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DIFFERENTIAL EQUATIONS CHARACTERIZING TIMELIKE AND SPACELIKE CURVES OF CONSTANT BREADTH IN MINKOWSKI 3-SPACE E_1^3

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Abstract. In this paper, we give the differential equations characterizing the timelike and spacelike curves of constant breadth in Minkowski 3-space E_1^3. Furthermore, we give a criterion for a timelike or spacelike curve to be the curve of constant breadth in E_1^3. As an example, the obtained results are applied to the case $\rho =$ const. and $k_2 =$ const., and are discussed.

1. Introduction

Constant breadth curves were introduced by L. Euler in 1778 [7]. He studied the constant breadth curves in the plane. After him, many geometers have shown increased interest in the properties of plane convex curves. A brief review of the most important publications on this subject has been published by Struik [18]. Also, a number of interesting properties of plane curves of constant breadth are included in the works of Ball [1], Barbier [2], Blaschke [3, 4] and Mellish [12]. A space curve of constant breadth was obtained by Fujiwara by taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve, and for which the distance $d(P,Q)$ is constant [8]. For such curves, PQ is also normal at Q. He also studied constant breadth surfaces. Furthermore, Blaschke defined the curve of constant breadth on the sphere [4]. Köse presented some concept for space curves of constant breadth in Euclidean 3-space in [10] and differential equations characterizing space curves of constant breadth were obtained by Sezer in [16]. Constant breadth curves in Euclidean 4-space were given by Mağden and Köse [11]. Corresponding characterizations for spacelike curves of constant breadth in Minkowski 4-space were given by Kazaz, Onder and Kocayiğit [9].

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Furthermore, Reuleaux studied the curves of constant breadth and gave the method related to these curves for the kinematics of machinery [14]. Then, constant breadth curves had an importance for engineering sciences, particularly, in com designs [19].

In this paper, we study the differential equations characterizing timelike and spacelike curves of constant breadth in Minkowski 3-space E_1^3. Moreover, we give the integral characterizations of these curves in E_1^3.

2. Preliminaries

The Minkowski 3-space E_1^3 is the real vector space $\mathbb{R}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike if $g(\vec{v}, \vec{v}) > 0$ or $\vec{v} = 0$, timelike if $g(\vec{v}, \vec{v}) < 0$ and null (lightlike) if $g(\vec{v}, \vec{v}) = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\alpha}'(s)$ are respectively spacelike, timelike or null (lightlike). We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in E_1^3, the vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = \begin{vmatrix} \vec{e}_1 & -\vec{e}_2 & -\vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2),$$

where $\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$ $\vec{e}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ and $\vec{e}_1 \times \vec{e}_2 = -\vec{e}_3$, $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$, $\vec{e}_3 \times \vec{e}_1 = -\vec{e}_2$.

The Lorentzian sphere and hyperbolic sphere of radius r and center 0 in E_1^3 are given by

$$S_1^2 = \left\{\vec{x} = (x_1, x_2, x_3) \in E_1^3 :\ \ g(\vec{x}, \vec{x}) = r^2\right\}$$

and

$$H_0^2 = \left\{\vec{x} = (x_1, x_2, x_3) \in E_1^3 :\ \ g(\vec{x}, \vec{x}) = -r^2\right\},$$

respectively (See for details [13]).

Denote by $\{\vec{T}, \vec{N}, \vec{B}\}$ the moving Frenet frame along the curve $\vec{\alpha}(s)$ in the Minkowski space E_1^3. For an arbitrary spacelike curve $\vec{\alpha}(s)$ in the space E_1^3, the following Frenet formulae are given,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix},$$

where $g(\vec{T},\vec{T}) = 1,\ g(\vec{N},\vec{N}) = \varepsilon = \pm 1,\ g(\vec{B},\vec{B}) = -\varepsilon,\ g(\vec{T},\vec{N}) = g(\vec{T},\vec{B}) = g(\vec{N},\vec{B}) = 0$ and k_1 and k_2 are curvature and torsion of the spacelike curve $\vec{\alpha}(s)$, respectively. Here, ε determines the kind of spacelike curve $\vec{\alpha}(s)$. If $\varepsilon = 1$, then $\vec{\alpha}(s)$ is a spacelike curve with spacelike principal normal $\vec{N}$ and timelike binormal $\vec{B}$. If $\varepsilon = -1$, then $\vec{\alpha}(s)$ is a spacelike curve with timelike principal normal $\vec{N}$ and spacelike binormal $\vec{B}$ [20].

Furthermore, for a timelike curve $\vec{\alpha}(s)$ in the space E_1^3, the following Frenet formulae are given

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix},$$

where $g(\vec{T},\vec{T}) = -1,\ g(\vec{N},\vec{N}) = g(\vec{B},\vec{B}) = 1,\ g(\vec{T},\vec{N}) = g(\vec{T},\vec{B}) = g(\vec{N},\vec{B}) = 0$ and k_1 and k_2 are curvature and torsion of the timelike curve $\vec{\alpha}(s)$ respectively [20].

3. Differential equations characterizing spacelike curves of constant breadth in E_1^3

In this section, we study the differential equations which characterize spacelike curves of constant breadth in Minkowski 3-space.

Let (C) be a unit speed spacelike curve of the class C^3 with nonzero curvature and torsion and assume that (C) has parallel tangents $\vec{T}$ and $\vec{T}^*$ in opposite direction at the opposite points α and α^* of the curve. If the chord joining the opposite points of (C) is a double-normal, then (C) has constant breadth, and conversely, if (C) is a spacelike curve of constant breadth, then every normal of (C) is a double-normal. A simple closed spacelike curve (C) of constant breadth having parallel tangents in opposite directions at opposite points may be represented by the equation

$$\vec{\alpha}^*(s) = \vec{\alpha}(s) + m_1(s)\vec{T}(s) + m_2(s)\vec{N}(s) + m_3(s)\vec{B}(s), \tag{1}$$

where $m_i(s),\ (1 \le i \le 3)$ are the differentiable functions of s which is arc length of (C). Differentiating this equation with respect to s and using the Frenet formulae of spacelike curve we obtain

$$\frac{d\vec{\alpha}^*}{ds} = \vec{T}^*\frac{ds^*}{ds} = \left(1 + \frac{dm_1}{ds} - \varepsilon m_2 k_1\right)\vec{T} + \left(m_1 k_1 + \frac{dm_2}{ds} + m_3 k_2\right)\vec{N} + \left(m_2 k_2 + \frac{dm_3}{ds}\right)\vec{B}.$$

