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EXISTENCE OF SOLUTIONS FOR DOUBLE PERTURBED IMPULSIVE NEUTRAL FUNCTIONAL EVOLUTION EQUATIONS

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ABSTRACT. In this paper, we study the existence of mild solutions for double perturbed impulsive neutral functional evolution equations with infinite delay in Banach spaces. The existence of mild solutions to such equations is obtained by using the theory of the Hausdorff measure of noncompactness and Darbo fixed point theorem, without the compactness assumption on associated evolution system. An example is provided to illustrate the theory.

1. INTRODUCTION

In recent years, the theory of impulsive differential equations has become an important area of investigation stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. Relative to this matter, we refer the reader to Bainov and Simeonov [6], Rogovchenko [25, 26] and Hernndez [21, 22, 23]. For other contributions on the impulsive problem see [2, 10, 13]. Dong [14, 15, 16] , Guedda [18], Banas [5], Heinz [20], Runping [27, 28] and Xue [30] studied some functional differential equations under the conditions in respect of the measure of noncompactness.

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades. The literature relative to ordinary and partial neutral functional differential equations is very extensive and we refer the reader to [3, 7, 8, 9, 11, 12] and the references therein.

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Recently Selma Baghli et al. [4] studied the existence of mild solutions partial perturbed evolution equation with infinite delay in Frechet spaces described in the form

$$\begin{aligned} y'(t) &= A(t)y(t) + f(t, y_t) + g(t, y_t), \quad \text{a.e } t \in J = [0, +\infty), \\ y_0 &= \varphi \in \mathcal{B}, \end{aligned}$$

and perturbed neutral evolution equation with infinite delay in Frechet spaces described in the form

$$\begin{aligned} \frac{d}{dt}(y(t) - h(t, y_t)) &= A(t)y(t) + f(t, y_t) + g(t, y_t), \quad \text{a.e } t \in J = [0, +\infty), \\ y_0 &= \varphi \in \mathcal{B}, \end{aligned}$$

by using the Avramescu Nonlinear Alternative theorem.

Very recently Runping Ye et al. [29] studied the existence of mild solutions for double perturbed neutral functional evolution equations with infinite delay described in the form

$$\begin{aligned} \frac{d}{dt}(x(t) - h(t, x_t)) &= A(t)x(t) + f(t, x_t) + g(t, x_t), \quad t \in I = [0, a], \\ x_0 &= \varphi \in \mathcal{B}, \end{aligned}$$

by using the Darbo fixed point theorem

In this paper, we study the existence of mild solutions for double perturbed impulsive neutral functional evolution equations with infinite delay described in the form

$$\frac{d}{dt}(x(t) - h(t, x_t)) = A(t)x(t) + f(t, x_t) + g(t, x_t), \quad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, ..., n, \tag{1.3}$$

where $\{A(t) : t \geq 0\}$ is a family of linear closed operators in a real Banach space X that generates an evolution system $\{U(t, s) : 0 \leq s \leq t < \infty\}$ and $D(A(t)) \subseteq X$ is dense in X; the history $x_t : (-\infty, 0] \to X$, $x_s(\theta) = x(s + \theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically; g, f, h, I_i are appropriate functions and the symbol $\Delta\xi(t)$ represents the jump of the function ξ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

In this paper, by using the tools involving the measure of noncompactness and fixed point theory, we obtain existence of mild solution of double perturbed impulsive neutral functional evolution equations (1.1)-(1.3) without the assumption of compactness or equicontinuity on the associated semigroup.

2. PRELIMINARIES

Now we introduce some definitions, notations and preliminary facts which are used throughout this paper.

Let $\{A(t) : t \geq 0\}$ is a family of linear closed operators in a real Banach space X into itself and $D(A(t)) = D$ is independent of t.

Definition 2.1 ([4]). *The family of bounded linear operators $\{U(t, s) : 0 \leq s \leq t < \infty\}$ on X is called an evolution system if the following properties are satisfied:*

(i) *$U(t, t) = I$ where I is the identity operator in X;*

(**ii**) *$U(t,s)U(s,r) = U(t,r)$ for every $0 \le r \le s \le t < +\infty$;*

(**iii**) *$U(t,s) \in B(X)$ the space of bounded linear operator on X, where for every $(t,s) \in \{(t,s) : 0 \le s \le t < +\infty\}$ and for each $x \in X$, the mapping $(t,s) \to U(t,s)x$ is continuous.*

Definition 2.2. *The evolution system $\{U(t,s) : 0 \le s \le t < \infty\}$ is said to be equicontinuous if for each bounded subset B in X, $\{s \to U(t,s)x : x \in B\}$ is equicontinuous for $t > 0$.*

For additional details about evolution families, we refer the reader to Pazy [24]. To consider the impulsive conditions, it is convenient to introduce some additional concepts and notations. We say that a function $u : [\sigma,\tau] \to X$ is a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $\mathcal{PC}([\sigma,\tau];X)$ the space formed by the normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all functions $u : [0,a] \to X$ such that u is continuous at $t \neq t_i$, $i = 1,...,n$. It is clear that $\mathcal{PC}$ endowed with the norm of uniform convergence is a Banach space.

