ by $D:\ell^2(\mathbb{Z})\to L^2(\mathbb{R})$ and $D_{\mathbf{c},\lambda}:\ell^2(\mathbb{Z})\to L^2(\mathbb{R})$, respectively, which are defined by

$$D(\mathbf{d}):=\sum_{n\in\mathbb{Z}}d_nf_n \quad \text{and} \quad D_{\mathbf{c},\lambda}(\mathbf{d}):=\sum_{n\in\mathbb{Z}}d_n\,\mathbf{c}*_\lambda f_n.$$

Then $S=DD^*$ and $S_{\mathbf{c},\lambda}=D_{\mathbf{c},\lambda}D^*_{\mathbf{c},\lambda}$ are the frame operators of $\{f_n\}$ and $\{\mathbf{c}*_\lambda f_n\}_{n\in\mathbb{Z}}$, respectively.

We are now ready to find a relation between the frame operators S and $S_{\mathbf{c},\lambda}$. Since $\{f_n\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$ with bounds (A,B), $\sup_{n\in\mathbb{Z}}\|f_n\|^2_{L^2(\mathbb{R})}\le B$ (see Proposition 12.15 in [5]). Hence for any $\mathbf{d}\in\ell^1(\mathbb{Z})$,

$$\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|d_n|\,|c_k|\,\|T_{k\lambda}f_n\|_{L^2(\mathbb{R})}\le\sqrt{B}\,\|\mathbf{c}\|_1\,\|\mathbf{d}\|_1<\infty$$

so that

$$D_{\mathbf{c},\lambda}(\mathbf{d})=\sum_{n\in\mathbb{Z}}d_n\,\mathbf{c}*_\lambda f_n=\sum_{n\in\mathbb{Z}}d_n\sum_{k\in\mathbb{Z}}c_kT_{k\lambda}f_n=\sum_{k\in\mathbb{Z}}c_kT_{k\lambda}(\sum_{n\in\mathbb{Z}}d_nf_n)=\big(\mathbf{c}*_\lambda D\big)(\mathbf{d}).$$

By the continuity of $D_{\mathbf{c},\lambda}$ and the fact that $\ell^1(\mathbb{Z})$ is dense in $\ell^2(\mathbb{Z})$, we can see that

$$D_{\mathbf{c},\lambda}(\mathbf{d})=\big(\mathbf{c}*_\lambda D\big)(\mathbf{d}),\ \ \mathbf{d}\in\ell^2(\mathbb{Z}). \tag{2.4}$$

Consequently, we have the following.

Theorem 2.10. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a frame of $L^2(\mathbb{R})$ and $\mathbf{c}\in\ell^1(\mathbb{Z})$ with $\min_{\mathbb{R}}|\widehat{\mathbf{c}}(\xi)|>0$. Then the frame operator $S_{\mathbf{c},\lambda}$ of $\{\mathbf{c}*_\lambda f_n\}_{n\in\mathbb{Z}}$ is given by*

$$S_{\mathbf{c},\lambda}=\mathbf{c}*_\lambda S*_\lambda\bar{\mathbf{c}},$$

where S is the frame operator of $\{f_n\}_{n\in\mathbb{Z}}$.

Proof. Since $D_{\mathbf{c},\lambda}=\mathbf{c}*_\lambda D$ by (2.4), we have

$$D^*_{\mathbf{c},\lambda}=(\sum_{k\in\mathbb{Z}}c_nT_{n\lambda}D)^*=\sum_{k\in\mathbb{Z}}\bar{c}_nD^*T^*_{n\lambda}=\sum_{k\in\mathbb{Z}}\bar{c}_nD^*T_{-n\lambda}.$$

Hence

$$\begin{aligned} S_{\mathbf{c},\lambda}(f) &= D_{\mathbf{c},\lambda} D^*_{\mathbf{c},\lambda}(f) \\ &= \sum_{k\in\mathbb{Z}} c_k T_{k\lambda} D \sum_{n\in\mathbb{Z}} \bar{c}_n D^* T_{-n\lambda}(f) \\ &= \sum_{k\in\mathbb{Z}} c_k T_{k\lambda} \sum_{n\in\mathbb{Z}} \bar{c}_n D D^* T_{-n\lambda}(f) \\ &= \sum_{k\in\mathbb{Z}} c_k T_{k\lambda} \sum_{n\in\mathbb{Z}} \bar{c}_n S T_{-n\lambda}(f) \\ &= \sum_{k\in\mathbb{Z}} c_k T_{k\lambda} (S *_\lambda \bar{\mathbf{c}}(f)) \\ &= \mathbf{c} *_\lambda S *_\lambda \bar{\mathbf{c}}(f),\ f \in L^2(\mathbb{R}), \end{aligned}$$

which completes the proof. □

Corollary 2.11. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a frame of $L^2(\mathbb{R})$ and $\mathbf{c} \in \ell^1(\mathbb{Z})$ with $\min_{\mathbb{R}} |\widehat{\mathbf{c}}(\xi)| > 0$. Then the canonical dual frame of $\{c *_\lambda f_n\}_{n\in\mathbb{Z}}$ is $\{\Phi^{-1}_{\tilde{\mathbf{c}},\lambda}(S^{-1} f_n)\}_{n\in\mathbb{Z}}$ where S is the frame operator of $\{f_n\}_{n\in\mathbb{Z}}$.*