Since $\vec{T} = -\vec{T}^*$ at the corresponding points of (C), we have

$$\begin{cases} 1 + \frac{dm_1}{ds} - \varepsilon m_2 k_1 = -\frac{ds^*}{ds}, \\ m_1 k_1 + \frac{dm_2}{ds} + m_3 k_2 = 0, \\ m_2 k_2 + \frac{dm_3}{ds} = 0. \end{cases} \tag{2}$$

It is well known that the curvature of (C) is $\lim(\Delta\varphi/\Delta s) = (d\varphi/ds) = k_1(s)$. Here $\varphi = \int_0^s k_1(s)ds$ is the angle between the tangent of the curve (C) and a given fixed direction at the point $\vec{\alpha}(s)$. The distance d between the opposite points α and α^* of the curve is the breadth of the curve and is constant, that is, $d^2 = \left\|\vec{d}\right\|^2 = \|\alpha^* - \alpha\|^2 = m_1^2 + \varepsilon m_2^2 - \varepsilon m_3^2 = \text{const}$. Also, the vector $\vec{d}$ is the double normal of the constant breadth curve (C). Hence (2) may be written as follows:

$$m_2 = \frac{1}{\varepsilon} f(\varphi), \;\; m_2' = -m_3\rho k_2, \;\; m_3' = -m_2\rho k_2, \;\; m_1 = 0, \tag{3}$$

or

$$m_1' = \varepsilon m_2, \;\; m_2' = -m_1 - m_3\rho k_2, \;\; m_3' = -m_2\rho k_2, \tag{4}$$

which are the systems describing the spacelike curve (1) and here and in what follows $(')$ denotes the differentiation with respect to φ. In (3) and (4), $f(\varphi) = \rho + \rho^*$ and, $\rho = \frac{1}{k_1}$ and $\rho^* = \frac{1}{k_1^*}$ denote the radius of curvatures at the points α and α^*, respectively.

Let us consider the system (3) with $m_1 = 0$. Here, eliminating first $f(\varphi)$, m_2 and their derivatives, and then $f(\varphi)$, m_3 and their derivatives, respectively, we obtain the following linear differential equations of second order

$$\begin{cases} (\rho k_2)\, m_2'' - (\rho k_2)'\, m_2' - (\rho k_2)^3\, m_2 = 0, \;\; \rho k_2 \neq 0, \\ (\rho k_2)\, m_3'' - (\rho k_2)'\, m_3' - (\rho k_2)^3\, m_3 = 0, \;\; \rho k_2 \neq 0. \end{cases} \tag{5}$$

By changing the variable φ of the form $\xi = \int_0^\varphi \rho(t)\, k_2(t)\, dt$, these equations can be transformed into the following differential equations with constant coefficients,

$$\frac{d^2 m_2}{d\xi^2} - m_2 = 0 \;\text{ and }\; \frac{d^2 m_3}{d\xi^2} - m_3 = 0 \tag{6}$$

(See [5]). Then, the general solutions of the differential equations (6), become, respectively,

$$\begin{cases} m_2 = A\cosh \int_0^\varphi \rho k_2 dt + B\sinh \int_0^\varphi \rho k_2 dt, \\ m_3 = C\cosh \int_0^\varphi \rho k_2 dt + D\sinh \int_0^\varphi \rho k_2 dt, \end{cases} \tag{7}$$

where A, B, C and D are constants. Substituting (7) into (3), we obtain $A = -B$, $C = -D$, and so, the set of the solutions of the system (3), in the form

$$\begin{cases} m_1 = 0, \\ m_2 = A\cosh \displaystyle\int_0^\varphi \rho k_2 dt + B\sinh \int_0^\varphi \rho k_2 dt, \\ m_3 = -\left(B\cosh \displaystyle\int_0^\varphi \rho k_2 dt + A\sinh \int_0^\varphi \rho k_2 dt\right). \end{cases} \tag{8}$$

Thus, the equation (1) is described.

Now, let us return to the system (4). Eliminating $m_2,\ m_3$ and their derivatives, we have the linear differential equation

$$(\rho k_2)\, m_1''' + (\rho k_2)'\, m_1'' - \Big((\rho k_2)^3 + (\rho k_2)\Big)\, m_1' - (\rho k_2)'\, m_1 = 0, \quad \rho k_2 \neq 0. \tag{9}$$

On the other hand, replacing $\rho k_2 = h(\varphi)$ into (9), we obtain Bernoulli's equation with unknown h. Solving the obtained equation, we can reduce (9) to the nonlinear differential equation

$$(\rho k_2)^{-2}\,(m_1 + m_1'')^2 + m_1^2 + (m_1')^2 = C, \tag{10}$$

where C is constant. Following the same way, for m_2 and m_3, we have the following differential equations:

(11)

$$\begin{aligned}&(\rho\tau)^2\, m_3''' - 2\,(\rho k_2)\,(\rho k_2)'\, m_3'' \\ &+ \left[\varepsilon\,(\rho k_2)^2 - (\rho k_2)^4 - (\rho k_2)\,(\rho k_2)'' + 2\left((\rho k_2)'\right)^2\right] m_3' - (\rho k_2)^3\,(\rho k_2)'\, m_3 = 0\end{aligned}$$

and

$$\begin{aligned}(\rho k_2)'\, m_2''' + (\rho k_2)''\, m_2'' + (\rho k_2)' \left[\varepsilon - (\rho k_2)^2\right] m_2' \\ + \left[3\,(\rho k_2)\,(\rho k_2)' + (\rho k_2)'' \left[\varepsilon - (\rho k_2)^2\right]\right] m_2 = 0.\end{aligned} \tag{12}$$

The statement (ρk_2) in these equations may be decided by means of the criterion in the next section. Hence, replacing the general solution of each one of equations (9), (11) and (12) into (4), separately, and then adjusting the arbitrary constants, we may find the solution set $\{m_1, m_2, m_3\}$.

The case $\rho k_2 = 0$ will be given in Example 3.1.

