J. Korean Math. Soc. **48** (2011), No. 4, pp. 823–835
DOI 10.4134/JKMS.2011.48.4.823

CONTINUITY OF (α,β)-DERIVATIONS OF OPERATOR ALGEBRAS

Chengjun Hou and Qing Meng

Abstract. We investigate the continuity of (α,β)-derivations on $B(\mathfrak{X})$ or C^*-algebras. We give some sufficient conditions on which (α,β)-derivations on $B(\mathfrak{X})$ are continuous and show that each (α,β)-derivation from a unital C^*-algebra into its a Banach module is continuous when α and β are continuous at zero. As an application, we also study the ultraweak continuity of (α,β)-derivations on von Neumann algebras.

1. Introduction

Let $\mathcal{B}$ be a complex algebra with a subalgebra $\mathcal{A}$, let α and β be two mappings from $\mathcal{A}$ into $\mathcal{B}$, let $\mathcal{M}$ be a $\mathcal{B}$-module and hence an $\mathcal{A}$-module. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called an (α,β)-derivation, if $\delta(AB)=\delta(A)\alpha(B)+\beta(A)\delta(B)$ holds for all A, $B\in\mathcal{A}$; moreover, δ is called inner, if there exists $M_0\in\mathcal{M}$ such that $\delta(A)=M_0\alpha(A)-\beta(A)M_0$ for each $A\in\mathcal{A}$. An (α,α)-derivation is called briefly an α-derivation. Clearly, an id-derivation is an ordinary linear derivation, where id denotes the embedding map from $\mathcal{A}$ into $\mathcal{B}$, and every endomorphism α on $\mathcal{A}$ is an $\frac{\alpha}{2}$-derivation on $\mathcal{A}$. Note that in our definition of an (α,β)-derivation, no extra assumptions on α and β, such as linearity, are required. The purpose of this note is to investigate the continuity of (α,β)-derivations from Banach algebras into their Banach modules.

In 1958, Kaplansky conjectured that every derivation on a C^*-algebra or a semisimple Banach algebra is continuous ([7, 8]). Sakai confirmed Kaplansky's conjecture for the C^*-algebra case in [14], and from this, Kadison deduced the ultraweak continuity of derivations when the C^*-algebras are represented on Hilbert spaces ([5]). In [13], Ringrose generalized these results to the derivations from C^*-algebras into their Banach modules. The conjecture on the continuity of derivations on semisimple Banach algebras by Kaplansky was confirmed by Johnson and Sinclair in [4]. For the detail on automatic continuity of derivations of Banach algebras, we refer to [3, 15].

Received April 15, 2010; Revised July 9, 2010.
2010 *Mathematics Subject Classification.* 46H40, 46L57.
Key words and phrases. automatic continuity, (α,β)-derivation, Banach algebra, C^*-algebra.

On the continuity of (α, β)-derivations on C^*-algebras, M. Mirzavazibri and S. Moslehian proved that each $*$-(α, β)-derivation from a C^*-algebra $\mathcal{A}$ acting on a Hilbert space $\mathfrak{H}$ into $B(\mathfrak{H})$ is continuous under the assumption that α and β are $*$-linear continuous mappings from $\mathcal{A}$ into $B(\mathfrak{H})$ ([10, 11, 12]).

For an (α, β)-derivation δ from a Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module $\mathcal{M}$, if both α and β are bounded algebraic homomorphisms from $\mathcal{A}$ into itself, then $\mathcal{M}$, equipped with the $\mathcal{A}$-module actions defined by $A \cdot M = \beta(A)M$, $M \cdot A = M\alpha(A)$, is also a Banach $\mathcal{A}$-module, denoted by $\mathcal{M}_{\alpha,\beta}$, and δ is indeed an ordinary derivation from $\mathcal{A}$ into $\mathcal{M}_{\alpha,\beta}$. Hence it is interesting to study the continuity of an (α, β)-derivation when at least one of α and β is not an algebraic homomorphism.

This note is organized as follows. In Section 2 we give two examples of continuous (α, β)-derivations for two nonlinear and non-continuous mappings α and β. In Section 3 we give some sufficient conditions on which an (α, β)-derivation on $B(\mathfrak{X})$, the algebra of all bounded linear operators on a complex Banach space $\mathfrak{X}$, is continuous. In particular, we show that if $\mathfrak{X}$ is simple and α, β are surjective and continuous at zero, then each (α, β)-derivation on $B(\mathfrak{X})$ is continuous. In Section 4, using a similar argument to the proof in [13], we show that every (α, β)-derivation of a unital C^*-algebra into its a Banach module is continuous if α, β are continuous at zero, which generalizes the main results in [10]. As corollaries, we also get the ultraweak continuity of (α, β)-derivations of von Neumann algebras when the ultraweak continuity and linearity on α and β are required.

For a complex Banach space $\mathfrak{X}$, we denote by $\mathfrak{X}^*$, $B(\mathfrak{X})$ and $K(\mathfrak{X})$, the Banach dual space of $\mathfrak{X}$, the algebra of all bounded linear operators on $\mathfrak{X}$ and the ideal of all compact operators in $B(\mathfrak{X})$, respectively. For nonzero vectors $\xi \in \mathfrak{X}$ and $f \in \mathfrak{X}^*$, we denote by $\xi \otimes f$ the rank operator defined by $(\xi \otimes f)(\eta) = f(\eta)\xi$ for each $\eta \in \mathfrak{X}$. Sometimes we write $\langle \eta, f \rangle$ in place of $f(\eta)$. Let $F_1(\mathfrak{X})$ denote the set of all rank one operators on $\mathfrak{X}$. Obviously, $A(\xi \otimes f)B = (A\xi) \otimes (B'f)$ for all $A, B \in B(\mathfrak{X})$, $\xi \in \mathfrak{X}$, $f \in \mathfrak{X}^*$, where B' denotes the transpose of the bounded linear operator B, defined by $\langle \xi, B'f \rangle = \langle B\xi, f \rangle$ for each $\xi \in \mathfrak{X}$ and $f \in \mathfrak{X}^*$.

2. Reduction and examples

Let $\mathcal{A}$ be a complex Banach algebra without identity. We take the direct sum $\mathcal{A} \oplus \mathbb{C}$ as a linear space $\mathcal{A}_I$. By a well-known fact, $\mathcal{A}_I$, endowed with a Banach algebra structure, is a unital Banach algebra. In addition, if $\mathcal{A}$ is a C^*-algebra, then there exists a (unique) norm on $\mathcal{A}_I$ which makes $\mathcal{A}_I$ be a unital C^*-algebra. If we identify each $A \in \mathcal{A}$ with $(A, 0) \in \mathcal{A}_I$, then $\mathcal{A}$ is a closed two-sided ideal of $\mathcal{A}_I$ ([6]). We write (A, λ) as $A + \lambda I$ for each $(A, \lambda) \in \mathcal{A}_I$. If $\mathcal{M}$ is a Banach $\mathcal{A}$-module, then it is a unital Banach $\mathcal{A}_I$-module under the module action given by $(A + \lambda)M = AM + \lambda M$ and $M(A + \lambda) = MA + \lambda M$ for every $A + \lambda \in \mathcal{A}_I$ and $M \in \mathcal{M}$.

