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THE CLASSIFICATION OF LOG ENRIQUES SURFACES OF RANK 18

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Abstract. Log Enriques surface is a generalization of K3 and Enriques surface. We will classify all the rational log Enriques surfaces of rank 18 by giving concrete models for the realizable types of these surfaces.

1. Introduction

A normal projective surface Z with at worst quotient singularities is called a *logarithmic* (abbr. *log*) *Enriques surface* if its canonical Weil divisor K_Z is numerically equivalent to zero, and if its irregularity $\dim H^1(Z, \mathcal{O}_Z) = 0$. By the abundance for surfaces, $K_Z \sim_{\mathbb{Q}} 0$.

Let Z be a log Enriques surface and define

$$I := I(Z) = \min\{n \in \mathbb{Z}^+ \mid \mathcal{O}_Z(nK_Z) \simeq \mathcal{O}_Z\}$$

to be the *canonical index* of Z. The *canonical cover* of Z is defined as

$$\pi : \bar{S} := \mathrm{Spec}_{\mathcal{O}_Z}\left(\bigoplus_{j=0}^{I-1} \mathcal{O}_Z(-jK_Z)\right) \to Z.$$

This is a Galois $\mathbb{Z}/I\mathbb{Z}$-cover. So $\bar{S}/(\mathbb{Z}/I\mathbb{Z}) = Z$.

Note that a log Enriques surface is irrational if and only if it is a K3 or Enriques surface with at worst Du Val singularities (cf. [8, Proposition 1.3]). More precisely, a log Enriques surface of index one is a K3 surface with at worst Du Val singularities, and a log Enriques surface of index two is an Enriques surface with at worst Du Val singularities or a rational surface. Therefore, the log Enriques surfaces can be viewed as generalizations of K3 surfaces and Enriques surfaces. More results about the canonical indices are studied in [8] and [9].

If a log Enriques surface Z has Du Val singularities, let $\widetilde{Z} \to Z$ be the partial minimal resolution of all Du Val singularities of Z, then $\widetilde{Z}$ is again a

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log Enriques surface of the same canonical index as Z. Therefore, we assume throughout this paper that Z has no Du Val singularities; otherwise we consider $\widetilde{Z}$ instead.

By the definition of the canonical cover and the classification result of surfaces, we have the following (cf. [8]).

1. $\bar{S}$ has at worst Du Val singularities, and its canonical divisor $K_{\bar{S}}$ is linearly equivalent to zero. So $\bar{S}$ is either an abelian surface or a projective K3 surface with at worst Du Val singularities.

2. $\pi : \bar{S} \to Z$ is a finite, cyclic Galois cover of degree $I = I(Z)$, and it is étale over $Z \setminus \operatorname{Sing} Z$.

3. $\operatorname{Gal}(\bar{S}/Z) \simeq \mathbb{Z}/I\mathbb{Z}$ acts faithfully on $H^0(\mathcal{O}_{\bar{S}}(K_{\bar{S}}))$. In other words, there is a generator g of $\operatorname{Gal}(\bar{S}/Z)$ such that $g^*\omega_{\bar{S}} = \zeta_I \omega_{\bar{S}}$, where ζ_I is the Ith primitive root of unity and $\omega_{\bar{S}}$ is a nowhere vanishing regular 2-form on $\bar{S}$.

Suppose $\operatorname{Sing} \bar{S} \neq \emptyset$. Let $\nu : S \to \bar{S}$ be the minimal resolution of $\bar{S}$, and Δ_S the exceptional divisor of ν. Then Δ_S is a disconnected sum of divisors of Dynkin's type:

$$(\oplus A_\alpha) \oplus (\oplus D_\beta) \oplus (\oplus E_\gamma)$$

Note that S is a K3 surface. The Chern map $c_1 : \operatorname{Pic}(S) \to H^2(S, \mathbb{Z})$ is injective. So $\operatorname{Pic}(S)$ is mapped isomorphically onto the Neron-Severi group $\operatorname{NS}(S)$. We can therefore define the *rank* of Δ_S to be the rank of the sublattice of the Néron Severi lattice $\operatorname{NS}(S) \simeq \operatorname{Pic}(S)$ generated by the irreducible components of Δ_S. In other words,

$$\operatorname{rank} \Delta_S = \sum \alpha + \sum \beta + \sum \gamma.$$

Moreover, let $\rho(S) := \operatorname{rank} \operatorname{Pic}(S)$ be the Picard number of S, then

$$\operatorname{rank} \Delta_S \leq \rho(S) - 1 \leq 20 - 1 = 19.$$

Since S is uniquely determined up to isomorphism, by abuse of language we also say Z is of type $(\oplus A_\alpha) \oplus (\oplus D_\beta) \oplus (\oplus E_\gamma)$, and call $\operatorname{rank} \Delta_S$ the *rank* of Z.