3.1. A criterion for spacelike curves of constant breadth in Minkowski 3-space E_1^3

Let us assume that (C) is a spacelike curve of constant breadth and $\vec{\alpha}(s)$ denotes the position vector of a generic point. Since (C) is a closed curve, the position vector $\vec{\alpha}(s)$ must be a periodic function of period $\omega = 2\pi$, where ω is the total length of (C). Then the curvature $k_1(s)$ and torsion $k_2(s)$ are also periodic of the same period. However, periodicity of $k_1(s)$ and $k_2(s)$ and closeness of the curve are not sufficient to guarantee a spacelike space curve to be constant breadth. That is, if a spacelike curve is closed curve (periodic), it may be the curve of constant breadth or not. Therefore, to guarantee that a spacelike curve is a constant breadth curve, we may use the system (4) characterizing a spacelike curve of constant breadth and follow the similar way given in [6].

For this purpose, first let us consider the following Frenet formulas at a generic point on the spacelike curve (C),

$$\frac{d\vec{T}}{ds} = k_1 \vec{N} \quad \frac{d\vec{N}}{ds} = -\varepsilon k_1 \vec{T} + k_2 \vec{B}, \quad \frac{d\vec{B}}{ds} = k_2 \vec{N}. \tag{13}$$

Writing the formulas (13) in terms of φ and allowing for $\frac{d\varphi}{ds} = k_1 = \frac{1}{\rho}$ we have

$$\frac{d\overrightarrow{T}}{d\varphi} = \vec{N}, \quad \frac{d\vec{N}}{d\varphi} = -\varepsilon\overrightarrow{T} + \rho k_2\vec{B}, \quad \frac{d\vec{B}}{d\varphi} = \rho k_2\vec{N}. \tag{14}$$

Furthermore we can write the Frenet vectors $\vec{T}$, $\vec{N}$, $\vec{B}$ in coordinate form as follows

$$\overrightarrow{T} = \sum_{i=1}^{3} t_i\vec{e}_i, \quad \overrightarrow{N} = \sum_{i=1}^{3} n_i\vec{e}_i, \quad \overrightarrow{B} = \sum_{i=1}^{3} b_i\vec{e}_i. \tag{15}$$

Since $\{\vec{T}, \vec{N}, \vec{B}\}$ is the orthonormal base in E_1^3, and putting (15) and their derivatives into (14), we have the systems of linear differential equations

$$\begin{cases} \frac{dt_1}{d\varphi} = n_1, & \frac{dt_2}{d\varphi} = n_2, & \frac{dt_3}{d\varphi} = n_3 \\ \frac{dn_1}{d\varphi} = -\varepsilon t_1 + \rho k_2 b_1, & \frac{dn_2}{d\varphi} = -\varepsilon t_2 + \rho k_2 b_2, & \frac{dn_3}{d\varphi} = -\varepsilon t_3 + \rho k_2 b_3 \\ \frac{db_1}{d\varphi} = \rho k_2 n_1, & \frac{db_2}{d\varphi} = \rho k_2 n_2, & \frac{db_3}{d\varphi} = \rho k_2 n_3. \end{cases} \tag{16}$$

From (16), we find that $\{t_1, n_1, b_1\}$, $\{t_2, n_2, b_2\}$, $\{t_3, n_3, b_3\}$ are three independent solutions of the following system differential equations:

$$\frac{d\psi_1}{d\varphi} = \psi_2, \quad \frac{d\psi_2}{d\varphi} = -\varepsilon\psi_1 + \rho k_2\psi_3, \quad \frac{d\psi_3}{d\varphi} = \rho k_2\psi_2. \tag{17}$$

If the spacelike curve (C) is the curve of constant breadth, the system (17) and (4) must be the same system. So, we observe that $\psi_1 = m_1$, $\psi_2 = m_2$, $\psi_3 = m_3$. For brevity, we can write (4) or (17) in the form

$$\frac{d\psi}{d\varphi} = A(\varphi)\psi, \tag{18}$$

where

$$\psi = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}, \quad A(\varphi) = \begin{bmatrix} 0 & 1 & 0 \\ -\varepsilon & 0 & \rho k_2 \\ 0 & \rho k_2 & 0 \end{bmatrix}.$$

Obviously, (18) is a special case of the general linear differential equations abbreviated to the form

$$\begin{cases} \frac{d\psi}{dt} = A(t)\psi, \\ \psi = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \ (3 \leq n) \end{cases} \tag{19}$$

where $a_{ij}(t)$ are assumed to be continuous and periodic of period ω (See [6, 15]). Let the initial conditions be $\psi_i(0) = x_i$, $(i = 1, 2, \ldots, n)$. Let us take $x = [x_1, x_2, \ldots, x_n]^T$ and

$$\psi(t, x) = [m_1(t, x)\, m_2(t, x) \cdots m_n(t, x)]^T.$$

Then the equation (19) may be written in the form $\frac{d\psi}{dt} = A(t)\psi$, $\psi(0) = x$ as is well known from [6], the solution $\psi(t,x)$ of this equation is periodic of period ω, if

$$\int_0^\omega A(\xi)\,\psi(\xi,x)\,d\xi = 0$$

and

$$(20)\qquad \begin{cases} \psi(t,x) & = \{E + M(t)\}\,x, \quad (E = \text{unit matrix}), \\ M(t) & = IA(t) + I^{(2)}A(t) + \cdots + I^{(n)}A(t) + \cdots, \\ (IA)(t) & \equiv I^{(I)}A(t) = \int_0^t A(\xi)\,d\xi, \\ \left(I^{(n)}A\right)(t) = \int_0^t A(\xi)\left(I^{(n-1)}A\right)(\xi)\,d\xi, & n > 1. \end{cases}$$

Furthermore, the following theorem is given in [6]:

Theorem 3.1. *The equations* $\frac{d\psi}{dt} = A(t)\psi$ *possess a non-vanishing periodic solution of period* ω*, if and only if* $\det(M(\omega)) = 0$*. In particular, in order that the equations* $\frac{d\psi}{dt} = A(t)\psi$ *possess* n *linearly independent periodic solutions of period* ω*, the necessary and sufficient condition is that* $M(\omega)$ *be a zero matrix.*

Now, let us apply this theorem to the system (18). If $M(\omega) = 0$, there exist the unit vector functions $\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}$ of period ω, such that each set of functions $\{t_i, n_i, b_i\}$, $(i = 1,2,3)$ form a solution of the equation (18) corresponding to the initial conditions (A_i, B_i, C_i). The spacelike curve (C) can be described

$$\vec{\alpha}(s) = \int_0^s \vec{T}(s)ds \quad \text{or} \quad \vec{\alpha}(\varphi) = \int_0^\varphi \rho(\varphi)\vec{T}(\varphi)d\varphi.$$