For a given mapping $\sigma : \mathcal{A} \to \mathcal{A}$, we can obtain its extension σ_I to $\mathcal{A}_I$ by $\sigma_I(A+\lambda) = \sigma(A) + \lambda$ for each $A + \lambda \in \mathcal{A}_I$. Then σ_I is linear if and only if so is σ, σ_I preserves the identity of $\mathcal{A}_I$ if and only if $\sigma(0) = 0$, and when $\mathcal{A}$ is a C^*-algebra, σ_I is a *-mapping if and only if so is σ. In both Banach algebra and C^*-algebra cases, σ_I is continuous at 0 if and only if so is σ.

For an (α,β)-derivation δ from $\mathcal{A}$ into $\mathcal{M}$, since δ is linear, we have $\delta(0) = 0$. Using the equation $0 = \delta(0) = \delta(A0) = \delta(0A)$, we have $\delta(A)\alpha(0) = 0$ and $\beta(0)\delta(A) = 0$ for each $A \in \mathcal{A}$. Hence if let $\alpha_0(A) = \alpha(A) - \alpha(0)$ and $\beta_0(A) = \beta(A) - \beta(0)$ for each $A \in \mathcal{A}$, then δ is a (α_0,β_0)-derivation. So, for an (α,β)-derivation δ, we sometimes can assume that $\alpha(0) = 0$ and $\beta(0) = 0$. Define the mapping δ_I from $\mathcal{A}_I$ into $\mathcal{M}$ by $\delta_I(A,\lambda) = \delta(A)$ for each $(A,\lambda) \in \mathcal{A}_I$. Then δ_I is an (α_I,β_I)-derivation from $\mathcal{A}_I$ into $\mathcal{M}$, and $\delta_I(0,1) = \delta(0) = 0$, $\alpha_I(0,1) = (0,1)$ and $\beta_I(0,1) = (0,1)$ by the assumptions that $\alpha(0) = 0$ and $\beta(0) = 0$. Obviously, δ is bounded if and only if so is δ_I.

Hence, to obtain the continuity of an (α,β)-derivation δ of a Banach algebra $\mathcal{A}$ into its a Banach module $\mathcal{M}$, sometimes we can assume that $\mathcal{A}$ is unital, $\mathcal{M}$ is a unital $\mathcal{A}$-module, $\delta(1) = 0$, $\alpha(0) = \beta(0) = 0$ and $\alpha(1) = \beta(1) = 1$, where 1 is the identity of $\mathcal{A}$.

The following examples yield continuous (α,β)-derivations without the assumption of linearity and continuity of α and β.

Example 2.1. Let $\mathcal{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, $\mathcal{A} \neq B(\mathfrak{H})$. Let α_0 and β_0 be bounded homomorphisms of $\mathcal{A}$ into itself, $f_0, g_0 : \mathcal{A} \to \mathbb{C}$ be two functionals without linearity and continuity, T_0 and S_0 be nonzero operators in $\mathcal{A}'$ with $T_0S_0 = S_0T_0 = 0$. Define the mappings α, β and δ from $\mathcal{A}$ into $B(\mathfrak{H})$ by

$$\alpha(A) = \alpha_0(A) + f_0(A)S_0,\ \beta(A) = \beta_0(A) + g_0(A)S_0,\ \delta(A) = T_0\alpha(A) - \beta(A)T_0$$

for each $A \in \mathcal{A}$. Then α and β are neither continuous nor linear, and δ is an (α,β)-derivation from $\mathcal{A}$ into $B(\mathfrak{H})$. By calculation, we have $\delta(A) = T_0\alpha_0(A) - \beta_0(A)T_0$ for each $A \in \mathcal{A}$, hence δ is continuous.

Example 2.2. Let $\mathfrak{H}$ be a separable infinite dimensional Hilbert space, $V \in B(\mathfrak{H})$ a partial isometry with $V^*V = I$, $VV^* = P \neq I$. Let $T_0 \in B(\mathfrak{H})$ be a self-adjoint operator with $T_0P = 0$, let $f_0 :\ B(\mathfrak{H}) \to \{V^*\}'$ be a mapping without linearity and continuity, (e.g., f is an nonlinear and non-continuous functional on $B(\mathfrak{H})$), where $\{V^*\}'$ is the commutant of $\{V^*\}$ in $B(\mathfrak{H})$. Define the mappings α, δ, : $B(\mathfrak{H}) \to B(\mathfrak{H})$ by $\alpha(A) = \frac{1}{2}(VAV^* + f_0(A)T_0)$ and $\delta(A) = VAV^*$ for each $A \in B(\mathfrak{H})$. Then α is neither linear nor continuous on $B(\mathfrak{H})$, but δ is a continuous α-derivation.

3. The $B(\mathfrak{X})$ case

In the following lemma, we list some properties given in [10] of an (α,β)-derivation.

Lemma 3.1 ([10])**.** *Let $\mathcal{B}$ be a complex algebra with a subalgebra $\mathcal{A}$ and let $\mathcal{M}$ be a $\mathcal{B}$-module. Let $\alpha, \beta : \mathcal{A} \to \mathcal{B}$ be two mappings. If δ is an (α, β)-derivation from $\mathcal{A}$ into $\mathcal{M}$, then, for each $\lambda, \mu \in \mathbb{C}$ and $A, B, C \in \mathcal{A}$, the following equations hold*:

(i) $\delta(A)\alpha(0) = \beta(0)\delta(A) = 0$;
(ii) $\delta(A)(\alpha(\lambda B + \mu C) - \lambda\alpha(B) - \mu\alpha(C)) = 0$;
(iii) $(\beta(\lambda A + \mu B) - \lambda\beta(A) - \mu\beta(B))\delta(C) = 0$;
(iv) $(\beta(AB) - \beta(A)\beta(B))\delta(C) = \delta(A)(\alpha(BC) - \alpha(B)\alpha(C))$. *In particular,* $(\beta(0) - \beta(0)\beta(B))\delta(C) = 0 = \delta(A)(\alpha(0) - \alpha(B)\alpha(0))$.

Proof. By the equation $0 = \delta(0) = \delta(A0) = \delta(0A)$, we can obtain (i). Using the linearity and the multiplicative rule of δ to expand the left of the following equations: $\delta(A(\lambda B+\mu C))-\lambda\delta(AB)-\mu\delta(AC) = 0$, $\delta((\lambda A+\mu B)C)-\lambda\delta(AC)-\mu\delta(BC) = 0$, $\delta((AB)C) - \delta(A(BC)) = 0$, we can get (ii), (iii) and (iv). □

Theorem 3.2. *Let $\mathfrak{X}$ be a complex Banach space, α and β be mappings from $B(\mathfrak{X})$ into itself. Let $\delta : B(\mathfrak{X}) \to B(\mathfrak{X})$ be an (α, β)-derivation. Suppose that α and β satisfy one of the following conditions*:

(i) *α is an automorphism, β is continuous at 0 and the set $\{\beta(T) : T \in F_1(\mathfrak{X})\}$ separates the points of $\mathfrak{X}$ in the sense that, for each pair ξ, $\eta \in \mathfrak{X}$ with $\xi \neq \eta$, there is a rank one operator T such that $\beta(T)\xi \neq \beta(T)\eta$, equivalently, the set $\{\beta(T) : T \in F_1(\mathfrak{X})\}$ has no nonzero right annihilators in $B(\mathfrak{X})$.*
(ii) *β is an automorphism, α is continuous at 0 and the set $\{\alpha(T) : T \in F_1(\mathfrak{X})\}$ has no nonzero left annihilators in $B(\mathfrak{X})$.*
(iii) *α and β are continuous at 0, span$\{\alpha(T)\xi : T \in F_1(\mathfrak{X}), \xi \in \mathfrak{X}\}$ is dense in $\mathfrak{X}$ and there is a rank one S such that $\beta(S)$ is injective.*