A rational log Enriques surface Z is called *extremal* if it is of rank 19, the maximal possible value 19. The extremal log Enriques surfaces are completely classified in [4]. In [3], the isomorphism classes of rational log Enriques surfaces of type A_{18} and D_{18} are determined. In this paper, we are going to classify all the rational log Enriques surfaces of rank 18 by proving the following theorem.

Main Theorem. *Let Z be a rational log Enriques surfaces of rank* 18 *without Du Val singularities. Let $\bar{S} \to Z$ be the canonical cover, and $S \to \bar{S}$ the minimal resolution with exceptional divisor Δ_S. Then we have the following assertions.*

1) *The canonical index $I(Z) = 2$, 3 or 4.*
2) *If $I(Z) = 2$, then $(S, g) \simeq (S_2, g_2)$, and Δ_S is of one of the following* 5 *types*:

$$A_1 \oplus A_{17}, \quad A_3 \oplus A_{15}, \quad A_5 \oplus A_{13}, \quad A_7 \oplus A_{11}, \quad A_9 \oplus A_9.$$

Moreover, all of them are realizable.

3) *If* $I(Z) = 3$, *then* $(S, g) \simeq (S_3, g_3)$, *and* Δ_S *is of one of the* 48 *possible types in Table 1, and from which* 40 *types have been realized.*
4) *If* $I(Z) = 4$, *then* $(S, g^2) \simeq (S_2, g_2)$, *and* Δ_S *is of one of the following 3 types*:
$$A_1 \oplus A_{17}, \quad A_5 \oplus A_{13}, \quad A_9 \oplus A_9.$$
Moreover, all of them are realizable.
5) *For each of the possible cases in* (2) *and* (3), *every irreducible curve in* Δ_S *is* g*-stable, and the action of* g *on* Δ *is uniquely determined, which are given in Table 2 and 1, respectively.*

Here (S_2, g_2) (*Definition 6*) *and* (S_3, g_3) (*Definition 3*) *are the Shioda-Inose's pairs of discriminants* 4 *and* 3 *respectively.*

2. Preliminaries

Definition 1. Let Z be a normal projective surface defined over the complex number field $\mathbb{C}$. It is called a *log Enriques surface* of *canonical index* I if

1) Z has at worst quotient singularities, and
2) IK_Z is linearly equivalent to zero for the minimum positive integer I, and
3) the irregularity $q(Z) := \dim H^1(Z, \mathcal{O}_Z) = 0$.

We will use the following notations in Section 3–4.

1. For each $I \in \mathbb{Z}^+$, $\zeta_I = \exp(2\pi\sqrt{-1}/I)$, a primitive Ith root of unity.
2. Let X be a variety, and G an automorphism group on X. For each $g \in X$, denote the fixed locus by $X^g = \{x \in X \mid g(x) = x\}$. Set $X^{[G]} = \bigcup_{g \in G \setminus \{\mathrm{id}\}} X^g$.
3. Let S be a surface and g an automorphism on S. A curve C on S is called *g-stable* if $g(C) = C$, and it is called *g-fixed* if $g(x) = x$ for every $x \in C$. A point $x \in S$ is an *isolated g-fixed point* if $g(x) = x$ and it is not contained in any g-fixed curve.

3. Log Enriques surfaces from Shioda-Inose's pairs

In this section, we assume that Z is a rational log Enriques surface of rank 18 and canonical index I without Du Val singularities. Let $\pi : \bar{S} \to Z$ be the canonical cover of Z, and $\nu : S \to \bar{S}$ the minimal resolution of $\bar{S}$ with exceptional divisor Δ_S. Then
$$20 \geq \rho(S) \geq \operatorname{rank} \Delta_S + 1 = 19.$$
Recall that S is a K3 surface. Let T_S denote the transcendental lattice of S, i.e., the orthogonal complement of $\operatorname{Pic}(S)$ in $H^2(S, \mathbb{Z})$. Then
$$\operatorname{rank} T_S = \dim H^2(S, \mathbb{Z}) - \rho(S) = 22 - \rho(S) = 2 \text{ or } 3.$$

Let g be the automorphism on S induced by a generator of $\operatorname{Gal}(\bar{S}/Z)$, and ω_S a nowhere vanishing holomorphic 2-form on S. Then $g^*\omega_S = \zeta_I \omega_S$. Note that $\omega_S \in T_S \otimes \mathbb{C}$. So ζ_I is an eigenvalue of g^* acting on T_S. Therefore, $\varphi(I) \leq \operatorname{rank} T_S \leq 3$, where φ is Euler's phi function. It follows that:

Lemma 2. *The canonical index* $I(Z) = 2, 3, 4$ *or* 6.