Here, to find $\vec{T}$, we can make use of the equation

$$(21)\qquad \begin{bmatrix} t_i \\ n_i \\ b_i \end{bmatrix} = \{E + M(\varphi)\} \begin{bmatrix} A_i \\ B_i \\ C_i \end{bmatrix}, \quad (i = 1,2,3)$$

which is established by (20). If we take the initial conditions as $t_i(0) = A_i$, $n_i(0) = B_i$, $b_i(0) = C_i$, $(i = 1,2,3)$ such that (A_1, A_2, A_3), (B_1, B_2, B_3), (C_1, C_2, C_3) form an orthonormal frame, then from (21) we obtain

$$(22)\qquad t_i = (1 + m_{11})A_i + m_{12}B_i + m_{13}C_i, \quad (i = 1,2,3).$$

When the spacelike curve (C) is a curve of constant breadth, which is also periodic of period ω, it is clear that

$$(23)\qquad \int_0^\omega \rho t_i d\varphi = 0.$$

Hence, form (22) and (23), we have

$$A_i \int_0^\omega \rho(1 + m_{11})\,d\varphi + B_i \int_0^\omega \rho m_{12} d\varphi + C_i \int_0^\omega \rho m_{13} d\varphi = 0; \quad (i = 1,2,3).$$

Since in this system the coefficient determinant $\Delta \neq 0$, we obtain the equalities

$$\int_0^\omega \rho\left(1+m_{11}\right) d\varphi = 0 = \int_0^\omega \rho m_{12} d\varphi = \int_0^\omega \rho m_{13} d\varphi, \tag{24}$$

which are the conditions for a spacelike curve to be constant breadth. Here, we can take the period $\omega = 2\pi$ because of $0 \leq \varphi \leq 2\pi$. Thus we establish the following result:

Corollary 3.1. *Let (C) be a spacelike curve in E_1^3 such that $\rho(\varphi) > 0$ and $k_2(\varphi)$ are continuous periodic functions of period ω. Then (C) is a spacelike curve of constant breadth, and also periodic of period ω, if and only if*

$$M\left(\omega\right) = 0, \quad \int_0^\omega \rho\left(1+m_{11}\right) d\varphi = 0 = \int_0^\omega \rho m_{12} d\varphi = \int_0^\omega \rho m_{13} d\varphi, \tag{25}$$

where

$$\begin{cases} M\left(t\right) = IA\left(t\right) + I^{(2)}A\left(t\right) + \cdots + I^{(n)}A\left(t\right) + \cdots, \\ A\left(t\right) = \begin{bmatrix} 0 & 1 & 0 \\ -\varepsilon & 0 & \rho k_2 \\ 0 & \rho k_2 & 0 \end{bmatrix} \end{cases} \tag{26}$$

and $m_{ij}\left(t\right)$ are the entries of the matrix $M\left(t\right)$.

By means of (20) and (26), the matrix $M\left(t\right)$ can be constructed and each m_{ij} involves infinitely many integrations. Hence, we can write the conditions (25) in the following forms:

(27)

$$\begin{cases} \int_0^\omega \rho(\varphi) d\varphi - \int_0^\omega \int_0^\varphi \int_0^t \rho(\varphi) dp dt d\varphi \\ + \int_0^\omega \int_0^\varphi \int_0^t \int_0^p \int_0^q \rho(\varphi)\left[\varepsilon - h^2\left(q\right)\right] dr dq dp dt d\varphi - \cdots = 0, \\ \int_0^\omega \int_0^\varphi \rho(\varphi) dt d\varphi - \int_0^\omega \int_0^\varphi \int_0^t \int_0^p \rho(\varphi)\left[\varepsilon - h\left(p\right) h\left(q\right)\right] dq dp dt d\varphi + \cdots = 0, \\ \int_0^\omega \int_0^\varphi \int_0^t \rho(\varphi) h\left(t\right) dp dt d\varphi \\ - \int_0^\omega \int_0^\varphi \int_0^t \int_0^p \int_0^q \rho(\varphi)\left[\varepsilon - h^2\left(q\right)\right] h\left(t\right) dr dq dp dt d\varphi + \cdots = 0, \end{cases}$$

where $h\left(\xi\right) = \rho\left(\xi\right)\tau\left(\xi\right)$.

Example 3.1. Let us consider the special case $\rho =$ const. and $k_2 =$ const. where $\rho k_2 = h =$ const. In this case, from (24), we have

$$\begin{cases} \omega - \varepsilon\frac{\omega^3}{3!} + \varepsilon\left(\varepsilon - \rho^2 k_2^2\right)\frac{\omega^5}{5!} - \varepsilon\left(\varepsilon - \rho^2 k_2^2\right)^2\frac{\omega^7}{7!}\cdots = 0, \\ \frac{\omega^2}{2!} - \left(\varepsilon - \rho^2 k_2^2\right)\frac{\omega^4}{4!} + \left(\varepsilon - \rho^2 k_2^2\right)^2\frac{\omega^6}{6!} - \cdots = 0, \\ k_2\left[\frac{\omega^3}{3!} - \left(\varepsilon - \rho^2 k_2^2\right)\frac{\omega^5}{5!} + \left(\varepsilon - \rho^2 k_2^2\right)^2\frac{\omega^7}{7!} - \cdots = 0\right], \end{cases} \tag{28}$$

or

$$(29)\qquad \begin{cases} \rho^2 k_2^2 \left(\varepsilon - \rho^2 k_2^2\right)^{\frac{1}{2}} \omega - \sin \varepsilon \left(\varepsilon - \rho^2 k_2^2\right)^{\frac{1}{2}} \omega = 0, \\ \cos \left(\varepsilon - \rho^2 k_2^2\right)^{\frac{1}{2}} \omega = 1 \quad \text{or} \quad \left(\varepsilon - \rho^2 k_2^2\right)^{\frac{1}{2}} \omega = 2k\pi, \ \ k \in \mathbb{Z} \\ k_2 \left[\sin \left(\varepsilon - \rho^2 k_2^2\right)^{\frac{1}{2}} \omega - \left(\varepsilon - \rho^2 k_2^2\right)^{\frac{1}{2}} \omega\right] = 0, \end{cases}$$

where $\omega = 2k\pi$ if $\varepsilon = 1$, and $\omega = -2k\pi i$ if $\varepsilon = -1$.