Then δ is continuous. Moreover, if (i)*, or when $\mathfrak{X}$ is reflexive and* (ii)*, holds, δ is inner.*

Proof. In order to obtain the continuity of δ, we use the closed graph theorem. Let $A_n \in B(\mathfrak{X})$, $n = 1, 2, \ldots$, with $A_n \to 0$ and $\delta(A_n) \to A$. For every $\xi \otimes f$, $\eta \otimes g \in F_1(\mathfrak{X})$ and $n = 1, 2, \ldots$, we have

$$\begin{aligned} & f(A_n\eta)\delta(\xi \otimes g) \\ = & \delta(\xi \otimes f \cdot A_n \cdot \eta \otimes g) \\ = & \delta(\xi \otimes f)\alpha(A_n\eta \otimes g) + \beta(\xi \otimes f)\delta(A_n)\alpha(\eta \otimes g) + \beta(\xi \otimes f)\beta(A_n)\delta(\eta \otimes g). \end{aligned}$$

If α and β are continuous at 0, letting $n \to \infty$, we have

$$\delta(\xi \otimes f)\alpha(0) + \beta(\xi \otimes f)A\alpha(\eta \otimes g) + \beta(\xi \otimes f)\beta(0)\delta(\eta \otimes g) = 0.$$

Using Lemma 3.1, we have

$$\beta(\xi \otimes f)A\alpha(\eta \otimes g) = 0$$

for every $\xi \otimes f, \eta \otimes g \in F_1(\mathfrak{X})$.

If (i) holds, then for each $\eta\otimes g\in F_1(\mathfrak{X})$, we have $A\alpha(\eta\otimes g)=0$. Since α is an automorphism, it is inner, i.e., there is an invertible bounded linear operator $T_0\in B(\mathfrak{X})$ such that $\alpha(T)=T_0TT_0^{-1}$ for each $T\in B(\mathfrak{X})$. So $(AT_0\eta)\otimes g=0$ for all $\eta\in\mathfrak{X}$ and $g\in\mathfrak{X}^*$. Hence $A=0$. Consequently, δ is continuous.

In this case, we can show that δ is inner. Choose $\xi_0\in\mathfrak{X}$ and $f_0\in\mathfrak{X}^*$ such that $f_0(\xi_0)=1$, and define the mapping $A_0:\mathfrak{X}\to\mathfrak{X}$ by $A_0\xi=\delta(T_0^{-1}\xi\otimes f_0)T_0\xi_0$ for each $\xi\in\mathfrak{X}$. Obviously, A_0 is linear and bounded. For each $T\in B(\mathfrak{X})$ and $\xi\in\mathfrak{X}$, we have

$$\begin{aligned}\delta((T\xi)\otimes f_0) &= \delta(T(\xi\otimes f_0))=\delta(T)\alpha(\xi\otimes f_0)+\beta(T)\delta(\xi\otimes f_0)\\ &= \delta(T)T_0(\xi\otimes f_0)T_0^{-1}+\beta(T)\delta(\xi\otimes f_0).\end{aligned}$$

Multiplying by the operator T_0, we have

$$\delta((T\xi)\otimes f_0)T_0=\delta(T)T_0(\xi\otimes f_0)+\beta(T)\delta(\xi\otimes f_0)T_0.$$

Applying mappings in two sides of the equation to ξ_0, we get $A_0(T_0T\xi)=\delta(T)T_0\xi+\beta(T)A_0(T_0\xi)$. Since ξ is arbitrary, we have $A_0T_0T=\delta(T)T_0+\beta(T)A_0T_0$, and hence $\delta(T)=A_0\alpha(T)-\beta(T)A_0$ for each $T\in B(\mathfrak{X})$. So δ is inner.

If (ii) holds, then by the equation $\beta(\xi\otimes f)A\alpha(\eta\otimes g)=0$, we have $\beta(\xi\otimes f)A=0$. Since β is an automorphism, there is an invertible bounded linear operator $S_0\in B(\mathfrak{X})$ such that $\beta(T)=S_0TS_0^{-1}$ for each $T\in B(\mathfrak{X})$. So $(\xi\otimes f)S_0^{-1}A=0$ for all $\xi\in\mathfrak{X}$ and $f\in\mathfrak{X}^*$. Hence $A=0$, so δ is continuous.

In this case, choose $\xi_0\in\mathfrak{X}$ and $f_0\in\mathfrak{X}^*$ such that $f_0(\xi_0)=1$. We define the mapping $B_0:\mathfrak{X}\to\mathfrak{X}$ by

$$\langle B_0\xi,f\rangle=\langle S_0^{-1}\delta((\xi_0\otimes f)S_0)\xi,f_0\rangle,\ \ \xi\in\mathfrak{X},\ f\in\mathfrak{X}^*.$$

Suppose that $\mathfrak{X}$ is reflexive. Then B_0 is well-defined. The continuity and linearity of δ imply the continuity and linearity of B_0. For each $\xi\in\mathfrak{X}$, $f\in\mathfrak{X}^*$ and $T\in B(\mathfrak{X})$, we have

$$\begin{aligned}&\langle(\beta(T)B_0-B_0\alpha(T))\xi,f\rangle\\ &=\langle B_0\xi,\beta(T)'f\rangle-\langle B_0\alpha(T)\xi,f\rangle\\ &=\langle S_0^{-1}\delta((\xi_0\otimes\beta(T)'f)S_0)\xi,f_0\rangle-\langle S_0^{-1}\delta((\xi_0\otimes f)S_0)\alpha(T)\xi,f_0\rangle\\ &=\langle S_0^{-1}\delta((\xi_0\otimes f)\beta(T)S_0)\xi,f_0\rangle-\langle S_0^{-1}\delta((\xi_0\otimes f)S_0)\alpha(T)\xi,f_0\rangle\\ &=\langle S_0^{-1}\delta((\xi_0\otimes f)S_0T)\xi,f_0\rangle-\langle S_0^{-1}\delta((\xi_0\otimes f)S_0)\alpha(T)\xi,f_0\rangle\\ &=\langle S_0^{-1}\beta((\xi_0\otimes f)S_0)\delta(T)\xi,f_0\rangle\\ &=\langle(\xi_0\otimes f)\delta(T)\xi,f_0\rangle\\ &=\langle\delta(T)\xi,f\rangle.\end{aligned}$$

Hence $\delta(T)=\beta(T)B_0-B_0\alpha(T)$ for each $T\in B(\mathfrak{X})$, and so, δ is inner.

