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Profile likelihood estimation of generalized half logistic distribution under progressively type-II censoring

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Abstract

The half logistic distribution has been used intensively in reliability and survival analysis especially when the data is censored. In this paper, we provide profile likelihood estimation of the shape parameter and scale parameter in the generalized half logistic distribution based on progressively Type-II censored data. We also introduce approximate maximum profile likelihood estimates for the scale parameter. As an illustration, we examine the validity of our estimation using real data and simulated data.

Keywords: Generalized half logistic distribution, profile likelihood estimation, progressively Type-II censoring.

1. Introduction

The probability density function (pdf) and cumulative distribution function (cdf) of the random variable X having the generalized half logistic distribution are given by

$$f(x;\lambda,\sigma) = \frac{\lambda}{\sigma}\left[\frac{2e^{-x/\sigma}}{1+e^{-x/\sigma}}\right]^{\lambda}\frac{1}{1+e^{-x/\sigma}} \tag{1.1}$$

and

$$F(x;\lambda,\sigma) = 1-\left[\frac{2e^{-x/\sigma}}{1+e^{-x/\sigma}}\right]^{\lambda}, \tag{1.2}$$

where λ is shape parameter and σ is scale parameter. In special case, when $\lambda = 1$, this distribution is the half logistic distribution. From (1.1), the reliability function of the generalized half logistic distribution with shape parameter λ is given by

$$R(t) = \left[\frac{2e^{-t/\sigma}}{1+e^{-t/\sigma}}\right]^{\lambda}, t>0, \lambda>0, \sigma>0. \tag{1.3}$$

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The half logistic distribution has been used quite extensively in reliability and survival analysis particularly when the data is censored. Inferences for the half logistic distribution were discussed by several authors. Balakrishnan and Puthenpura (1986) introduced the best linear unbiased estimators of location and scale parameters of the half logistic distribution through linear functions of order statistics. Balakrishnan and Wong (1991) obtained approximate maximum likelihood estimates (AMLEs) for the location and scale parameters of the half logistic distribution with Type-II right censored sample. Recently, Kang and Park (2005) derived the AMLE of the scale parameter of the half logistic distribution based on multiply Type-II censored samples. Kang *et al.* (2009) proposed the AMLEs of the scale parameter in a half logistic distribution based on double hybrid censored samples.

The progressive censoring appears to be of great importance in planning duration experiments in reliability studies. In many life testing and industrial experiments, experiments have to be terminated early and also the number of failures must be limited for various reasons. Progressively Type-II censoring is a generalization of Type-II censoring. In this case, the first r_1 failures in a life test of n items are observed; then n_1 of the remaining $n - r_1$ unfailed items are removed from the experiment, leaving $n - r_1 - n_1$ items still present. When a further r_2 items have failed, n_2 of the still unfailed items are removed, and so on. The experiment terminates after some prearranged series of repetitions of this procedures. Also note that if $r_1 = r_2 = \cdots = r_m = 0$, so that $m = n$, progressively Type-II censoring scheme reduces to the case of no censoring, that is, complete data. Balakrishnan *et al.* (2003) suggested point and interval estimation for Gaussian distribution based on progressively Type-II censored samples. Balakrishnan *et al.* (2004) studied point and interval estimation for the extreme value distribution under progressively Type-II censored sample. Seo and Kang (2007) derived AMLEs for Rayleigh distribution based on progressively Type-II censored data. Kang *et al.* (2008) derived the AMLEs and maximum likelihood estimator of the scale parameter in a half logistic distribution based on progressively Type-II censored samples (also see Kang and Jung, 2009).

Profile likelihood can be used in all circumstance by profiling out the nuisance parameter while marginal likelihood and conditional likelihood are avaiable only in very special case. Profile likelihood behaves like ordinary likelihood in that it has a quadratic expansion. In parametric model, Patefield (1977) showed that the inverse of the observed profile information is equal to the parameter aspect of the full observed inverse information. Further discussion in the parametric context was given by Barndorff-Nielsen and Cox (1994). Thus it appears that the profile likelihood can be used and visualized in the same fashion as an ordinary parametric likelihood. Ferrari *et al.* (2007) discussed adjusted profile likelihoods for the Weibull shape parameter. Recently, Ventura *et al.* (2010) proposed and discussed the construction of a default prior distribution for a scalar parameter of interest to be used together with quasi-profile likelihood functions.

In this paper, we propose a profile likelihood estimation for the shape parameter and location parameter of generalized half logistic distribution under progressively type-II censoring. We also derive approximated maximum profile likelihood estimators (approximated profile MLEs).