In what follows, we put $t_0 = 0$, $t_{n+1} = a$, and for $u \in \mathcal{PC}$, we denote by $\widetilde{u}_i \in C([t_i,t_{i+1}];X)$, $i = 0,1,...n$, the function given by

$$\widetilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases}$$

Moreover, for $\mathcal{B} \subseteq \mathcal{PC}$, we employ the notation $\widetilde{B}_i$, $i = 0,1,...,n$, for the sets $\widetilde{B}_i = \{\widetilde{u}_i : u \in B\}$.

We employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to that introduced by Hale and Kato [19] and it is appropriated to treat impulsive neutral functional evolution equations.

Definition 2.3 ([19])**.** *The phase $\mathcal{B}$ is a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and satisfying the following axioms:*

(**A**) *If $x : (-\infty,\sigma+b] \to X$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma,\sigma+b] : X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+b)$ the following conditions hold:*

(**i**) *x_t is in $\mathcal{B}$,*

(**ii**) *$\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$,*

(**iii**) *$\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K,M : [0,\infty) \to [1,\infty)$, K is continuous, M is locally bounded and H,K,M are independent of $x(\cdot)$.*

(**B**) *The space $\mathcal{B}$ is complete.*

Definition 2.4 ([5])**.** *The Hausdorff's measure of noncompactness χ_Y is defined by*

$$\chi_Y(B) = \inf\{r > 0,\ B \text{ can be covered by finite number of balls with radii } r\},$$

for bounded set B in any Banach space Y.

Lemma 2.1 ([5])**.** *Let Y be a real Banach space and $B,C \subseteq Y$ be bounded, the following properties are satisfied:*

(1) *B is pre-compact if and only if $\chi_Y(B) = 0$;*

(2) *$\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(convB)$, where $\overline{B}$ and $convB$ are the closure and the convex hull of B respectively;*

(3) $\chi_Y(B) \le \chi_Y(C)$ *when* $B \subseteq C$*;*
(4) $\chi_Y(B+C) \le \chi_Y(B) + \chi_Y(C)$ *where* $B + C = \{x + y;\ x \in B,\ y \in C\}$*;*
(5) $\chi_Y(B \cup C) = \max\{\chi_Y(B), \chi_Y(C)\}$*;*
(6) $\chi_Y(\lambda B) = |\lambda|\chi_Y(C)$ *for any* $\lambda \in R$*;*
(7) *If the map* $Q : D(Q) \subseteq Y \to Z$ *is Lipschitz continuous with constant* k *then* $\chi_Z(QB) \le k\chi_Y(B)$ *for any bounded subset* $B \subseteq D(Q)$*, where* Z *is a Banach space;*
(8) *If* $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of bounded closed nonempty subset of* Y *and* $\lim_{n\to\infty}\chi_Y(W_n) = 0$*, then* $\bigcap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* Y*.*

Definition 2.5 ([26])**.** *The map* $Q : W \subseteq Y \to Y$ *is said to be a* χ_Y*-contraction if there exists a positive constant* $k < 1$ *such that* $\chi_Y(Q(C)) \le k\chi_Y(C)$ *for any bounded close subset* $C \subseteq W$ *where* Y *is a Banach space.*

Lemma 2.2 ((Darbo) [1])**.** *If* $W \subseteq Y$ *is closed and convex and* $0 \in W$*, the continuous map* $Q : W \to W$ *is a* χ_Y*-contraction, if the set* $\{x \in W = \lambda\Gamma x\}$ *is bounded for* $0 < \lambda < 1$*, then the map* Q *has at least one fixed point in* W*.*

Lemma 2.3 ((Darbo-Sadovskii) [5])**.** *If* $W \subseteq Y$ *is bounded closed and convex, the continuous map* $Q : W \to W$ *is a* χ_Y*-contraction, then the map* Q *has at least one fixed point in* W*.*

In this paper we denote by χ the Hausdorff's measure of noncompactness of X, by χ_C the Hausdorff's measure of noncompactness of $C([0,a];X)$ and by $\chi_{\mathcal{PC}}$ the Hausdorff's measure of noncompactness of $\mathcal{PC}([0,a];X)$. To discuss the existence we need the following auxiliary results.