Obviously, if (iii) holds, then $A=0$, which yields the continuity of δ. □

For a linear mapping T from a Banach space $\mathfrak{E}$ into a Banach space $\mathfrak{F}$, the separating space $\mathfrak{S}(T)$ is defined to be the set of elements ξ in $\mathfrak{F}$ such that there is a sequence $\{\xi_n\}$ in $\mathfrak{E}$ with $\xi_n \to 0$ in $\mathfrak{E}$ and $T(\xi_n) \to \xi$ in $\mathfrak{F}$. Clearly, $\mathfrak{S}(T) = \cap_{n=1}^{\infty} \overline{\{T(\eta) : \|\eta\| < \frac{1}{n}\}}$, hence is a closed linear subspace of $\mathfrak{F}$, and by the closed graph theorem, T is continuous if and only if $\mathfrak{S}(T) = \{0\}$.

Recall that a Banach space $\mathfrak{X}$ is called simple, if $B(\mathfrak{X})$ has a unique nontrivial norm-closed two-sided ideal. For example, l^p $(1 \leq p < \infty)$, c_0 and a separable infinite dimensional Hilbert space $\mathfrak{H}$ are simple. In this case, the norm closure of all the finite rank operators is the ideal of compact operators, which is wot-dense in $B(\mathfrak{X})$ and is the unique nontrivial norm-closed two-sided ideal of $B(\mathfrak{X})$.

Proposition 3.3. *Suppose that $\mathfrak{X}$ is a simple complex Banach space, $\mathcal{A}$ is a unital norm closed subalgebra of $B(\mathfrak{X})$, α, $\beta : \mathcal{A} \to B(\mathfrak{X})$ are surjective and continuous at 0. If at least one of α and β is not an algebraic homomorphism, then every (α, β)-derivation δ from $\mathcal{A}$ into $B(\mathfrak{X})$ is automatically continuous.*

Proof. We first show that $\mathfrak{S}(\delta)$ is a closed two-sided ideal of $B(\mathfrak{X})$. For arbitrary $A \in \mathfrak{S}(\delta)$ and $B \in B(\mathfrak{X})$, there is a sequence $\{A_n\}$ in $\mathcal{A}$ with $A_n \to 0$ and $\delta(A_n) \to A$. Since α and β are surjective, there are S, $T \in \mathcal{A}$ such that $B = \alpha(S) = \beta(T)$. Hence $TA_n \to 0$, $A_nS \to 0$. Also since α and β are continuous at 0, using Lemma 3.1, we have $\delta(TA_n) = \delta(T)\alpha(A_n) + \beta(T)\delta(A_n) \to \delta(T)\alpha(0) + BA = BA$ and $\delta(A_nS) = \delta(A_n)\alpha(S) + \beta(A_n)\delta(S) \to AB + \beta(0)\delta(S) = AB$. Hence AB, $BA \in \mathfrak{S}(\delta)$. So $\mathfrak{S}(\delta)$ is a closed two-sided ideal of $B(\mathfrak{X})$. Since $\mathfrak{X}$ is simple, we have $\mathfrak{S}(\delta) = 0, B(\mathfrak{X}), \text{or } K(\mathfrak{X})$.

For an arbitrary $A \in \mathfrak{S}(\delta)$, let $\{A_n\}$ be a sequence in $\mathcal{A}$ with $A_n \to 0$ and $\delta(A_n) \to A$. For all pairs $B, C \in \mathcal{A}$, using (iv) of Lemma 3.1, we have $(\beta(A_nB) - \beta(A_n)\beta(B))\delta(C) = \delta(A_n)(\alpha(BC) - \alpha(B)\alpha(C))$ and $(\beta(BC) - \beta(B)\beta(C))\delta(A_n) = \delta(B)(\alpha(CA_n) - \alpha(C)\alpha(A_n))$. The continuity of α and β at 0 implies that $(\beta(0) - \beta(0)\beta(B))\delta(C) = A(\alpha(BC) - \alpha(B)\alpha(C))$ and $(\beta(BC) - \beta(B)\beta(C))A = \delta(B)(\alpha(0) - \alpha(C)\alpha(0))$. Using (iv) of Lemma 3.1, we have

$$A(\alpha(BC) - \alpha(B)\alpha(C)) = 0, \tag{1}$$

$$(\beta(BC) - \beta(B)\beta(C))A = 0 \tag{2}$$

for each $A \in \mathfrak{S}(\delta)$ and $B, C \in \mathcal{A}$. Similarly, using (ii) and (iii) of Lemma 3.1, for each $A \in \mathfrak{S}(\delta)$, $B, C \in \mathcal{A}$, $\lambda, \mu \in \mathbb{C}$, we get that

$$A(\alpha(\lambda B + \mu C) - \lambda\alpha(B) - \mu\alpha(C)) = 0, \tag{3}$$

$$(\beta(\lambda B + \mu C) - \lambda\beta(B) - \mu\beta(C))A = 0. \tag{4}$$

Suppose that $\mathfrak{S}(\delta) = B(\mathfrak{X})$ or $\mathfrak{S}(\delta) = K(\mathfrak{X})$. Then it follows from (1), (2), (3) and (4) that both α and β are algebraic homomorphisms, which is a contradiction. Hence $\mathfrak{S}(\delta) = 0$, and so, δ is continuous. $\square$

Theorem 3.4. *Let $\mathfrak{X}$ be a simple Banach space, δ an (α, β)-derivation from $B(\mathfrak{X})$ into itself. Suppose that α, $\beta : B(\mathfrak{X}) \to B(\mathfrak{X})$ are surjective and continuous at* 0*. Then δ is continuous.*

Proof. If at least one of α and β is not an algebraic homomorphism, then Proposition 3.3 yields the continuity of δ. If both α and β are algebraic homomorphisms, then they are bounded automorphisms of $B(\mathfrak{X})$. Theorem 3.2 implies that δ is continuous. □

Removing the continuity in above theorem, we have the following results.

Theorem 3.5. *For a complex Banach space $\mathfrak{X}$ and two mappings α, β on $B(\mathfrak{X})$, assume that α and β are surjective and multiplicative and there are rank one operators T_0 and S_0 such that $\alpha(T_0) \neq 0$ and $\beta(S_0) \neq 0$. Then every (α, β)-derivation from $B(\mathfrak{X})$ into itself is continuous.*

Proof. It suffices to show that α and β are (bounded) automorphisms of $B(\mathfrak{X})$. Assume that $\delta \neq 0$.

Since α and β are surjective and multiplicative, it is not difficult to show that, for each $\lambda \in \mathbb{C}$, there are scales $f(\lambda)$ and $g(\lambda)$ such that $\alpha(\lambda I) = f(\lambda)I$, $\beta(\lambda I) = g(\lambda)I$ and $\alpha(I) = I$, $\beta(I) = I$. Note that $\delta(I) = \delta(I)\alpha(I) + \beta(I)\delta(I) = 2\delta(I)$, which yields $\delta(I) = 0$. Hence for $\lambda \in \mathbb{C}$ and $T \in B(\mathfrak{X})$, $\lambda\delta(T) = \delta(T \cdot \lambda I) = \delta(T)\alpha(\lambda I) + \beta(T)\delta(\lambda I) = \delta(T)\alpha(\lambda I) = f(\lambda)\delta(T)$, which implies that $f(\lambda) = \lambda$, and thus, $\alpha(\lambda I) = \lambda I$. Hence α is homogeneous. Similarly, using $\lambda\delta(T) = \delta(\lambda I \cdot T)$, we can get that β is also homogeneous.

Now we show that α and β are injective.