2. Profile likelihood estimation

We begin by deriving the maximum likelihood estimator of the shape parameter based on progressively Type-II censored sample for a fixed scale parameter σ. Let $x_{1:m:n}, \ldots, x_{m:m:n}$ denote such a sample with $(r_1, \ldots, r_m)$ being the progressive Type-II censoring scheme. The likelihood function based on the sample, $x_{1:m:n}, \ldots, x_{m:m:n}$, is then given by

$$L(\lambda) = C\left(\frac{\lambda}{\sigma}\right)^m \prod_{i=1}^{m}\left[\frac{2e^{-x_{i:m:n}/\sigma}}{1+e^{-x_{i:m:n}/\sigma}}\right]^{\lambda(1+r_i)} \frac{1}{1+e^{-x_{i:m:n}/\sigma}}, \tag{2.1}$$

where $C = n \times (n-1-r_1) \times (n-2-r_1-r_2) \times \cdots \times (n-m+1-r_1-\cdots-r_{m-1})$. It is easy to obtain the maximum likelihood estimator (MLE) of λ for the fixed σ to be

$$\hat{\lambda}(\sigma) = \frac{m}{T_1}, \tag{2.2}$$

where

$$T_1 = \sum_{i=1}^{m}(1+r_i)\left[\frac{x_{i:m:n}}{\sigma} + \ln\left(\frac{1+e^{-x_{i:m:n}/\sigma}}{2}\right)\right]. \tag{2.3}$$

The profile likelihood function of the scale parameter σ is $L_p(\sigma) = L(\sigma, \hat{\lambda}(\sigma))$. Thus, the profile log likelihood function of σ is given by

$$l_p(\sigma) = \ln C - m - \sum_{i=1}^{m}\ln(1+e^{-x_{i:m:n}/\sigma}) \tag{2.4}$$

$$+\, m\left[\ln m - \ln\left\{\sum_{i=1}^{m}(1+r_i)\left(\frac{x_{i:m:n}}{\sigma} + \ln\frac{1+e^{-x_{i:m:n}/\sigma}}{2}\right)\right\}\right] - m\ln\sigma. \tag{2.5}$$

The profile MLE of σ, denote by $\hat{\sigma}$, is given by

$$\frac{1}{\hat{\sigma}}\left[\sum_{i=1}^{m}\left(\frac{e^{-x_{i:m:n}/\hat{\sigma}}}{1+e^{-x_{i:m:n}/\hat{\sigma}}}\frac{x_{i:m:n}}{\hat{\sigma}}\right) + m\right]$$

$$= \frac{m}{\hat{\sigma}}\left[\frac{\sum_{i=1}^{m}\frac{(1+r_i)}{1+e^{-x_{i:m:n}/\hat{\sigma}}}\frac{x_{i:m:n}}{\hat{\sigma}}}{\sum_{i=1}^{m}(1+r_i)\left(\frac{x_{i:m:n}}{\hat{\sigma}} + \ln\frac{1+e^{-x_{i:m:n}/\hat{\sigma}}}{2}\right)}\right] \tag{2.6}$$

This Equation is in implicit form, so it may be solved by using numerical method such as Newton-Raphson or Bisection. The MLE of shape parameter λ is obtained by replacing σ by $\hat{\sigma}$ in Equation (2.2).

Instead of numerical approach, we may consider approximated MLE (AMLE) of the scale parameter based on the profile likelihood function. Here we expand the functions $\frac{e^{-z_{i:m:n}}}{1+e^{-z_{i:m:n}}}z_{i:m:n}$, $\frac{1}{1+e^{-z_{i:m:n}}}z_{i:m:n}$, and $\ln\left(\frac{2e^{-z_{i:m:n}}}{1+e^{-z_{i:m:n}}}\right)$ in Taylor series around the points ξ_i, where $z_{i:m:n} = \frac{x_{i:m:n}}{\sigma}$, $\xi_i = F^{-1}(p_{i:m:n}) = \ln\left(\frac{2-q_{i:m:n}^{1/\lambda_0}}{q_{i:m:n}^{1/\lambda_0}}\right)$, $q_{i:m:n} = 1 - p_{i:m:n}$, and is given by the book of Balakrishnan and Aggarwala (2000),

$$p_{i:m:n} = 1 - \prod_{j=m-i+1}^{m}\frac{j + r_{m-j+1} + \cdots + r_m}{j + 1 + r_{m-j+1} + \cdots + r_m}, i = 1, ..., m. \tag{2.7}$$