It is seen that all of the equalities (28) or (29) are satisfied, simultaneously, if and only if $\rho k_2 = 0$, that is, $\rho = \text{const.} > 0$ and $k_2 = 0$. Therefore, only ones with $\rho = \text{const.} > 0$ and $k_2 = 0$ of the spacelike curves with $\rho = \text{const.} > 0$ and $k_2 = \text{const.}$ are curves of constant breadth, which are circles.

Now let us construct the relation characterizing these circles. Since $\rho k_2 = 0$, system (4) becomes

$$(30)\qquad m_1' = \varepsilon m_2, \ m_2' = -m_1 - \rho k_2 m_3, \ m_3' = -\rho k_2 m_2.$$

From (30), the equations with the unknowns m_1, m_2 and m_3 can be written as follows

$$(31)\qquad \varepsilon m_1 + m_1'' = 0, \ \varepsilon m_2 + m_2'' = 0, \ m_3' = 0.$$

Let now find the general solutions of the system (31) which has two general solutions according to the sign of ε.

If $\varepsilon = 1$, the general solution is

$$(32)\qquad \begin{cases} m_1 = A\cos\varphi + B\sin\varphi \\ m_2 = D\cos\varphi + E\sin\varphi \\ m_3 = C, \end{cases}$$

where A, B, C, D and E are arbitrary constants.

Replacing (32) into (30), we have$A = -E, \ B = D$, and thus

$$(33)\qquad \{m_1 = A\cos\varphi + B\sin\varphi, \ m_2 = B\cos\varphi - A\sin\varphi, \ m_3 = C\}$$

which is the solution set of the system (31) with $\varepsilon = 1$.

Consequently, replacing (33) into (1), we obtain the equation

$$\vec{\alpha}^*(\varphi) = \vec{\alpha}(\varphi) + (A\cos\varphi + B\sin\varphi)\,\vec{T} + (B\sin\varphi - E\cos\varphi)\,\vec{N} + C\vec{B},$$

which represents the circles with the diameter $d = \|\alpha^* - \alpha\| = \left(\left|D^2 + E^2 - C^2\right|\right)^{\frac{1}{2}}$. In this case, a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ in $0 \le \varphi \le 2\pi$.

If $\varepsilon = -1$, the general solution of the system (31) is

$$(34)\qquad \begin{cases} m_1 = A\cosh\varphi + B\sinh\varphi \\ m_2 = D\cosh\varphi + E\sinh\varphi \\ m_3 = C, \end{cases}$$

where A, B, C, D and E are arbitrary constants.

Replacing (34) into (30), we have$A = -E, \ B = -D$ and thus

$$(35)\qquad \{m_1 = A\cosh\varphi + B\sinh\varphi, \ m_2 = -B\cosh\varphi - A\sinh\varphi, \ m_3 = C\},$$

which is the solution set of the system (31) with $\varepsilon = -1$.

Consequently, replacing (35) into (1), we obtain the equation

$$\vec{\alpha}^*(\varphi) = \vec{\alpha}(\varphi) + (A\cosh\varphi + B\sinh\varphi)\,\vec{T} + (-A\cosh\varphi - B\sinh\varphi)\,\vec{N} + C\vec{B},$$

which represents the circles with the diameter $d = \|\alpha^* - \alpha\| = \left(\left|A^2 - B^2 + C^2\right|\right)^{\frac{1}{2}}$. In this case, a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ in $0 \leq \varphi \leq 2\pi$.

4. Differential equations characterizing timelike curves of constant breadth in E_1^3

In this section, we study the differential equations which characterize timelike curves of constant breadth in Minkowski 3-space E_1^3.

Let (C) be a unit speed timelike curve of the class C^3 with nonzero curvature and torsion and assume that (C) has parallel tangents $\vec{T}$ and $\vec{T}^*$ in opposite direction at the opposite points α and α^* of the curve. If the chord joining the opposite points of (C) is a double-normal, then (C) has constant breadth, and conversely, if (C) is a timelike curve of constant breadth, then every normal of (C) is a double-normal. A simple closed timelike curve (C) of constant breadth having parallel tangents in opposite directions at opposite points may be represented by the equation

$$\vec{\alpha}^*(s) = \vec{\alpha}(s) + m_1(s)\vec{T}(s) + m_2(s)\vec{N}(s) + m_3(s)\vec{B}(s), \tag{36}$$

where $m_i(s)$ $(1 \leq i \leq 3)$ are the differentiable functions of s which is arc length of (C). Differentiating this equation with respect to s and using the Frenet formulae of timelike curve we obtain

$$\begin{aligned}\frac{d\vec{\alpha}^*}{ds} = \vec{T}^*\frac{ds^*}{ds} = &\left(1 - \frac{dm_1}{ds} + m_2k_1\right)\vec{T} + \left(m_1k_1 + \frac{dm_2}{ds} - m_3k_2\right)\vec{N}\\ &+ \left(m_2k_2 + \frac{dm_3}{ds}\right)\vec{B}.\end{aligned}$$

Since $\vec{T} = -\vec{T}^*$ at the corresponding points of (C) we have

$$\begin{cases} 1 - \frac{dm_1}{ds} + m_2k_1 = -\frac{ds^*}{ds}, \\ m_1k_1 + \frac{dm_2}{ds} - m_3k_2 = 0, \\ m_2k_2 + \frac{dm_3}{ds} = 0. \end{cases} \tag{37}$$

By considering the curvature of (C) defined by $\lim(\Delta\varphi/\Delta s) = (d\varphi/ds) = k_1(s)$, we have $\varphi = \int_0^s k_1(s)ds$ which is the angle between the tangent of the curve (C) and a given fixed direction at the point $\alpha(s)$. The distance d between the opposite points α and α^* of the curve is the breadth of the curve and is constant, that is, $d^2 = \|\vec{d}\|^2 = \|\alpha^* - \alpha\|^2 = -m_1^2 + m_2^2 + m_3^2 = \text{const.}$ Furthermore, since the normal vector $\vec{d}$ is a spacelike vector, from last equality, we have $m_2^2+m_3^2>m_1^2$. Also, the vector $\vec{d}$ is the double normal of the constant breadth curve (C). Hence (37) may be written as follows:

$$m_2=-f(\varphi)\,,\;\; m_2'=m_3\rho k_2,\;\; m_3'=-m_2\rho k_2,\;\; m_1=0, \tag{38}$$

or

$$m_1'=-m_2,\;\; m_2'=-m_1+m_3\rho k_2,\;\; m_3'=-m_2\rho k_2, \tag{39}$$

which are the systems describing the curve (36) and here and in what follows $(')$ denotes the differentiation with respect to φ. In (35) and (36), $f(\varphi)=\rho+\rho^*$ and, $\rho=\frac{1}{k_1}$ and $\rho^*=\frac{1}{k_1^*}$ denote the radius of curvatures at the points α and α^*, respectively.