Let $T_0 = \xi_0 \otimes f_0$ be the rank one operator such that $\alpha(T_0) \neq 0$. For each rank one operator $\xi \otimes f$, choose $g_0 \in \mathfrak{X}^*$ and $\eta_0 \in \mathfrak{X}$ such that $g_0(\xi) = f(\eta_0) = 1$. Then $\xi_0 \otimes f_0 = (\xi_0 \otimes g_0)(\xi \otimes f)(\eta_0 \otimes f_0)$, which implies $\alpha(\xi \otimes f) \neq 0$ for all rank one operators $\xi \otimes f$.

If $\alpha(T) = 0$, then $T = 0$. For, otherwise, there exists $\xi \in \mathfrak{X}$ with $T\xi \neq 0$. For $f_0 \in \mathfrak{X}^*$, $f_0 \neq 0$, we have $T\xi \otimes f_0$ is a rank one operator, but $\alpha(T\xi \otimes f_0) = \alpha(T)\alpha(\xi \otimes f_0) = 0$, which is a contradiction with above argument.

Next we show α is injective on the set of all rank one operators. Let $R = \xi_0 \otimes f_0$ and $S = \eta_0 \otimes g_0$ be two arbitrary rank one operators with $\alpha(R) = \alpha(S)$. If R and S are linearly independent, and ξ_0 and η_0 are linearly dependent, then f_0 and g_0 are linearly independent. Choosing $\xi \in \mathfrak{X}$ with $f_0(\xi) = 1$ and $g_0(\xi) = 0$, we have $R(\xi \otimes f_0) = \xi_0 \otimes f_0$ and $S(\xi \otimes f_0) = 0$, which is impossible, for $0 \neq \alpha(R(\xi \otimes f_0)) = \alpha(S(\xi \otimes f_0)) = 0$. If R and S are linearly independent, and ξ_0 and η_0 are linearly independent, then we can choose $h_0 \in \mathfrak{X}^*$ such that $h_0(\xi_0) = 0$ and $h_0(\eta_0) = 1$. Hence $(\xi_0 \otimes h_0)R = (\xi_0 \otimes h_0)(\xi_0 \otimes f_0) = 0$ and $(\xi_0 \otimes h_0)S = (\xi_0 \otimes h_0)(\eta_0 \otimes g_0) = \xi_0 \otimes g_0 \neq 0$. So $0 = \alpha((\xi_0 \otimes h_0)R) = \alpha((\xi_0 \otimes h_0)S) \neq 0$, which is a contradiction. Hence R and S are linearly dependent. The homogeneity of α yields $R = S$.

If $\alpha(T) = \alpha(S)$ for $T, S \in B(\mathfrak{X})$, then for each nonzero vectors $\xi \in \mathfrak{X}$ and $f \in \mathfrak{X}^*$, we have $\alpha(S\xi \otimes f) = \alpha(T\xi \otimes f)$. Obviously, $S\xi = 0$ if and only if

$T\xi = 0$. If $S\xi \neq 0$, using the injectivity of α on the set of all rank one operators and the arbitrariness of f, we have $S\xi = T\xi$. Hence $S = T$. We have shown that α is injective. Similarly, we can show the injectivity of β.

Hence α and β are multiplicative bijections on $B(\mathfrak{X})$. By the celebrated result of Martindale in [9], α and β are additive. Consequently, α and β are surjective algebraic homomorphisms, hence are automorphisms on $B(\mathfrak{X})$. By Theorem 3.2, δ is continuous. □

4. The C^*-algebra case

In this section we study the continuity of (α, β)-derivations of C^*-algebras into their Banach modules. Inspiring the proof of the related results on the ordinary derivations in [13], we have the following Theorem 4.4. We start with some lemmas which can be found in Ex 4.6.39, Ex 4.6.13 and Ex 4.6.20 in [6] (see also Lemma 1 and the proof of Theorem 3 in [13]).

Lemma 4.1 ([6, 13])**.** *Let $\mathcal{J}$ be a closed two-sided ideal in a unital C^*-algebra $\mathcal{A}$, $B \in \mathcal{J}$ a positive element with $\|B\| \leq 1$, $A \in \mathcal{J}$ with $AA^* \leq B^4$. Then $A = BC$ for some C in $\mathcal{J}$ with $\|C\| \leq 1$.*

Lemma 4.2 ([6, 13])**.** *Suppose that $\mathcal{D}$ is an infinite dimensional unital C^*-algebra. Then there is an infinite sequence $\{A_1, A_2, \ldots\}$ of nonzero positive elements in $\mathcal{D}$ such that $A_j A_k = 0$ for $j \neq k$.*

Lemma 4.3 ([6, 13])**.** *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and φ is a $*$-homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. For each sequence $\{B_1, B_2, \ldots\}$ of positive elements of $\mathcal{B}$ such that $B_j B_k = 0$ when $j \neq k$, there is a sequence $\{A_1, A_2, \ldots\}$ of positive elements of $\mathcal{A}$ such that $A_j A_k = 0$ when $j \neq k$ and $\varphi(A_j) = B_j$ for each $j = 1, 2, \ldots$.*

Theorem 4.4. *Let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{B}$ be a unital Banach algebra containing $\mathcal{A}$ as a unital Banach subalgebra, and let $\mathcal{M}$ be a Banach $\mathcal{B}$-module. Suppose that α, $\beta : \mathcal{A} \to \mathcal{B}$ are continuous at 0. Then every (α, β)-derivation δ from $\mathcal{A}$ into $\mathcal{M}$ is continuous.*

Proof. For each A in $\mathcal{A}$, we define the mappings L_A, S_A, γ_A, $\sigma_A : \mathcal{A} \to \mathcal{M}$ by

$$L_A(T) = \delta(AT),\ S_A(T) = \beta(A)\delta(T),\ \gamma_A(T) = \delta(A)\alpha(T),\ \sigma_A(T) = \beta(T)\delta(A)$$

for each T in $\mathcal{A}$. Since δ is linear, we have L_A and S_A are linear, hence $\gamma_A = L_A - S_A$ is linear. It follows from (iii) of Lemma 3.1 that σ_A is linear. The continuity of α and β at 0 implies the continuity of γ_A and σ_A at 0, hence at every $T \in \mathcal{A}$. Hence γ_A and σ_A are bounded. Let $\mathcal{J} = \{A \in \mathcal{A} : L_A \text{ is bounded}\}$. Obviously, $0 \in \mathcal{J}$, $\mathcal{J}$ is a subspace of $\mathcal{A}$, and $\mathcal{J} = \{A \in \mathcal{A} : S_A \text{ is bounded}\}$. Firstly, we claim that $\mathcal{J}$ is a norm closed two-sided ideal of $\mathcal{A}$.

For each J in $\mathcal{J}$ and A in $\mathcal{A}$, since L_{JA} is the composition of the bounded mapping L_J and the (bounded) multiplication on the left by A: $T \to AT$ from

$\mathcal{A}$ into itself, we have that JA is in $\mathcal{J}$; on the other hand, by Lemma 3.1(iv), we have

$$S_{AJ}(T)=\beta(AJ)\delta(T)=\beta(A)S_J(T)+\gamma_A(JT)-\gamma_A(J)\alpha(T)$$

for each $T\in\mathcal{A}$. Hence S_{AJ} is continuous at 0, which yields that S_{AJ} is continuous. Hence $AJ\in\mathcal{J}$, so $\mathcal{J}$ is a two-sided ideal of $\mathcal{A}$.