We have indicated that all the realizable rational log Enriques surfaces listed in Main Theorem can be constructed from the Shioda-Inose's pairs (S_2, g_2) or (S_3, g_3) (cf. [5]). Precisely, if $I(Z) = 2$, then $(S, g) \simeq (S_2, g_2)$; if $I(Z) = 3$, then $(S, g) \simeq (S_3, g_3)$; if $I(Z) = 4$, then $(S, g^2) \simeq (S_2, g_2)$; we will also show that $I \neq 6$.

Definition 3. Let $\zeta_3 := \exp(2\pi\sqrt{-1}/3)$, and $E_{\zeta_3} := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta_3)$ the elliptic curve of period ζ_3. Let $\bar{S}_3 := E_{\zeta_3}^2/\langle \operatorname{diag}(\zeta_3, \zeta_3^2)\rangle$ be the quotient surface, and $S_3 \to \bar{S}_3$ the minimal resolution of $\bar{S}_3$. Let g_3 be the automorphism of S_3 induced by the action $\operatorname{diag}(\zeta_3, 1)$ on $E_{\zeta_3}^2$. Then (S_3, g_3) is called the *Shioda-Inose's pair of discriminant* 3.

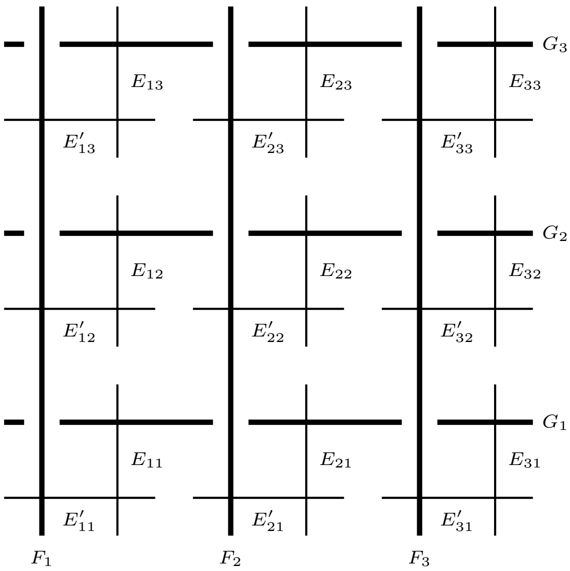

FIGURE 1. (S_3, g_3)

It is proved in [6] and [4] that:

Proposition 4. *Let* (S_3, g_3) *be the Shioda-Inose's pair of discriminant* 3. *Then*

1) S_3 *contains* 24 *rational curves*: F_1, F_2, F_3 *coming from* $(E_{\zeta_3})^{\zeta_3} \times E_{\zeta_3}$; G_1, G_2, G_3 *coming from* $E_{\zeta_3} \times (E_{\zeta_3})^{\zeta_3}$; *and* E_{ij}, E'_{ij} $(i, j = 1, 2, 3)$ *the exceptional curves arising from the* 9 *Du Val singular points of* $\bar{S}_3$ (*Figure. 1*);

2) $g_3^*\omega_{S_3} = \zeta_3\omega_3$, *where* ω_{S_3} *is a nowhere vanishing holomorphic* 2*-form on* S_3, *and* $g_3^*|_{\mathrm{Pic}(S_3)} = \mathrm{id}$; *so each of the* 24 *curves is* g_3*-stable*;
3) $S_3^{g_3} = (\coprod_{i=1}^3 F_i) \coprod (\coprod_{j=1}^3 G_j) \coprod (\coprod_{i,j=1}^3 \{P_{ij}\})$, *where* $\{P_{ij}\} = E_{ij} \cap E'_{ij}$;
4) $g_3 \circ \varphi = \varphi \circ g_3$ *for all* $\varphi \in \mathrm{Aut}(S_3)$.