Let us consider the system (38) with $m_1=0$. Here, eliminating first $f(\varphi)$, m_2 and their derivatives, and then $f(\varphi)$, m_3 and their derivatives, respectively, we obtain the following linear differential equations of second order

$$\begin{cases}(\rho k_2)\,m_2''+(\rho k_2)^3\,m_2-(\rho k_2)'\,m_2'=0, & \rho k_2\neq 0,\\ (\rho k_2)\,m_3''+(\rho k_2)^3\,m_3-(\rho k_2)'\,m_3'=0, & \rho k_2\neq 0.\end{cases} \tag{40}$$

By changing the variable φ of the form $\xi=\int_0^\varphi\rho\,(t)\,\tau\,(t)\,dt$, these equations can be transformed into the following differential equations with constant coefficients,

$$\frac{d^2m_2}{d\xi^2}+m_2=0\ \text{ and }\ \frac{d^2m_3}{d\xi^2}+m_3=0 \tag{41}$$

(See [5]). Then, the general solutions of the differential equations (41), become, respectively,

$$\begin{cases}m_2=A\cos\int_0^\varphi\rho k_2dt+B\sin\int_0^\varphi\rho k_2dt,\\ m_3=C\cos\int_0^\varphi\rho k_2dt+D\sin\int_0^\varphi\rho k_2dt,\end{cases} \tag{42}$$

where A,B,C and D are constants. Substituting (42) into (38), we obtain $A=-D,\;\; B=C$ and so, the set of the solutions of the system (38), in the form

$$\begin{cases}m_1=0,\\ m_2=A\cosh\displaystyle\int_0^\varphi\rho k_2dt+B\sinh\int_0^\varphi\rho k_2dt,\\ m_3=\left(B\cosh\displaystyle\int_0^\varphi\rho k_2dt-A\sinh\int_0^\varphi\rho k_2dt\right).\end{cases} \tag{43}$$

Thus, the equation (36) is described.

Now, let us return to the system (42). Eliminating $m_2,\; m_3$ and their derivatives, we have the linear differential equation

$$(\rho k_2)\,m_1'''-(\rho k_2)'\,m_1''+\left((\rho k_2)^3-(\rho k_2)\right)m_1'+(\rho k_2)'\,m_1=0;\quad \rho k_2\neq 0. \tag{44}$$

On the other hand, replacing $\rho k_2 = h(\varphi)$ into (44), we obtain Bernoulli's equation with unknown h. Solving the obtained equation, we can reduce (44) to the nonlinear differential equation

$$(\rho k_2)^{-2}(m_1'' - m_1)^2 + m_1^2 - (m_1')^2 = C, \tag{45}$$

where C is constant. Following the same way for m_2 and m_3, we have the following differential equations:

(46)

$$\begin{aligned}&(\rho k_2)^2 m_3''' - 2(\rho k_2)(\rho k_2)' m_3'' \\ &\quad + \left[(\rho k_2)^2 + (\rho k_2)^4 - (\rho k_2)(\rho k_2)'' + 2\left((\rho k_2)'\right)^2\right] m_3' + (\rho k_2)^3 (\rho k_2)' m_3 = 0\end{aligned}$$

and

$$\begin{aligned}&(\rho k_2)' m_2''' - (\rho k_2)'' m_2'' + (\rho k_2)'\left[(\rho k_2)^2 - 1\right] m_2' \\ &\qquad + \left[3(\rho k_2)\left((\rho k_2)'\right)^2 + (\rho k_2)''\left[1 + (\rho k_2)^2\right]\right] m_2 = 0.\end{aligned} \tag{47}$$

The statement (ρk_2) in these equations may be decided by means of the criterion in the next section. Hence, replacing the general solution of each one of equations (44), (46) and (47) into (39), separately, and then adjusting the arbitrary constants, we may find the solution set $\{m_1, m_2, m_3\}$.

4.1. A criterion for timelike curves of constant breadth in Minkowski 3-space

Let us assume that (C) is a timelike curve of constant breadth and $\vec{\alpha}(s)$ denotes the position vector of a generic point. Since (C) is a closed curve, the position vector $\vec{\alpha}(s)$ must be a periodic function of period $\omega = 2\pi$, where ω is the total length of (C). Then the curvature $k_1(s)$ and torsion $k_2(s)$ are also periodic of the same period. However, similar to the case given for spacelike curves, periodicity of $k_1(s)$ and $k_2(s)$ and closeness of the curve are not sufficient to guarantee a timelike space curve to be constant breadth. Therefore, to guarantee a timelike curve to be constant breadth, we may use the system (39) characterizing a timelike curve of constant breadth and follow the similar way as [6].

For this purpose, first let us consider the Frenet formulas of timelike curve given by

$$\frac{d\vec{T}}{ds} = k_1\vec{N} \quad \frac{d\vec{N}}{ds} = k_1\vec{T} + k_2\vec{B}, \quad \frac{d\vec{B}}{ds} = -k_2\vec{N}, \tag{48}$$

at a generic point on the timelike curve (C). Writing the formulas (48) in terms of φ and taking $\frac{d\varphi}{ds} = k_1 = \frac{1}{\rho}$, we have

$$\frac{d\vec{T}}{d\varphi} = \vec{N}, \quad \frac{d\vec{N}}{d\varphi} = \vec{T} + \rho k_2\vec{B}, \quad \frac{d\vec{B}}{d\varphi} = -\rho k_2\vec{N}. \tag{49}$$

Let write $\vec{T}$, $\vec{N}$, $\vec{B}$ in coordinate forms as follows,

$$\overrightarrow{T} = \sum_{i=1}^{3} t_i \vec{e}_i, \quad \overrightarrow{N} = \sum_{i=1}^{3} n_i \vec{e}_i, \quad \overrightarrow{B} = \sum_{i=1}^{3} b_i \vec{e}_i. \tag{50}$$