First, we can approximate these functions by

$$\frac{1}{1+e^{-z_{i:m:n}}} z_{i:m:n} \approx \alpha_i + \beta_i z_{i:m:n}, \tag{2.8}$$

$$\frac{e^{-z_{i:m:n}}}{1+e^{-z_{i:m:n}}} z_{i:m:n} \approx -\alpha_i + (1-\beta_i) z_{i:m:n} \tag{2.9}$$

and

$$\ln\left(\frac{1+e^{-z_{i:m:n}}}{2e^{-z_{i:m:n}}}\right) \approx \gamma_i + \eta_i z_{i:m:n}, \tag{2.10}$$

where

$$\alpha_i = -\frac{e^{-\xi_i}}{(1+e^{-\xi_i})^2}\xi_i^2, \qquad \beta_i = \frac{1}{1+e^{-\xi_i}} + \frac{e^{-\xi_i}}{(1+e^{-\xi_i})^2}\xi_i, \tag{2.11}$$

$$\gamma_i = -\ln\left(\frac{2e^{-\xi_i}}{1+e^{-\xi_i}}\right) - \frac{1}{(1+e^{-\xi_i})}\xi_i \quad \text{and} \quad \eta_i = \frac{1}{(1+e^{-\xi_i})}. \tag{2.12}$$

This approach requires a initial value for λ_0, which is obtained by

$$\lambda_0 = \frac{m}{\sum_{i=1}^m (1+r_i)\left[\ln\left(\frac{1+e^{-x_{i:m:n}/\sigma}}{2e^{-x_{i:m:n}/\sigma}}\right)\right]_{\sigma=1}}, \tag{2.13}$$

We may need to repeat this procedure by updating σ value. For more stable approximation, let

$$\phi_1 = \sum_{i=1}^m (1+r_i)\alpha_i, \quad \phi_2 = \sum_{i=1}^m (1+r_i)\beta_i x_{i:m:n}, \tag{2.14}$$

$$\varphi_1 = \sum_{i=1}^m (1+r_i)\gamma_i, \quad \varphi_2 = \sum_{i=1}^m (1+r_i)\eta_i x_{i:m:n}, \tag{2.15}$$

and

$$\psi_1 = m - \sum_{i=1}^m \alpha_i, \quad \psi_2 = \sum_{i=1}^m (1-\beta_i) x_{i:m:n}. \tag{2.16}$$

For a reasonable initial value ζ, approximating the right side of (2.6) by

$$\frac{\phi_1+\phi_2\zeta}{\varphi_1+\varphi_2\zeta} - \frac{(\phi_2\varphi_1-\phi_1\varphi_2)\zeta}{(\varphi_1+\varphi_2\zeta)^2} + \frac{\varphi_2(\phi_2\varphi_1-\phi_1\varphi_2)\zeta^2}{(\varphi_1+\varphi_2\zeta)^3} \tag{2.17}$$

$$+\left(\frac{(\phi_2\varphi_1-\phi_1\varphi_2)}{(\varphi_1+\varphi_2\zeta)^2} - \frac{2\varphi_2(\phi_2\varphi_1-\phi_1\varphi_2)\zeta}{(\varphi_1+\varphi_2\zeta)^3}\right)\frac{1}{\sigma} + \frac{\varphi_2(\phi_2\varphi_1-\phi_1\varphi_2)}{(\varphi_1+\varphi_2\zeta)^3}\frac{1}{\sigma^2}, \tag{2.18}$$

we can derive an approximate profile MLE of σ as follows:

$$\hat{\sigma} = \frac{\sqrt{B^2-4AC}-B}{2A}, \tag{2.19}$$

where

$$A = \frac{\phi_1+\phi_2\zeta}{\varphi_1+\varphi_2\zeta} - \frac{(\phi_2\varphi_1-\phi_1\varphi_2)\zeta}{(\varphi_1+\varphi_2\zeta)^2} + \frac{\varphi_2(\phi_2\varphi_1-\phi_1\varphi_2)\zeta^2}{(\varphi_1+\varphi_2\zeta)^3} - \psi_1, \tag{2.20}$$

$$B = \frac{(\phi_2\varphi_1 - \phi_1\varphi_2)}{(\varphi_1 + \varphi_2\zeta)^2} - \frac{2\varphi_2(\phi_2\varphi_1 - \phi_1\varphi_2)\zeta}{(\varphi_1 + \varphi_2\zeta)^3} - \psi_2 \tag{2.21}$$

and

$$C = \frac{\varphi_2(\phi_2\varphi_1 - \phi_1\varphi_2)}{(\varphi_1 + \varphi_2\zeta)^3}. \tag{2.22}$$

3. Illustrative examples

In this section, we present two examples to illustrate our estimation methods discussed in the previous sections.