Lemma 2.4 ([5])**.** *(1) If* $W \subset C([a,b];X)$ *is bounded, then* $\chi(W(t)) \le \chi_C(W)$*, for any* $t \in [a,b]$*, where* $W(t) = \{u(t) : u \in W\} \subseteq X$*;*

(2) *If* W *is equicontinuous on* $[a,b]$*, then* $\chi(W(t))$ *is continuous for* $t \in [a,b]$*, and*
$$\chi_C(W) = \sup\{\chi(W(t)),\ t \in [a,b]\};$$
(3) *If* $W \subset C([a,b];X)$ *is bounded and equicontinuous, then* $\chi(W(t))$ *is continuous for* $t \in [a,b]$*, and*
$$\chi\Big(\int_a^t W(s)ds\Big) \le \int_a^t \chi(W(s))ds, \text{ for all } t \in [a,b],$$
where $\int_a^t W(s)ds = \Big\{\int_a^t x(s)ds :\ x \in W\Big\}$*.*

Lemma 2.5 (Lemma 2.9 [27])**.** *(1) If* $W \subset \mathcal{PC}$ *is bounded, then* $\chi(W(t)) \le \chi_{\mathcal{PC}}(W)$*, for any* $t \in [a,b]$*, where* $W(t) = \{u(t) : u \in W\} \subseteq X$*;*

(2) *If* W *is piecewise equicontinuous on* $[a,b]$*, then* $\chi(W(t))$ *is piecewise continuous for* $t \in [a,b]$*, and*
$$\chi_{\mathcal{PC}}(W) = \sup\{\chi(W(t)),\ t \in [a,b]\};$$
(3) *If* $W \subset \mathcal{PC}$ *is bounded and piecewise equicontinuous, then* $\chi(W(t))$ *is piecewise continuous for* $t \in [a,b]$*, and*
$$\chi\Big(\int_a^t W(s)ds\Big) \le \int_a^t \chi\Big(W(s)\Big)ds, \text{ for all } t \in [a,b],$$

where $\int_a^t W(s)ds = \Big\{ \int_a^t x(s)ds : \ x \in W \Big\}$.

Lemma 2.6. *If the evolution system* $\{U(t,s) \,:\, 0 \leq s \leq t < \infty\}$ *is equicontinuous and* $\eta \in L([0,a];\mathbb{R}^+)$*, then the set* $\{\int_0^t U(t,s)x(s)ds : \|x(s)\| \leq \eta(s)$ *for a.e.* $0 \leq \ s \leq t \leq a\}$ *is equicontinuous.*

3. Main Results

For the system (1.1)-(1.3), we assume the following hypotheses are satisfied.

$\mathbf{H_1}$ The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:

(**i**) For each $x : (-\infty, a] \to X$, $x_0 \in \mathcal{B}$ and $x|_I \in \mathcal{PC}$, the function $f(\cdot, x)$ is strongly measurable for all $x \in \mathcal{B}$ and $f(t, \cdot)$ is continuous for a.e. $t \in [0, b]$.

(**ii**) There exists an integrable function $m : I \to [0, +\infty)$ and a monotone continuous nondecreasing function $W : [0, +\infty) \to (0, +\infty)$ such that

$$\|f(t,\psi)\| \leq m(t)W(\|\psi\|_{\mathcal{B}}), \quad (t,\psi) \in I \times \mathcal{B}.$$

(**iii**) There exists an integrable function $\eta : I \to [0, +\infty)$, such that

$$\chi\Big(U(t,s)f(t,D)\Big) \leq \eta(t) \sup_{-\infty \leq \theta \leq 0} \chi(D(\theta)) \text{ for a.e. } s,t \in I,$$

where $D(\theta) = \{v(\theta) : v \in D\}$.

$\mathbf{H_2}$ There exists an continuous function $L : [0, a] \to \mathbb{R}^+$, such that

$$\|g(t,u) - g(t,v)\| \leq L(t)\|u - v\|_{\mathcal{B}}, \quad \forall u, v \in \mathcal{B}.$$

$\mathbf{H_3}$ There exist positive constants L_{h1}, L_{h2} and L_{h*}, such that

$\|A(t)h(t,v)\| \leq L_{h1}\|v\|_{\mathcal{B}} + L_{h2},\ (t,v) \in I \times \mathcal{B}$ and

$\|A(t_1)h(t_1,v_1) - A(t_2)h(t_2,v_2)\| \leq L_{h*}(\|t_1 - t_2\| + \|v_1 - v_2\|_{\mathcal{B}}),\ (t_i, v_i) \in I \times \mathcal{B},\ i = 1, 2.$

$\mathbf{H_4}$ There exist positive constants L_i such that

$$\|I_i(\psi_1) - I_i(\psi_2)\| \leq L_i\|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad \psi_j \in \mathcal{B}, \quad j = 1, 2, \quad i = 1, 2, ..., n.$$