Proof. In view of (2.2) and (2.3), $S_{\mathbf{c},\lambda} = \Phi_{\mathbf{c},\lambda} \circ S \circ \Phi_{\tilde{\mathbf{c}},\lambda}$ so that

$$S^{-1}_{\mathbf{c},\lambda}(c *_\lambda f) = \Phi^{-1}_{\tilde{\mathbf{c}},\lambda} \circ S^{-1} \circ \Phi^{-1}_{\mathbf{c},\lambda}\big(\Phi_{\mathbf{c},\lambda}(f)\big) = \Phi^{-1}_{\tilde{\mathbf{c}},\lambda}(S^{-1} f),\ f \in L^2(\mathbb{R}).$$

Hence $\{S^{-1}_{\mathbf{c},\lambda}(c *_\lambda f_n)\}_{n\in\mathbb{Z}} = \{\Phi^{-1}_{\tilde{\mathbf{c}},\lambda}(S^{-1} f_n)\}_{n\in\mathbb{Z}}$. □

3. A NOTE ON THE CONVOLUTION ON $L^2(\mathbb{R})$

All the previous results are concerned with the discrete-continuous convolution product of frames (Riesz bases, respectively) with a sequence in $\ell^1(\mathbb{Z})$. However, analogous arguments do not work for the convolution of functions which is given by $\big(f * g\big)(t) := \int_{\mathbb{R}} f(s) g(t-s) ds$.

Proposition 3.1. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a frame of $L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\{f_n * g\}_{n\in\mathbb{Z}}$ is a Bessel sequence of $L^2(\mathbb{R})$ but not a frame of $L^2(\mathbb{R})$.*

Proof. By Young's inequality([9]), $\|f_n * g\|_{L^2(\mathbb{R})} \le \|f_n\|_{L^2(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$ so that $f_n * g \in L^2(\mathbb{R}), n \in \mathbb{Z}$.

Since $\widehat{f_n * g} = \widehat{f_n}\, \widehat{g}$ in $L^2(\mathbb{R})$ (see Theorem 5.8 in [9]) and $\widehat{\widehat{g}} = \overline{\overline{g}}$,

$$\sum_{n\in\mathbb{Z}} |\langle h, f_n * g\rangle|^2 = \sum_{n\in\mathbb{Z}} |\langle \widetilde{g} * h, f_n\rangle|^2,\ h \in L^2(\mathbb{R}),$$

where $\widetilde{g}(t) := \overline{g(-t)}$. Hence

$$A\,\|\widetilde{g} * h\|^2_{L^2(\mathbb{R})} \le \sum_{n\in\mathbb{Z}} |\langle h, f_n * g\rangle|^2 \le B\,\|\widetilde{g} * h\|^2_{L^2(\mathbb{R})} \le B\,\|\widetilde{g}\|_{L^1(\mathbb{R})}\,\|h\|_{L^2(\mathbb{R})},\ h \in L^2(\mathbb{R}), \tag{3.1}$$

where (A, B) are the frame bounds of $\{f_n\}_{n\in\mathbb{Z}}$, so that $\{f_n * g\}_{n\in\mathbb{Z}}$ is a Bessel sequence of $L^2(\mathbb{R})$ with Bessel bound $B\,\|g\|_{L^1(\mathbb{R})}$.

To show that $\{f_n * g\}_{n\in\mathbb{Z}}$ is not a frame of $L^2(\mathbb{R})$, let $h_k(t) := \frac{1}{\sqrt{k}}\frac{\sin k\pi t}{\pi t}$ for $k \geq 1$. Then $h_k \in L^2(\mathbb{R})$ with $\|h_k\|_{L^2(\mathbb{R})} = 1$ and $\widehat{h_k}(\xi) = \frac{1}{\sqrt{k}}\chi_{[-k\pi,k\pi]}(\xi)$. On the other hand, $\|\widetilde{g} * h_k\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\left\|\overline{\widehat{g}(\xi)}\widehat{h_k}(\xi)\right\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi k}\int_{-k\pi}^{k\pi}|\widehat{g}(\xi)|^2\,d\xi \to 0$ as $k \to \infty$. Therefore $\sum_{n\in\mathbb{Z}}|\langle h_k, f_n * g\rangle|^2 \to 0$ as $k \to \infty$ by (3.1), while $\|h_k\|_{L^2(\mathbb{R})} = 1, k \geq 1$. This implies that $\{f_n * g\}_{n\in\mathbb{Z}}$ is not a frame of $L^2(\mathbb{R})$. □

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