3.1. Real data

Consider the data given by Nelson (1982), represents failure log times to breakdown of an insulating fluid testing experiment (see Table 3.1). This data has been mainly used for an illustration of the extreme value distribution. A progressively Type-II censored sample are generated from this data by Viverous and Balakrishnan (1994). The observations and censoring scheme are given in Table 3.2. In this example, we have $n = 16$ and $m = 8$. The profile MLEs are obtained by solving the nonlinear equations (2.6) using Newton-Raphson method ($\hat{\lambda} = 2.1104$ and $\hat{\sigma} = 1.2849$), which can be compared to approximated profile MLEs $\tilde{\lambda} = 1.9872$ and $\tilde{\sigma} = 1.3425$.

Table 3.1 Failure log times to breakdown of an insulating fluid testing experiment

0.270027	1.02245	1.15057	1.42311	1.54116	1.57898	1.8718	1.9947
2.08069	2.11263	2.48989	3.45789	3.48186	3.52371	3.60305	4.28895

Table 3.2 Progressively type-II censored data

i	1	2	3	4	5	6	7	8
x	0.270027	1.02245	1.15057	1.57898	2.11263	2.48989	3.60305	4.28895
r_i	0	0	2	3	0	3	0	0

3.2. Simulated data

To assess the performance of the profile likelihood estimation of the shape parameter λ and scale parameter σ, we explore the mean squared errors of profile MLEs through Monte Carlo simulation method. The progressively Type-II censored data from the generalized half logistic distribution are generated for $(\lambda, \sigma) = \{(0.8, 1.0), (1.0, 1.0), (1.2, 1.0)\}$ with various sample sizes and censoring schemes. As discussed in section 2, The exact and approximated profile MLEs of the shape parameter λ and scale parameter σ are obtained and their the mean squared errors are computed by Monte Carlo method based on 10,000 runs for sample sizes $n = 10, 20, 30$ and proposed censoring under progressively Type-II censored samples with $\sigma = 1$. For simplicity in notation, we denote the scheme $(0, 0, \ldots, n - m)$ as $((m - 1) \times 0, n - m)$, for example, (10×0) and $(3 \times 0, 2, 2, 0)$ denote the progressively censoring schemes $(0, 0, \ldots, 0)$ and $0, 0, 0, 2, 2, 0)$, respectively. These values are given in Tables 3.3. For a large sample size, approximated profile estimation provides quite good performance compared to exact profile estimation.

Table 3.3 The relative mean squared errors for profile MLEs of shape parameter λ and scale parameter σ

n	m	censoring scheme	$\lambda = 0.8$		$\lambda = 1.2$		$\lambda = 1.7$	
			mse of σ	mse of λ	mse of σ	mse of λ	mse of σ	mse of λ
10	10	(10×0)	0.11823	0.12273	0.14406	0.41024	0.17534	0.85829
	8	(2 7×0)	0.12948	0.14574	0.15262	0.52845	0.18126	1.01519
	6	(4 5×0)	0.14077	0.20444	0.15960	0.76850	0.18946	1.33681
20	20	(20×0)	0.10842	0.08608	0.11297	0.28116	0.16904	0.66221
	15	(8×0,5,6×0)	0.11532	0.10069	0.12151	0.32931	0.17149	0.73411
	10	(2×0,1,0,2,0,2,2×0,5)	0.12972	0.18065	0.13479	0.59800	0.17506	1.03727
	10	(2×0,3,0,2,0,2,2×0,3)	0.12460	0.16634	0.13947	0.57956	0.17425	1.04968
30	30	(30×0)	0.08454	0.08001	0.11166	0.24784	0.15944	0.61048
	25	(2,12×0,2,1,10×0)	0.08889	0.08509	0.11548	0.26455	0.16297	0.64100
	20	(9×0,10,10×0)	0.09983	0.09110	0.11967	0.29807	0.17024	0.69226
	15	(10,6×0,5,7×0)	0.10925	0.10040	0.11899	0.32965	0.16854	0.72533