$\mathbf{H_5}$ There exist positive constants C_i^j, $j = 1, 2$, $i = 1, 2, ..., n$, such that

$$\|I_i(\psi)\| \leq C_i^1\|\psi\|_{\mathcal{B}} + C_i^2, \quad \text{for every } \psi \in \mathcal{B}.$$

$\mathbf{H_6}$ (**a**) The evolution system $\{U(t,s) : 0 \leq s \leq t < \infty\}$ is equicontinuous and there exist a positive constant M such that

$$\|U(t,s)\| \leq M \text{ for } 0 \leq s \leq t \leq a.$$

(**b**) $\frac{K_aM}{1-\mu_2}\int_0^t \max\{m(t), L(t)\}ds < \int_c^{+\infty} \frac{ds}{W(s)+s}$, where

$$\mu_1 = K_bM\|h(0,\varphi)\| + (MK_aH + M_a)\|\varphi\|_{\mathcal{B}} + K_aL_{h2}(M_0 + M_b) + K_aM\sum_{0<t_i<t} C_i^2,$$

$$\mu_2 = K_aL_{h1}(M_0 + M_b) + K_aM\sum_{0<t_i<t} C_i^1 < 1, \quad c = \frac{\mu_1}{1-\mu_2};$$

$K_a = \sup\limits_{t\in[0,a]} K(t)$ and $M_a = \sup\limits_{t\in[0,a]} M(t)$.

(**c**) $M\int_0^a (L_{h*}+L(s))K(s)ds + K_aM\sum_{i=1}^n L_i + \int_0^a \eta(s)ds < 1$.

(**d**) $0 \in \rho(A(t))$ for $t \in I$, and there exist a constant $M_0 > 0$, such that $\|A^{-1}(t)\| \le M_0$ for $t \in I$.

Let $y : (-\infty, 0] \to X$ be the function defined by $y_0 = \varphi$ such that $y(t) = U(t,0)(\phi(0) - h(0,\varphi))$ on I. Clearly, $\|y\|_{\mathcal{B}} \le (K_bMH + M_a)\|\varphi\|_{\mathcal{B}} + K_bM\|h(0,\varphi)\|$.

We now define the mild solution for the initial value problem (1.1)-(1.3).

Definition 3.6. *A function $x : (-\infty, a] \to X$ is called a mild solution of the initial value problem (1.1)-(1.3) if $x_0 = \varphi$; $x_{(\cdot)}|_{[0,a]} \in \mathcal{PC}$ and*

$$\begin{aligned} x(t) =& U(t,0)(\varphi(0) - h(0,\varphi)) + h(t,x_t) + \int_0^t U(t,s)A(s)h(s,x_s)ds \\ &+ \int_0^t U(t,s)(f(s,x_s) + g(s,x_s))ds + \sum_{0<t_i<t} U(t,t_i)I_i(x_{t_i}), \quad t \in I = [0,a]. \end{aligned}$$

Now we are in a position to establish our main results.

Theorem 3.1. *If the hypotheses $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{H_3}$, $\mathbf{H_4}$, $\mathbf{H_5}$ and $\mathbf{H_6}$ are satisfied, then there exists at least one mild solution to the initial value problem (1.1)-(1.3).*

Proof. Let $S(a)$ be the space $S(a) = \Big\{x : (-\infty, a] \to X;\ x_0 = 0,\ x|_I \in \mathcal{PC}\Big\}$ endowed with the supremum norm $\|\cdot\|_a$. Let $\Gamma : S(a) \to S(a)$ be the map defined by

$$\Gamma x(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ h(t, x_t + y_t) + \int_0^t U(t,s)A(s)h(s, x_s + y_s)ds & \\ + \int_0^t U(t,s)(f(s, x_s + y_s) + g(s, x_s + y_s))ds & \\ + \sum_{0<t_i<t} U(t,t_i)I_i(x_{t_i} + y_{t_i}), & t \in I. \end{cases} \tag{3.1}$$

Due to the fact that $\|x_t + y_t\|_{\mathcal{B}} \le K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_a\|\ x\|_t$, where $\|x\|_t = \sup_{0\le s\le t}\|x(s)\|$, Γ is well defined with values in $S(a)$. It is easy to see that if x is a fixed point of Γ, then $x + y$ is a mild to the initial value problem (1.1)-(1.3). In the sequel we will prove that there exists a fixed point of Γ by Darbo's fixed point theorem.