Let $\{A_n\}$ be a sequence in $\mathcal{J}$ with $A_n\to A\in\mathcal{A}$. For each T in $\mathcal{A}$, noting that σ_T is bounded, we have $\lim_{n\to\infty}S_{A_n}(T)=\lim_{n\to\infty}\beta(A_n)\delta(T)=\lim_{n\to\infty}\sigma_T(A_n)=\sigma_T(A)=\beta(A)\delta(T)=S_A(T)$. Since $\{S_{A_n}\}$ is a sequence in $B(\mathcal{A},\mathcal{M})$, the set of all the bounded linear mappings from $\mathcal{A}$ into $\mathcal{M}$, using the principle of uniform boundedness, we have S_A is also bounded, hence A belongs to $\mathcal{J}$. We have established the claim.

Now we show that the restriction $\delta|_{\mathcal{J}}$ to $\mathcal{J}$ of δ is bounded. For otherwise, choose a sequence $A_1,A_2,\ldots$ in $\mathcal{J}$ such that for each n, $\|A_n\|^2\le\frac{1}{2^n}$ and $\|\delta(A_n)\|\ge n$. Let $B=(\sum_{n=1}^{\infty}A_nA_n^*)^{\frac14}$. Then B is a positive element in $\mathcal{J}$ with $\|B\|\le 1$ and $A_nA_n^*\le B^4$ for each n. By Lemma 4.1, for each n, there exists C_n in $\mathcal{J}$ such that $\|C_n\|\le 1$ and $A_n=BC_n$. Hence $\|L_B(C_n)\|=\|\delta(A_n)\|\ge n$ for each n, which contradicts the continuity of L_B. This proves that $\delta|_{\mathcal{J}}$ is bounded.

Let $\pi:\mathcal{A}\to\mathcal{A}/\mathcal{J}$ be the canonical quotient mapping which is a surjective $*$-homomorphism. We claim that $\mathcal{A}/\mathcal{J}$ is finite dimensional. On the contrary, using Lemma 4.2, we choose an infinite sequence $\{\widetilde{A_1},\widetilde{A_2},\ldots\}$ of nonzero positive elements in $\mathcal{A}/\mathcal{J}$ such that $\widetilde{A_j}\widetilde{A_k}=0$ when $j\ne k$. By Lemma 4.3, there is an infinite sequence $\{A_1,A_2,\ldots\}$ of nonzero positive elements in $\mathcal{A}$ such that $A_jA_k=0$ when $j\ne k$ and $\pi(A_j)=\widetilde{A_j}$ for each j. Since $\widetilde{A_j}$ is nonzero in $\mathcal{A}/\mathcal{J}$, we have that A_j, and hence, A_j^2 is not in $\mathcal{J}$, which implies that $L_{A_j^2}$ is unbounded. Consequently, we have constructed a sequence $A_1,A_2,\ldots$ of positive elements in $\mathcal{A}$ such that $A_j^2\notin\mathcal{J}$ and $A_jA_k\ne 0$ when $j\ne k$. Replacing A_j by an appropriate scalar multiple, we may assume also that $\|A_j\|\le 1$ for each j. Since $L_{A_j^2}$ is unbounded, there is T_j in $\mathcal{A}$ such that $\|T_j\|\le 2^{-j}$ and $\|L_{A_j^2}(T_j)\|=\|\delta(A_j^2T_j)\|\ge\|\gamma_{A_j}\|+j$. Let $A=\sum_{j=1}^{\infty}A_jT_j$. Then $A\in\mathcal{A}$, $\|A\|\le 1$ and $A_jA=A_j^2T_j$. Hence, for each $j=1,2,\ldots$,

$$\begin{aligned}\|\sigma_A\| &\ge \|\sigma_A(A_j)\|=\|\beta(A_j)\delta(A)\|=\|\delta(A_jA)-\delta(A_j)\alpha(A)\|\\ &= \|\delta(A_j^2T_j)-\gamma_{A_j}(A)\|\ge\|\delta(A_j^2T_j)\|-\|\gamma_{A_j}(A)\|\\ &\ge \|\delta(A_j^2T_j)\|-\|\gamma_{A_j}\|\ge j,\end{aligned}$$

which is impossible. Hence $\mathcal{A}/\mathcal{J}$ is finite dimensional.

Since $\delta|_{\mathcal{J}}$ is norm continuous and $\mathcal{J}$ has finite codimension in $\mathcal{A}$, it follows that δ is norm continuous. □

Remark. When $\mathcal{A}$ in above theorem is only a Banach algebra, we can also get the closed two-sided ideal $\mathcal{J}$ of $\mathcal{A}$. Using the closed graph theorem, we can show that if $\mathcal{J}$ has a bounded left approximate identity, then the restriction

$\delta|_{\mathcal{J}}$ of δ to $\mathcal{J}$ is bounded. Firstly, we recall the Cohen's factorization theorem ([1], Corollary 12 in Chapter 1), which tells us if $\mathcal{B}$ is a Banach algebra with a bounded left approximate identity, then for each sequence $\{A_n\}$ in $\mathcal{B}$ with $A_n \to 0$, there exist $A, B_n \in \mathcal{B}$ with $A_n = AB_n$, $(n = 1, 2, \ldots)$ and $B_n \to 0$. Now we show the boundedness of $\delta|_{\mathcal{J}}$. Let $A_n \in \mathcal{J}$ $(n = 1, 2, \ldots)$ with $A_n \to 0$ and $\delta(A_n) \to J$. It follows from the Cohen's factorization theorem that there exist $A, B_n \in \mathcal{J}$ with $A_n = AB_n$, $(n = 1, 2, \ldots)$ and $B_n \to 0$. Since $\delta(A_n) = \delta(A)\alpha(B_n) + \beta(A)\delta(B_n)$ for each n and $A \in \mathcal{J}$, by Lemma 3.1(i), the boundedness of S_A and the continuity of α at 0 yield $J = 0$. Hence $\delta|_{\mathcal{J}}$ is continuous.

Let $\mathcal{S}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, let $\mathcal{M}$ be a dual normal $\mathcal{S}$-module. If $\mathcal{M}_*$ is the predual of $\mathcal{M}$, we write $\langle M, \omega\rangle$ in place of $M(\omega)$ for each $M \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$. Then $\mathcal{M}_*$ is a Banach $\mathcal{S}$-module under the following module actions determined by

$$\langle M, \omega A\rangle = \langle AM, \omega\rangle,\ \langle M, A\omega\rangle = \langle MA, \omega\rangle$$

for $\omega \in \mathcal{M}_*$, $A \in \mathcal{S}, M \in \mathcal{M}$. In [13], using the properties of the Mackey topologies on $\mathcal{M}_*$ and $\mathcal{S}$, Ringrose proved that the mappings $A \to \omega A$, $A \to A\omega$ are continuous from the unit ball of $\mathcal{S}$ (with strong* topology) into $\mathcal{M}_*$ (with norm topology). Hence, for a C^*-subalgebra $\mathcal{A}$ of $\mathcal{S}$ and a pair of ultraweakly and strong* continuous linear mappings α, β from $\mathcal{A}$ into $\mathcal{S}$, the mappings $A \to \alpha(A)\omega$, $A \to \omega\beta(A)$ are strong*-norm continuous from the unit ball of $\mathcal{A}$ into $\mathcal{M}_*$. We have the following corollary.