Table 3.4 The relative mean squared errors for approximated profile MLEs of shape parameter λ and scale parameter σ

n	m	censoring scheme	$\lambda = 0.8$		$\lambda = 1.2$		$\lambda = 1.7$	
			mse of σ	mse of λ	mse of σ	mse of λ	mse of σ	mse of λ
10	10	(10×0)	0.14482	0.16463	0.18149	0.65445	0.21095	1.27344
	8	(2 7×0)	0.16573	0.20925	0.20945	0.79477	0.23579	1.49058
	6	(4 5×0)	0.17964	0.27445	0.24864	0.98351	0.28534	1.82753
20	20	(20×0)	0.12521	0.11476	0.13958	0.32345	0.18257	0.71754
	15	(8×0,5,6×0)	0.14346	0.14583	0.15456	0.41334	0.19124	0.81657
	10	(2×0,1,0,2,0,2,2×0,5)	0.15853	0.19656	0.17963	0.63873	0.21738	1.35748
	10	(2×0,3,0,2,0,2,2×0,3)	0.15368	0.20227	0.18028	0.62347	0.20935	1.33575
30	30	(30×0)	0.09347	0.08346	0.12399	0.26346	0.16537	0.69432
	25	(2,12×0,2,1,10×0)	0.10153	0.08996	0.13645	0.29352	0.16936	0.71035
	20	(9×0,10,10×0)	0.10966	0.09734	0.14954	0.32305	0.18886	0.73423
	15	(10,6×0,5,7×0)	0.11282	0.10975	0.15475	0.33145	0.18157	0.75236

4. Concluding remarks

In this paper we present the profile MLEs and approximated profile MLEs of the shpae and scale parameters of generalized half logistic distribution under progressively Type-II censoring. As might be expected, the MSE of all estimators decreases as sample size n increases. For fixed sample size, the MSE increases generally as the number of unobserved or missing data $n-m$ increase. Approximate profile MLE of scale parameter is also introduced. The approximate profile MLEs of λ and σ are evaluated in terms of estimated MSE. We can see that maximum profile likelihood estimation provides a quite efficient and plausible results over all.

References

Balakrishnan, N., Kannan, N., Lin, C. T. and Wu, S. J. S. (2004). Inference for the extreme value distribution under progressively Type-II censoring. *Journal of Statistical Computation and Simulation*, **74**, 25-45.

Balakrishnan, N., Kannan, N., Lin, C. T. and Ng, H. K. T. (2003). Point and interval estimation for Gaussian distribution based on progressively Type-II censored samples. *IEEE Transactions on Reliability*, **52**, 90-95.

Balakrishnan, N. and Wong, K. H. T. (1991). Approximate MLEs for the location and scale parameters of the half logistic distribution with Type-II right-censoring. *IEEE Transactions on Reliability*, **40**, 140-145.

Balakrishnan, N. and Puthenpura, S. (1986). Best linear unbiased estimators of location and scale parameters of the half logistic distribution. *Journal of Statistical Computation and Simulation*, **25**, 193-204.

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive censoring: Theory, methods and applications*, Bircäuser, Boston.

Barndorff-Nielsen, 0. E., and Cox, D. R. (1994). *Inference and asymptotic*, Chapman and Hall, London.

Ferrari, S. L. P., Da Silva, M. F. and Crabari-Neto, F. (2007). Adjusted profile likelihoods for the Weibull shape parameter. *Journal of Statistical Computation and Simulation*, **77**, 531-548.

Kang, S. B., Cho, Y. S. and Han, J. T. (2008). Estimation for the half logistic distribution under progressive Type-II censoring. *Communications of the Korean Statistical Society*, **15**, 815-823.

Kang, S. B., Cho, Y. S. and Han, J. T. (2009). Estimation for the half logistic distribution based on double hybrid censored samples. *Communications of the Korean Statistical Society*, **16**, 1055-1066.

Kang, S. B. and Jung, W. T. (2009). Estimation for the double Rayleigh distribution based on progressive Type-II censored samples. *Journal of the Korean Data & Information Science Society*, **20**, 1199-1206.

Kang, S. B. and Park, Y. K. (2005). Estimation for the half-logistic distribution based on multiply Type-II censored samples. *Journal of the Korean Data & Information Science Society*, **16**, 145-156.

Nelson, W. B (1982). *Applied life data analysis*, John Willey & Sons, New York.

Patefield, W. M. (1977). On the maximized likelihood function. *Sankhyā, Series B*, **39**, 92-96.

Seo, E. H. and Kang, S. B. (2007). AMLEs for Rayleigh distribution based on progressively Type-II censored data. *The Korean Communications in Statistics*, **14**, 329-344.

Ventura, L., Cabras, S. and Racugno, W. (2010). Default prior distributions from quasi- and quasi-profile likelihoods. *Journal of Statisical Planning and Inference*, **140**, 2937-2942.

Viverous, R. and Balakrishnan, N. (1994). Interval estimation of parameters of life from progressively censored data. *Technometrics*, **36**, 84-91.