We first note that, Γ is continuous on the basis of the axioms of phase space, the Lebesgue Dominated Convergence Theorem and the conditions $\mathbf{H_1}$, $\mathbf{H_2}$ and $\mathbf{H_3}$, we assert that the function $s \to U(t,s)A(s)h(s, x_s + y_s)$ and $s \to (f(s, x_s + y_s) + g(s, x_s + y_s))$ are integrable on $[0,t]$ for every $t > 0$ and every bounded $x \in S(a)$. In fact, let $\|x\|_a < r$, where r is a positive constant. In view of $\mathbf{H_6}$, we have

$$\begin{aligned} \|U(t,s)A(s)h(s, x_s + y_s)\| \le& M(L_{h1}\|x_s + y_s\|_{\mathcal{B}} + L_{h2}) \\ \le& M(L_{h1}(K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_ar) + L_{h2}), \end{aligned}$$

$$\begin{aligned} &\|U(t,s)(f(s, x_s + y_s) + g(s, x_s + y_s))\| \\ &\quad \le M(m(t)W(\|x_s + y_s\|_{\mathcal{B}}) + L(t)\|x_s + y_s\|_{\mathcal{B}} + \|g(s,0)\|) \\ &\quad \le M(m(t)W(K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_ar) \\ &\quad + L_g(K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_ar) + \|g(s,0)) \end{aligned}$$

where $L_g = \max_{t \in I} L(t)$. Hence the function $s \to U(t,s)A(s)h(s, x_s + y_s)$ and $s \to U(t,s)$ $(f(s, x_s + y_s) + g(s, x_s + y_s))$ are integrable on $[0, t]$ for every $t > 0$.

Next, we show that the set $\{x \in \mathcal{PC} : x = \lambda \Gamma x\}$ is bounded for $0 < \lambda < 1$. Let x^λ be a solution of $x = \lambda \Gamma x$ for $0 < \lambda < 1$. Then

$$\| x_t^\lambda + y_t\|_{\mathcal{B}} \leq K_a M \|h(0, \varphi)\| + (K_a M H + M_a)\|\varphi\|_{\mathcal{B}} + K_a \| x^\lambda \|_t.$$

Let $v^\lambda(t) = K_a M \|h(0, \varphi)\| + (K_a M H + M_a)\|\varphi\|_{\mathcal{B}} + K_a \| x^\lambda \|_t$, for each $t \in I$, then

$$\begin{aligned}
\|x^\lambda(t)\| = \|\lambda \Gamma x^\lambda(t)\| &\leq \lambda \|\Gamma x^\lambda(t)\| \\
&\leq M \int_0^t (m(s) W(\|x_s^\lambda + y_s\|_{\mathcal{B}}) + L(s)\|x_s^\lambda + y_s\|_{\mathcal{B}}) ds \\
&\quad + L_{h1}(M_0 + Ma) v^\lambda(t) + L_{h2}(M_0 + Ma) + M \int_0^t \|g(s, 0)\| ds \\
&\quad + M \sum_{0 < t_i < t} (C_i^1 v^\lambda(t) + C_i^2)
\end{aligned}$$

$$\begin{aligned}
\|x^\lambda\|_t \leq{}& M \int_0^t (m(s) W(\|x_s^\lambda + y_s\|_{\mathcal{B}}) + L(s)\|x_s^\lambda + y_s\|_{\mathcal{B}}) ds + M \sum_{0 < t_i < t} C_i^2 \\
&+ \Big(L_{h1}(M_0 + Ma) + M \sum_{0 < t_i < t} C_i^1\Big) v^\lambda(t) + L_{h2}(M_0 + Ma) + M \int_0^t \|g(s, 0)\| ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
v^\lambda(t) \leq{}& K_a M \|h(0, \varphi)\| + (K_a M H + M_a)\|\varphi\|_{\mathcal{B}} + K_a \Big(M \int_0^t (m(s) W(\|x_s^\lambda + y_s\|_{\mathcal{B}}) \\
&+ L(s)\|x_s^\lambda + y_s\|_{\mathcal{B}}) ds + M \sum_{0 < t_i < t} C_i^2 + \Big(L_{h1}(M_0 + Ma) + M \sum_{0 < t_i < t} C_i^1\Big) v^\lambda(t) \\
&+ L_{h2}(M_0 + Ma) + M \int_0^t \|g(s, 0)\| ds \Big).
\end{aligned}$$

Consequently,

$$v^\lambda(t) \leq c + \frac{K_a M}{1 - \mu_2} \int_0^t (m(s) W(v^\lambda(s)) + L(s) v^\lambda(s)) ds. \tag{3.2}$$

Denoting by $\beta_\lambda(t)$ the right-hand side of (3.2), we get

$$\begin{aligned}
\beta_\lambda'(t) &= \frac{K_a M}{1 - \mu_2} (m(s) W(v^\lambda(s)) + L(s) v^\lambda(s)) \\
&\leq \frac{K_a M}{1 - \mu_2} \max\{\alpha(t), L(t)\} (W(\beta^\lambda(t)) + \beta^\lambda(t)),
\end{aligned}$$

therefore,

$$\frac{\beta_\lambda'(t)}{W(\beta^\lambda(t)) + \beta^\lambda(t)} \leq \frac{K_a M}{1 - \mu_2} \max\{\alpha(t), L(t)\}, \tag{3.3}$$

Integrating (3.3) and applying our hypothesis $\mathbf{H_6(a)}$, we obtain

$$\int_c^{\beta_\lambda(t)} \frac{ds}{W(s) + s} \leq \frac{K_a M}{1 - \mu_2} \int_0^t \max\{\alpha(s), L(s)\} ds < \int_c^{+\infty} \frac{ds}{W(s) + s},$$

which implies that $\beta_\lambda(t)$ is bounded in I. Thus, $v^\lambda(t)$ is bounded on I, and $x^\lambda(\cdot)$ is also bounded on I.