Corollary 4.5. *Let $\mathcal{S}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, and let $\mathcal{A}$ be a unital C^*-subalgebra of $\mathcal{S}$, with the weak closure $\mathcal{R}$. Suppose that $\mathcal{M}$ is a dual normal $\mathcal{S}$-module and α, β are two ultraweakly and strong* continuous linear mappings from $\mathcal{A}$ into $\mathcal{S}$. Then every (α, β)-derivation δ from $\mathcal{A}$ into $\mathcal{M}$ is ultraweakly-weak* continuous, and extends to an ultraweakly-weak* continuous $(\overline{\alpha}, \overline{\beta})$-derivation of $\mathcal{R}$, where $\overline{\alpha}$ and $\overline{\beta}$ are the extension of α and β to $\mathcal{R}$, respectively.*

Proof. By Theorem 4.4, δ is norm continuous. To establish the ultraweak-weak* continuity of δ, it suffices to show that, for each ω in $\mathcal{M}_*$, the linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, defined by $\varphi(A) = \langle \delta(A), \omega\rangle$ for each $A \in \mathcal{A}$, is ultraweakly continuous (equivalently, show that φ is continuous on the unit ball of $\mathcal{A}$ under the weak operator topology). By Lemma 7.1.3 in [6], we only need to prove that the restriction of φ to $\mathcal{A}_1^+$, the set of all positive elements in the unit ball of $\mathcal{A}$, is strongly continuous at 0.

Let $\{T_\iota\}$ be a net converging strongly to 0 in $\mathcal{A}_1^+$. Then $\{T_\iota^{1/2}\}$ converges strongly, and hence under the strong* topology, to 0. Since α and β are ultraweakly and strong* continuous, by the previous argument of Corollary 4.5, both $\{\|\alpha(T_\iota^{1/2})\omega\|\}$ and $\{\|\omega\beta(T_\iota^{1/2})\|\}$ converge to 0. It follows that

$$\begin{aligned} |\varphi(T_\iota)| &= \|\langle\delta(T_\iota^{1/2}T_\iota^{1/2}),\omega\rangle\| \\ &= \|\langle\delta(T_\iota^{1/2})\alpha(T_\iota^{1/2})+\beta(T_\iota^{1/2})\delta(T_\iota^{1/2}),\omega\rangle\| \\ &= \|\langle\delta(T_\iota^{1/2}),\alpha(T_\iota^{1/2})\omega+\omega\beta(T_\iota^{1/2})\rangle\| \\ &\le \|\delta\|(\|\alpha(T_\iota^{1/2})\omega\|+\|\omega\beta(T_\iota^{1/2})\|)\longrightarrow 0. \end{aligned}$$

Hence we have proved that δ is ultraweakly-weak* continuous. Since by Kaplansky density theorem, the unit ball of $\mathcal{A}$ is weakly dense in the unit ball of $\mathcal{R}$, and the unit ball $\mathcal{M}$ is *weak** compact, we have that δ can extend without increase in norm to an ultraweak-*weak** continuous linear mapping, denoted by $\overline{\delta}$, from $\mathcal{R}$ into $\mathcal{M}$.

Now, we claim that $\overline{\delta}$ is an $(\overline{\alpha},\overline{\beta})$-derivation. For a given arbitrary element $\omega\in\mathcal{M}_*$, define a bilinear form $F_\omega:\mathcal{R}\times\mathcal{R}\to\mathbb{C}$ by $F_\omega(A,B)=\langle\overline{\delta}(AB)-\overline{\delta}(A)\overline{\alpha}(B)-\overline{\beta}(A)\overline{\delta}(B),\omega\rangle$ for each pair A, $B\in\mathcal{R}$. Clearly, $F_\omega(A,B)=0$ when A and B are in $\mathcal{A}$. For self-adjoint operators $A,B\in\mathcal{R}$, by Kaplansky density theorem, we choose self-adjoint element $\{A_\iota\}$ and $\{B_\iota\}$ in $\mathcal{A}$ which converges strongly to A and B, respectively and $\|A_\iota\|\le\|A\|$, $\|B_\iota\|\le\|B\|$ for each ι. Also since the joint multiplication is strongly continuous on the bounded sets of self-adjoint elements, we have $\{A_\iota B_\iota\}$ converges strongly to AB, and hence $F_\omega(A,B)=\lim_\iota F_\omega(A_\iota,B_\iota)=0$. Since ω is arbitrary, we have $\overline{\delta}(AB)-\overline{\delta}(A)\overline{\alpha}(B)-\overline{\beta}(A)\overline{\delta}(B)=0$ for arbitrary self-adjoint operators, and hence for any elements, in $\mathcal{R}$. Consequently, δ is an $(\overline{\alpha},\overline{\beta})$ derivation. □

The following corollary is a direct result of Corollary 4.5.

Corollary 4.6. *Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras acting on a separable Hilbert space $\mathfrak{H}$, $\mathcal{R}\subseteq\mathcal{S}$ and let $\mathcal{M}$ be a dual normal $\mathcal{S}$-module. For two given ultraweakly and strong* continuous linear mappings $\alpha,\beta:\mathcal{R}\to\mathcal{S}$, every (α,β)-derivation $\delta:\mathcal{R}\to\mathcal{M}$ is ultraweakly-weak* continuous.*

Corollary 4.7. *Let $\mathcal{S}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, $\mathcal{A}$ be an ultraweakly closed unital subalgebra of $\mathcal{S}$. Suppose that $\mathcal{M}$ is a dual normal $\mathcal{S}$-module, α, $\beta:\mathcal{A}\to\mathcal{S}$ are ultraweakly and strong* continuous linear mappings. Then for each (α,β)-derivation $\delta:\mathcal{A}\to\mathcal{M}$, there is a central projection P in $\mathcal{A}\cap\mathcal{A}^*$ such that $(\mathcal{A}\cap\mathcal{A}^*)(I-P)$ is finite dimensional and the mapping $A\to\delta(PA)$ from $\mathcal{A}$ into $\mathcal{M}$ is norm continuous.*

Proof. Let $\mathcal{R}=\mathcal{A}\cap\mathcal{A}^*$. Then $\mathcal{R}$ is a von Neumann algebra. As in the proof in Theorem 4.4, set $\mathcal{J}=\{A\in\mathcal{R}:L_A$ is bounded from $\mathcal{A}$ into $\mathcal{M}\}$. By the same argument, one can see that $\mathcal{J}$ is a two-sided ideal of $\mathcal{R}$. Now we show that $\mathcal{J}$ is ultraweakly closed. Let $\{A_\iota\}$ be a net of elements in $\mathcal{J}$ converging ultraweakly to A. Since $\mathcal{J}$ is a two-sided ideal of a von Neumann algebra, it is selfadjoint, for let $J\in\mathcal{J}$ and $J=W|J|$ be its polar decomposition, we have $W\in\mathcal{R}$ and $J^*=|J|W^*=WJW^*\in\mathcal{J}$. Using Kaplansky density theorem, we assume that $\|A_\iota\|\le\|A\|$ for each ι. By Corollary 4.6, the restriction $\delta|_{\mathcal{R}}$ of