Since $\{\vec{T}, \vec{N}, \vec{B}\}$ is the orthonormal base in E_1^3, putting (50) and their derivatives into (49), we have the systems of linear differential equations

$$\begin{cases} \frac{dt_1}{d\varphi} = n_1, & \frac{dt_2}{d\varphi} = n_2, & \frac{dt_3}{d\varphi} = n_3 \\ \frac{dn_1}{d\varphi} = t_1 + \rho k_2 b_1, & \frac{dn_2}{d\varphi} = t_2 + \rho k_2 b_2, & \frac{dn_3}{d\varphi} = t_3 + \rho k_2 b_3 \\ \frac{db_1}{d\varphi} = -\rho k_2 n_1, & \frac{db_2}{d\varphi} = -\rho k_2 n_2, & \frac{db_3}{d\varphi} = -\rho k_2 n_3. \end{cases} \tag{51}$$

From (51), we find that $\{t_1, n_1, b_1\}$, $\{t_2, n_2, b_2\}$, $\{t_3, n_3, b_3\}$ are three independent solutions of the following system of differential equations:

$$\frac{d\psi_1}{d\varphi} = \psi_2, \quad \frac{d\psi_2}{d\varphi} = \psi_1 + \rho k_2 \psi_3, \quad \frac{d\psi_3}{d\varphi} = -\rho k_2 \psi_2. \tag{52}$$

If the timelike curve (C) is the curve of constant breadth, the system (52) and (39) must be the same system; so we observe that $\psi_1 = m_1$, $\psi_2 = m_2$, $\psi_3 = m_3$. For brevity, we can write (39) or (52) in the form

$$\frac{d\psi}{d\varphi} = A(\varphi)\,\psi, \tag{53}$$

where

$$\psi = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}, \quad A(\varphi) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \rho k_2 \\ 0 & -\rho k_2 & 0 \end{bmatrix}.$$

Obviously, (53) is a special case of the general linear differential equations abbreviated to the form

$$\begin{cases} \frac{d\psi}{dt} = A(t)\,\psi, \\ \psi = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \ (3 \le n) \end{cases} \tag{54}$$

where $a_{ij}(t)$ are assumed to be continuous and periodic of period ω (See [6, 15]). Let the initial conditions be $\psi_i(0) = x_i$, $(i = 1, 2, \ldots, n)$. Let us take $x = [x_1, x_2, \ldots, x_n]^T$ and

$$\psi(t, x) = [m_1(t, x)\, m_2(t, x) \cdots m_n(t, x)]^T.$$

Then the equation (54) may be written in the form $\frac{d\psi}{dt} = A(t)\,\psi$, $\psi(0) = x$ as is well known from [6], the solution $\psi(t, x)$ of this equation is periodic of period ω, if

$$\int_0^{\omega} A(\xi)\,\psi(\xi, x)\, d\xi = 0$$

and

$$\left\{\begin{array}{ll} \psi(t,x) & = \{E+M(t)\}\,x, \quad (E=\text{unit matrix}), \\ M(t) & = IA(t)+I^{(2)}A(t)+\cdots+I^{(n)}A(t)+\cdots, \\ (IA)(t) & \equiv I^{(I)}A(t)=\int_0^t A(\xi)\,d\xi, \\ \left(I^{(n)}A\right)(t) & =\int_0^t A(\xi)\left(I^{(n-1)}A\right)(\xi)\,d\xi, \quad n>1. \end{array}\right. \tag{55}$$

Furthermore, from [6] we have Theorem 3.1 again.

Now let us apply Theorem 3.1 to the system (53). If $M(\omega)=0$, there exist the unit vector functions $\overrightarrow{T},\overrightarrow{N},\overrightarrow{B}$ of period ω, such that each set of functions $\{t_i,n_i,b_i\}$, $(i=1,2,3)$ form a solution of the equation (53) corresponding to the initial conditions (A_i,B_i,C_i). The timelike curve (C) can be described

$$\vec{\alpha}(s)=\int_0^s \vec{T}(s)ds \quad \text{or} \quad \vec{\alpha}(\varphi)=\int_0^\varphi \rho(\varphi)\vec{T}(\varphi)d\varphi.$$

Here, to find $\vec{T}$, we can make use of the equation

$$\begin{bmatrix} t_i \\ n_i \\ b_i \end{bmatrix}=\{E+M(\varphi)\}\begin{bmatrix} A_i \\ B_i \\ C_i \end{bmatrix}, \quad (i=1,2,3) \tag{56}$$

which is established by (55). If we take the initial conditions as $t_i(0)=A_i$, $n_i(0)=B_i$, $b_i(0)=C_i$; $(i=1,2,3)$ such that $(A_1,A_2,A_3),(B_1,B_2,B_3)$, (C_1,C_2,C_3) form an orthonormal frame, then from (56) we obtain

$$t_i=(1+m_{11})A_i+m_{12}B_i+m_{13}C_i, \quad (i=1,2,3). \tag{57}$$

When the timelike curve (C) is a curve of constant breadth, which is also periodic of period ω, it is clear that

$$\int_0^\omega \rho t_i d\varphi=0. \tag{58}$$

Hence, form (57) and (58), we have

$$A_i\int_0^\omega \rho(1+m_{11})\,d\varphi+B_i\int_0^\omega \rho m_{12}d\varphi+C_i\int_0^\omega \rho m_{13}d\varphi=0; \quad (i=1,2,3).$$

Since in this system the coefficient determinant $\Delta\neq 0$, we obtain the equalities

$$\int_0^\omega \rho(1+m_{11})\,d\varphi=0=\int_0^\omega \rho m_{12}d\varphi=\int_0^\omega \rho m_{13}d\varphi, \tag{59}$$

which are the conditions for a timelike curve to be constant breadth. Here, we can take the period $\omega=2\pi$ because of $0\leq\varphi\leq 2\pi$. Thus we establish the following result:

Corollary 4.1. *Let (C) be a timelike curve in E_1^3, such that $\rho(\varphi)>0$ and $k_2(\varphi)$ are continuous periodic functions of period ω. Then (C) is a timelike curve of constant breadth, and also periodic of period ω, if and only if*

$$M(\omega)=0, \quad \int_0^\omega \rho(1+m_{11})\,d\varphi=0=\int_0^\omega \rho m_{12}d\varphi=\int_0^\omega \rho m_{13}d\varphi, \tag{60}$$

where

$$
(61)\qquad \begin{cases} M(t) = IA(t) + I^{(2)}A(t) + \cdots + I^{(n)}A(t) + \cdots, \\ A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \rho k_2 \\ 0 & -\rho k_2 & 0 \end{bmatrix} \end{cases}
$$

and $m_{ij}(t)$ *are the entries of the matrix* $M(t)$.