Now, we show that Γ is $\chi-contraction$. To clarify this, we decompose Γ in the form $\Gamma = \Gamma_1+\Gamma_2$, for $t \geq 0$, where

$$\begin{aligned}\Gamma_1 x(t) &= h(t, x_t + y_t) + \int_0^t U(t,s)A(s)h(s, x_s + y_s)ds + \int_0^t U(t,s)g(s, x_s + y_s)ds \\ &\quad + \sum_{0<t_i<t} U(t, t_i)I_i(x_{t_i} + y_{t_i}), \qquad t \in [0,a], \\ \Gamma_2 x(t) &= \int_0^t U(t,s)(f(s, x_s + y_s)ds, \quad t \in [0,a].\end{aligned}$$

First, we show that Γ_1 is Lipschitzian on $S(a)$. Take $x_1, x_2 \in S(a)$ arbitrary, on account of definition 2.1 and hypotheses, we get that

$$\begin{aligned}\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\| &\leq \| \int_0^t U(t,s)A(s)(h(s, x_{1s} + y_s) - h(s, x_{2s} + y_s))ds\| \\ &\quad + \| \int_0^t U(t,s)(g(s, x_{1s} + y_s) - g(s, x_{2s} + y_s))ds\| \\ &\quad + M \sum_{0<t_i<t} L_i \|x_{1t_i} - x_{2t_i}\|_{\mathcal{B}} \\ &\leq M \int_0^t L_{h*}K(s) \sup_{0\leq\tau\leq a} \|x_1(\tau) - x_2(\tau)\|ds \\ &\quad + M \int_0^t L(s)K(s) \sup_{0\leq\tau\leq a} \|x_1(\tau) - x_2(\tau)\|ds \\ &\quad + MK_a \sum_{0<t_i<t} L_i \|x_1 - x_2\|_a \\ &\leq M \int_0^a \Big((L_{h*} + L(s))K(s)ds + K_a \sum_{i=1}^n L_i\Big)\|x_1 - x_2\|_a.\end{aligned}$$

Therefore,

$$\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\|_a \leq M \int_0^a \Big((L_{h*} + L(s))K(s)ds + K_a \sum_{i=1}^n L_i\Big)\|x_1 - x_2\|_a.$$

for any $x_1, x_2 \in S(a)$, and Γ_1 is Lipschitzian on $S(a)$ with Lipschitz constant

$$L' = M \int_0^a \Big((L_{h*} + L(s))K(s)ds + K_a \sum_{i=1}^n L_i\Big).$$

Next, take bounded subset $W \subset S(a)$ arbitrary. The hypothesis that $U(t,s)$ is equicontinuous implies the equicontinuity of the set $U(t,s)f(s, W_s + y_s)$. In view of Lemma 2.1, Lemma 2.6 and

$\mathbf{H_1(iii)}$, we obtain that

$$\begin{aligned}\chi\Big(\Gamma_2 w(t)\Big) &= \chi\Big(\int_0^t U(t,s)f(s,W_s+y_s)ds\Big)\\ &\le \int_0^t \eta(s)\sup_{-\infty<\theta\le 0}\chi\Big(W(s+\theta)+y(s+\theta)\Big)ds\\ &\le \int_0^t \eta(s)\sup_{0\le\tau\le s}\chi(W(\tau))ds\\ &\le \chi_{\mathcal{PC}}(W)\int_0^t \eta(s)ds,\end{aligned}$$

and hence

$$\begin{aligned}\chi_{\mathcal{PC}}(\Gamma_2 w) &\le \chi_{\mathcal{PC}}(W)\int_0^t \eta(s)ds\\ &\le \chi_{\mathcal{PC}}(W)\int_0^a \eta(s)ds\end{aligned}$$

for each bounded set $W \in \mathcal{PC}([0,a];X)$, we get from (3.4),(3.5) and Lemma 2.1 that

$$\begin{aligned}\chi_{PC}(\Gamma W) &= \chi_{\mathcal{PC}}\Big(\Gamma_1 W+\Gamma_2 W\Big)\\ &\le \chi_{\mathcal{PC}}(\Gamma_1 W)+\chi_{\mathcal{PC}}(\Gamma_2 W)\\ &\le \Big(L'+\int_0^t \eta(s)ds\Big)\chi_{\mathcal{PC}}(W)\\ &\le \chi_{\mathcal{PC}}(W).\end{aligned}$$

The hypothesis $\mathbf{H_6(b)}$ implies that Γ is $\chi-contraction.$ In view of Lemma 2.2, (Darbo fixed point theorem), we conclude that Γ has at least one fixed point of Γ in $S(a)$. Let x be a fixed of Γ on $S(a)$, then $z=x+y$ is a mild solution of (1.1)-(1.3), which completes the proof.