δ to $\mathcal{R}$ is bounded and ultraweakly-$weak^*$ continuous. Hence, for each $T \in \mathcal{A}$, we have $L_A(T) = \delta(AT) = \delta(A)\alpha(T) + \beta(A)\delta(T) = weak^*\text{-}\lim_\iota \delta(A_\iota)\alpha(T) + \beta(A_\iota)\delta(T) = weak^*\text{-}\lim_\iota \delta(A_\iota T) = weak^*\text{-}\lim_\iota L_{A_\iota}(T)$; and moreover, for each ι, we have

$$\begin{aligned} \|L_{A_\iota}(T)\| &= \|\delta(A_\iota)\alpha(T) + \beta(A_\iota)\delta(T)\| \\ &\leq \|\delta|_{\mathcal{R}}\|\|A_\iota\|\|\alpha\|\|T\| + \|\beta\|\|A_\iota\|\|\delta(T)\| \\ &\leq \|\delta|_{\mathcal{R}}\|\|\alpha\|\|A\|\|T\| + \|\beta\|\|A\|\|\delta(T)\|. \end{aligned}$$

Using the principle of uniform boundedness, we have $\{\|L_{A_\iota}\|\}$ is bounded. So L_A, as the pointwise limit of the net $\{L_{A_\iota}\}$ of continuous mappings from $\mathcal{A}$ into $\mathcal{M}$, is continuous, and thus $A \in \mathcal{J}$. Hence $\mathcal{J}$ is an ultraweakly two-sided ideal of $\mathcal{R}$, so there is a unique central projection P in $\mathcal{R}$ such that $\mathcal{J} = \mathcal{R}P$.

Now we claim that $\mathcal{R}(I - P)$ is finite dimensional. For, otherwise, there is a sequence of nontrivial pairwise orthogonal projections $\{Q_n\}$ in $\mathcal{R}$ with sum $I - P$. Since for each n, the mapping L_{Q_n} is unbounded, there exists A_n in $\mathcal{A}$ such that $\|A_n\| \leq 2^{-n}$ and $\|\delta(Q_nA_n)\| > 2^n$. Let $A = \sum_{n=1}^\infty Q_nA_n$. Then $\|A\| \leq 1$ and $Q_nA = Q_nA_n$ for each n. Consequently, $2^n \leq \|\delta(Q_nA_n)\| = \|\delta(Q_nA)\| \leq \|\delta|_{\mathcal{R}}\|\|\alpha(A)\| + \|\beta\|\|\delta(A)\|$ for each n, which is impossible. Hence $\mathcal{R}(I - P)$ is finite dimensional. □

Remark. Applying Corollary 4.7 to $\delta^*(A) = \delta(A^*)^*$ on $\mathcal{A}^*$, we have that there is a central projection Q in $\mathcal{A}\cap\mathcal{A}^*$ such that $(\mathcal{A}\cap\mathcal{A}^*)(I-Q)$ is finite dimensional and the mapping $A \to \delta(AQ)$ from $\mathcal{A}$ into $\mathcal{M}$ is norm continuous.

Corollary 4.8. *Suppose that $\mathcal{A}$ is a CSL algebra acting on a separable Hilbert space $\mathfrak{H}$, i.e., $\mathcal{A}$ is a reflexive algebra whose lattice $Lat(\mathcal{A})$ of invariant projections is commutative. If $\alpha, \beta : \mathcal{A} \to B(\mathfrak{H})$ are ultraweakly and strong* continuous linear mappings, then every (α, β)-derivation from $\mathcal{A}$ into $B(\mathfrak{H})$ is bounded.*

Proof. The proof is the same as that of Corollary 2.3 in [2], we describe it briefly. Let $\mathcal{L} = Lat(\mathcal{A})$ and $\mathcal{R} = \mathcal{A} \cap \mathcal{A}^*$. Then $\mathcal{R} = \mathcal{L}'$ with center $\mathcal{L}''$. By Corollary 4.7 and its remark, there are projections P and Q in $\mathcal{L}''$ such that $\mathcal{R}P^\perp$ and $\mathcal{R}Q^\perp$ are finite dimensional, and the mappings $L_P : A \in \mathcal{A} \to \delta(PA) \in B(\mathfrak{H})$ and $R_Q : A \in \mathcal{A} \to \delta(AQ) \in B(\mathfrak{H})$ are continuous. Let $P^\perp = \sum_{i=1}^k P_i$ and $Q^\perp = \sum_{j=1}^l Q_i$ be the sum of minimal projections in $\mathcal{L}''$, each of which is finite rank, for $\mathcal{R}P^\perp = \sum_{i=1}^k \oplus B(P_i\mathfrak{H})$ and $\mathcal{R}Q^\perp = \sum_{j=1}^l \oplus B(Q_j\mathfrak{H})$ are finite dimensional. Hence for each $A \in \mathcal{A}$, we have $\delta(A) = \delta(PA) + \delta(P^\perp AQ) + \sum_{i,j} \delta(P_iAQ_j)$. Since $P_i\mathcal{A}Q_j$ is finite dimensional and L_P, R_Q are continuous, we have δ is continuous. □

Acknowledgements. This work was supported by National Natural Science Foundation of China (No. 10971117) and Natural Science Foundation of Shandong Province (No. Y2006A03, ZR2009AQ005).

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York-Heidelberg, 1973.

[2] E. Christensen, *Derivations of nest algebras*, Math. Ann. **229** (1977), no. 2, 155–161.

[3] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs. New Series, 24. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.

[4] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 1067–1073.

[5] R. V. Kadison, *Derivations of operator algebras*, Ann. of Math. (2) **83** (1966), 280–293.

[6] R. V. Kadison and J. Ringrose, *Fundamentals of the Theory of Operators, Vol 1: Elementary Theory; Vol 2: Advanced Theory*, Graduate Studies in Mathematics, Vol. 15, Vol. 16, American Mathematical Society, 1997.

[7] I. Kaplansky, *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858.

[8] ———, *Derivation of Banach algebras*, Seminar on Analytic Functions, Vol. II. Institute for Advanced Study, Prnceton, 1958.

[9] W. S. Martindale III, *When are multiplicative mappings additive?*, Proc. Amer. Math. Soc. **21** (1969), 695–698.

[10] M. Mirzavaziri and M. S. Moslehian, *Automatic continuity of σ-derivations on C^*-algebras*, Proc. Amer. Math. Soc. **134** (2006), no. 11, 3319–3327.

[11] ———, *σ-derivations in Banach algebras*, Bull. Iranian Math. Soc. **32** (2006), no. 1, 65–78.

[12] M. S. Moslehian, *Hyers-Ulam-Rassias stability of generalized derivations*, Int. J. Math. Math. Sci. **2006** (2006), Art. ID 93942, 8 pp.

[13] J. R. Ringrose, *Automatic continuity of derivations of operator algebras*, J. London Math. Soc. (2) **5** (1972), 432–438.

[14] S. Sakai, *On a conjecture of Kaplansky*, Tohoku Math. J. (2) **12** (1960), 31–33.

[15] A. R. Villena, *Automatic continuity in associative and nonassociative context*, Irish Math. Soc. Bull. No. **46** (2001), 43–76.

Chengjun Hou
Institute of Operations Research
Qufu Normal University
Shandong 273165, P. R. China
E-mail address: cjhou@mail.qfnu.edu.cn

Qing Meng
School of Mathematical Sciences
Qufu Normal University
Shandong 273165, P. R. China
E-mail address: mengqing80@163.com