By means of (55) and (61), the matrix $M(t)$ can be constructed and each m_{ij} involves infinitely many integrations. Hence we can write the conditions (60) in the following forms:

(62)

$$
\begin{cases} \int_0^\omega \rho(\varphi)d\varphi + \int_0^\omega \int_0^\varphi \int_0^t \rho(\varphi)dpdtd\varphi \\ + \int_0^\omega \int_0^\varphi \int_0^t \int_0^p \int_0^q \rho(\varphi)\left[1 - h^2(q)\right] drdqdpdtd\varphi - \cdots = 0, \\ \int_0^\omega \int_0^\varphi \rho(\varphi)dtd\varphi + \int_0^\omega \int_0^\varphi \int_0^t \int_0^p \rho(\varphi)\left[1 - h(p)h(q)\right]dqdpdtd\varphi + \cdots = 0, \\ \int_0^\omega \int_0^\varphi \int_0^t \rho(\varphi)h(t)\,dpdtd\varphi \\ + \int_0^\omega \int_0^\varphi \int_0^t \int_0^p \int_0^q \rho(\varphi)\left(1 - h^2(q)\right)h(t)\,drdqdpdtd\varphi + \cdots = 0, \end{cases}
$$

where $h(\xi) = \rho(\xi)\,k_2(\xi)$.

Example 4.1. Let us consider the case $\rho =$ const. and $k_2 =$ const., where $\rho k_2 = h =$ const. In this case, from (59), we have

$$
(63)\qquad \begin{cases} \omega + \frac{\omega^3}{3!} + \left(1 - \rho^2 k_2^2\right)\frac{\omega^5}{5!} + \left(1 - \rho^2 k_2^2\right)^2 \frac{\omega^7}{7!} \ldots = 0 \\ \frac{\omega^2}{2!} + \left(1 - \rho^2 k_2^2\right)\frac{\omega^4}{4!} + \left(1 - \rho^2 k_2^2\right)^2 \frac{\omega^6}{6!} + \ldots = 0 \\ k_2\left[\frac{\omega^3}{3!} + \left(1 - \rho^2 k_2^2\right)\frac{\omega^5}{5!} + \left(1 - \rho^2 k_2^2\right)^2 \frac{\omega^7}{7!} - \ldots = 0\right] \end{cases}
$$

or

$$
(64)\qquad \begin{cases} \rho^2 k_2^2 \left(1 - \rho^2 k_2^2\right)^{\frac{1}{2}}\omega - \sinh\left(1 - \rho^2 k_2^2\right)^{\frac{1}{2}}\omega = 0, \\ \cosh\left(1 - \rho^2 k_2^2\right)^{\frac{1}{2}}\omega = 1 \text{ or } \left(1 - \rho^2 k_2^2\right)^{\frac{1}{2}}\omega = 0, \\ k_2\left[\sinh\left(1 - \rho^2 k_2^2\right)^{\frac{1}{2}}\omega - \left(1 - \rho^2 k_2^2\right)^{\frac{1}{2}}\omega\right] = 0, \end{cases}
$$

where $k_2 \neq 0$, $\omega \neq 0$ and $\rho = \pm\frac{1}{k_2}$.

It is seen that all of the equalities (63) or (64) are satisfied, simultaneously, if and only if $k_2 \neq 0$, $\omega \neq 0$ and $\rho = \pm\frac{1}{k_2}$. Therefore, only ones with $\rho = \pm\frac{1}{k_2} =$ const. of the timelike curves are curves of constant breadth, which are timelike helices with the property that $k_1 = \pm k_2 =$ const.

Now let us construct the relation characterizing these helices.

When $\rho = \frac{1}{k_2} = $ const., the system (39) becomes

$$m_1' = -m_2, \quad m_2' = -m_1 + m_3, \quad m_3' = -m_2, \tag{65}$$

The general solutions of them are, respectively,

$$\begin{cases} m_1 = \frac{c_1}{2}\varphi^2 + c_2\varphi + c_3 \\ m_2 = -c_1\varphi - c_2 \\ m_3 = \frac{c_1}{2}\varphi^2 + c_2\varphi + (c_3 - c_1), \end{cases} \tag{66}$$

where $c_1,\ c_2,\ c_3$ are arbitrary constants.

Consequently, replacing (66) into (36), we obtain the equation

$$\begin{aligned} \vec{\alpha}^*(\varphi) = \vec{\alpha}(\varphi) &+ \left(\frac{c_1}{2}\varphi^2 + c_2\varphi + c_3\right)\vec{T} \\ &+ (-c_1\varphi - c_2)\,\vec{N} + \left(\frac{c_1}{2}\varphi^2 + c_2\varphi + (c_3 - c_1)\right)\vec{B}, \end{aligned}$$

which gives the constant distance $d = \left(\left|c_1^2 + c_2^2 - 2c_1c_3\right|\right)^{\frac{1}{2}}$ between the points $\alpha^*(\varphi)$ and $\alpha(\varphi)$.

When $\rho = -\frac{1}{k_2} = $ const., the system (39) becomes

$$m_1' = -m_2, \quad m_2' = -m_1 - m_3, \quad m_3' = m_2. \tag{67}$$

The general solutions of them are

$$\begin{cases} m_1 = -\frac{c_1}{2}\varphi^2 - c_2\varphi \\ m_2 = c_1\varphi + c_2 \\ m_3 = \frac{c_1}{2}\varphi^2 + c_2\varphi - c_1 \end{cases} \tag{68}$$

respectively, where $c_1,\ c_2,\ c_3$ are arbitrary constants.

Consequently, replacing (68) into (36), we obtain the equation

$$\vec{\alpha}^*(\varphi) = \vec{\alpha}(\varphi) + \left(-\frac{c_1}{2}\varphi^2 - c_2\varphi\right)\vec{T} + (c_1\varphi + c_2)\,\vec{N} + \left(\frac{c_1}{2}\varphi^2 + c_2\varphi - c_1\right)\vec{B},$$

which gives the constant distance $d = \left(c_1^2 + c_2^2\right)^{\frac{1}{2}}$ between the points $\alpha^*(\varphi)$ and $\alpha(\varphi)$.

5. Conclusion

In this paper, the differential equations characterizing the spacelike and timelike curves of constant breadth in Minkowski 3-space E_1^3 are obtained. Furthermore, a criterion for a spacelike or a timelike curve to be the curve of constant breadth in E_1^3 is given. As an example, the obtained results are applied to the case $\rho = $ const. and $k_2 = $ const., and are discussed.

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