Theorem 3.2. *Assume that the hypotheses* $\mathbf{H_1}$*,* $\mathbf{H_2}$*,* $\mathbf{H_3}$*,* $\mathbf{H_4}$*,* $\mathbf{H_5}$*,* $\mathbf{H_6(a)}$*,* $\mathbf{H_6(c)}$ *and* $\mathbf{H_6(d)}$ *are satisfied. Furthermore, we suppose*

$$K_a\Big(L_{h1}(M_0+Ma)+M\sum_{i=1}^n C_i^1+\int_0^a L(s)ds+M\int_0^a m(s)ds\lim_{\tau\to\infty}\sup\frac{(W(\tau))}{\tau}\Big)<1$$

Then there exists a mild solution of (1.1)-(1.3).

Proof. Proceeding as in the proof of the Theorem 3.1, we infer that the map Γ given by (3.1) is continuous from $S(a)$ into $S(a)$. Furthermore, there exists $r>0$ such that $\Gamma(B_r)\subset B_r$, where $B_r=\Big\{x\in S(a):\|x\|_a\le r\Big\}$. In fact, if we assume that the assertion is false, then for $r>0$ there exists $x^r \in B_r$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r)\|$. This yields that

$$\begin{aligned}
r < \|\Gamma x^r(t^r)\| \leq{}& M\int_0^t (m(s)W(\|x^r_{s^r} + y_{s^r}\|_{\mathcal{B}}) + L(s)\|x^r_{s^r} + y_{s^r}\|_{\mathcal{B}})ds\\
&+ L_{h1}(M_0 + Ma)(\|x^r_{t^r} + y_{t^r}\|) + L_{h2}(M_0 + Ma) + M\int_0^t \|g(s,0)\|ds\\
&+ M\sum_{0<t_i<t}(C_i^1(\|x^r_{t^r} + y_{t^r}\|) + C_i^2)\\
\leq{}& M\int_0^a \Big(m(s)W(K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_ar)\\
&+ L(s)(K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_ar)\Big)ds\\
&+ L_{h1}(M_0 + Ma)(K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_ar)\\
&+ M\sum_{i=1}^n (C_i^1(K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_ar) + C_i^2)\\
&+ L_{h2}(M_0 + Ma) + M\int_0^a \|g(s,0)\|ds
\end{aligned}$$

which implies that

$$\begin{aligned}
1 \leq{}& K_a\Big(L_{h1}(M_0 + Ma) + M\sum_{i=1}^n C_i^1 + \int_0^a L(s)ds\Big)\\
&+ M\int_0^a m(s)ds \lim_{r\to\infty}\sup\frac{(W(K_aM\|h(0,\varphi)\| + (K_aMH + M_a)\|\varphi\|_{\mathcal{B}} + K_ar)}{r}\\
\leq{}& K_a\Big(L_{h1}(M_0 + Ma) + M\sum_{i=1}^n C_i^1 + \int_0^a L(s)ds + M\int_0^a m(s)ds \lim_{\tau\to\infty}\sup\frac{(W(\tau))}{\tau}\Big),
\end{aligned}$$

which is contradiction to our assumption.

By means of Lemma (2.3), as in the proof of Theorem 3.1, we conclude that (1.1)-(1.3) has a mild solution.

4. Example

Consider the model

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big[x(t,\xi) - \int_{-\infty}^0 T(\theta)u(t, x(t+\theta,\xi))d\theta\Big] = b(t,\xi)\frac{\partial^2}{\partial\xi^2}x(t,\xi)\\
&\quad + \int_{-\infty}^0 P(\theta)r(t, x(t+\theta,\xi))d\theta + \int_{-\infty}^0 Q(\theta)s(t, x(t+\theta,\xi))d\theta,\ t\in I,\ \xi\in[0,\pi] \qquad (4.1)\\
&x(t,0) = x(t,\pi) = 0,\ t\in I, \qquad (4.2)\\
&x(\tau,\xi) = x_0(\tau,\xi),\ \tau\leq 0,\ 0\leq\xi\leq\pi \qquad (4.3)\\
&\Delta x(t_j,\xi) = \int_{-\infty}^{t_j}\gamma_j(t_j - s)x(s,\xi)ds,\ j = 1,2,...,n \qquad (4.4)
\end{aligned}$$

where $b(t,\xi)$ is a continuous function and is uniformly Holder continuous in t: $T, P, Q : (-\infty, 0] \to \mathbb{R}$; $u, r, z : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $x_0 : (-\infty, 0] \times [0,\pi] \to \mathbb{R}$ are continuous functions.

Let $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = b(t,\xi)w''$ with domain $D(A) = \{w \in E : w, w'$ are absolutely continuous $w'' \in E,\ w(0) = w(\pi) = 0\}$. Then $A(t)$ generates an evolution system $U(t,s)$ satisfying the assumptions $(\mathbf{H1})$ and $(\mathbf{H2})$ (see [17]).

For the phase space $\mathcal{B}$, we choose the well known space $BU\mathcal{PC}(\mathbb{R}^-, E)$: the space of uniformly bounded continuous functions endowed with the norm $\|\varphi\| = \sup_{\theta\le 0}|\varphi(\theta)|$ for $\varphi \in B$. If we put for $\varphi \in BU\mathcal{PC}(\mathbb{R}^-, E)$ and $\xi \in [0,\pi]$,

$$
\begin{aligned}
&v(t)\xi = x(t,\xi),\ t \in I,\ \xi \in [0,\pi],\\
&\phi(\theta)(\xi) = x_0(\theta,\xi),\ -\infty < \theta \le 0,\ \xi \in [0,\pi],\\
&h(t,\varphi)(\xi) = \int_{-\infty}^{0} T(\theta)u(t,\psi(\theta)(\xi))d\theta,\ -\infty < \theta \le 0,\ \xi \in [0,\pi],\\
&f(t,\varphi)(\xi) = \int_{-\infty}^{0} P(\theta)r(t,\psi(\theta)(\xi))d\theta,\ -\infty < \theta \le 0,\ \xi \in [0,\pi],\\
&g(t,\varphi)(\xi) = \int_{-\infty}^{0} Q(\theta)z(t,\psi(\theta)(\xi))d\theta,\ -\infty < \theta \le 0,\ \xi \in [0,\pi],\\
&I_j(\psi) = \int_{-\infty}^{0} \gamma_j(-s)\psi(s,\xi)ds, \quad j = 1,2,...,n,
\end{aligned}
$$

Then, (4.1)-(4.4) takes the abstract neutral perturbed evolution form (1.1)-(1.3). To show the existence of the mild solution to (4.1)-(4.4), we assume the following hypotheses:

$(\mathbf{1})$ The functions x and s are Lipschitz with respect to its second argument, and constants lip(x) and lip(s) respectively.

$(\mathbf{2})$ There exist $p \in L^1([0,+\infty),\mathbb{R}^+)$ and a nondecreasing continuous function $\psi : [0,+\infty) \to (0,\infty)$ such that $|r(t,u)| \le p(t)\psi(|x|)$, for $t \in [0,\infty)$, $u \in \mathbb{R}$.

$(\mathbf{3})$ T, P and Q are integrable on $(-\infty, 0]$.

By the Dominated Convergence Theorem, one can show that f is a continuous function from $\mathcal{B}$ to E. Moreover the mapping h, g and γ_j are Lipschitz continuous in its second argument, in fact, we have

$$
\begin{aligned}
&|g(t,\varphi_1) - g(t,\varphi_2)| \le lip(s)\int_{-\infty}^{0}|Q(\theta)|d\theta|\varphi_1 - \varphi_2|, \quad \varphi_1,\varphi_2 \in \mathcal{B},\\
&|h(t,\varphi_1) - h(t,\varphi_2)| \le M_0\ lip(x)\int_{-\infty}^{0}|T(\theta)|d\theta|\varphi_1 - \varphi_2|, \quad \varphi_1,\varphi_2 \in \mathcal{B},\\
&|\gamma_j(\xi,s) - \gamma_j(\xi,\overline{s})| \le L_i|s - \overline{s}|, \quad \xi \in [0,\pi],\ s,\overline{s} \in \mathbb{R}.
\end{aligned}
$$

On the other hand, for $\varphi \in \mathcal{B}$ and $\xi \in [0,\pi]$, we have

$$|f(t,\varphi)(\xi)| \le \int_{-\infty}^{0}|p(t)P(\theta)|\psi(|\varphi(\theta)(\xi)|)d\theta.$$

Since the function is nondecreasing, it follows that

$$|f(t,\varphi)| \le p(t)\int_{-\infty}^{0}|P(\theta)|d\theta\psi(|\varphi|),\ \text{for } \varphi \in \mathcal{B}.$$

Proposition 4.1. *Assume the above hypotheses and the condition* $\mathbf{H_6}$ *in Theorem 3.2 hold,* $\varphi \in \mathcal{B}$*, then there exists at least one mild solution to system (4.1)-(4.4).*

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