Proposition 5. *Let* (S, g) *be a pair of a smooth K3 surface* S *and an automorphism of* g *on* S. *Assume that*

1) $g^3 = \mathrm{id}$, *the identity on* S;
2) $g^*\omega_S = \zeta_3\omega_S$, *where* ω_S *is a nowhere vanishing holomorphic* 2*-form on* S;
3) S^g *consists of only rational curves and isolated points*;
4) S^g *contains at least* 6 *rational curves.*

Then $(S, g) \simeq (S_3, g_3)$. *Moreover,* S^g *consists of exactly* 6 *rational curves and* 9 *isolated points.*

Definition 6. Let $E_{\zeta_4} := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$ be the elliptic curve of period $\zeta_4 = \sqrt{-1}$. Let $\bar{S}_2 := E_{\zeta_4}^2/\langle \mathrm{diag}(\zeta_4, \zeta_4^3)\rangle$ be the quotient surface and $S_2 \to \bar{S}_2$ the minimal resolution of $\bar{S}_2$. Let g_2 be the involution of S_2 induced by the action $\mathrm{diag}(-1, 1)$ on $E_{\zeta_4}^2$. Then (S_2, g_2) is called the *Shioda-Inose's pair of discriminant* 4.

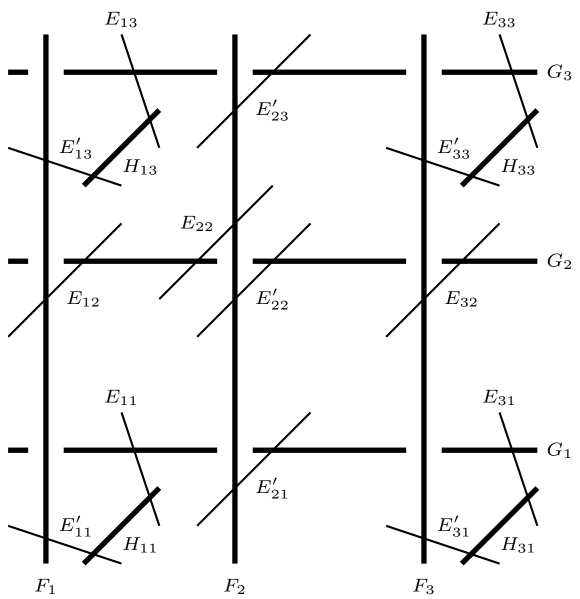

FIGURE 2. (S_2, g_2)

It is also proved in [6] and [4] that:

Proposition 7. *Let (S_2, g_2) be the Shioda-Inose's pair of discriminant* 4. *Then*

1) *S_2 contains* 24 *rational curves: F_1, F_2, F_3 coming from $(E_{\zeta_4})^{[\langle\zeta_4\rangle]} \times E_{\zeta_4}$; G_1, G_2, G_3 coming from $E_{\zeta_4} \times (E_{\zeta_4})^{[\langle\zeta_4\rangle]}$; and $E'_{ij} + H_{ij} + E_{ij}$, $i, j \in \{1, 3\}$, the exceptional curves arising from the* 4 *Du Val singular points of Dynkin type A_3; and $E_{12}, E_{22}, E_{32}, E'_{21}, E'_{22}, E'_{23}$, the exceptional curves arising from the* 6 *Du Val singular points of Dynkin type A_1* (*Figure. 2*);
2) *$g_2^* \omega_{S_2} = -\omega_{S_2}$, where ω_{S_2} is a nowhere vanishing holomorphic* 2*-form on S_2, and $g_2^*|_{\mathrm{Pic}(S)} = \mathrm{id}$; so each of the* 24 *curves is g_2-stable*;
3) $S_2^{g_2} = (\coprod_{i=1}^3 F_i) \coprod (\coprod_{j=1}^3 G_j) \coprod (\coprod_{i,j\in\{1,3\}} H_{ij})$;
4) *$g_2 \circ \varphi = \varphi \circ g_2$ for all $\varphi \in \mathrm{Aut}(S_2)$.*

Proposition 8. *Let (S, g) be a pair of a smooth K3 surface S and an automorphism g of S. Assume that*

1) *$g^2 = \mathrm{id}$, the identity on S*;
2) *$g^* \omega_S = -\omega_S$, where ω_S is a nowhere vanishing holomorphic* 2*-form on S*;
3) *S^g consists of only rational curves*;
4) *S^g contains at least* 10 *rational curves.*

Then $(S, g) \simeq (S_2, g_2)$. Moreover, S^g consists of exactly 10 *rational curves.*

4. The classification

In this section, we assume that Z is a log Enriques surface of rank 18 without Du Val singularities. Let $\pi : \bar{S} \to Z$ be the canonical cover, and $\nu : S \to \bar{S}$ the minimal resolution with exceptional divisor $\Delta := \Delta_S$. Since the canonical cover $\bar{S} \to Z$ is unramified in codimension one, every curve in $S^{[\langle g\rangle]}$ is contained in Δ. In particular, $S^{[\langle g\rangle]}$ consists of only smooth rational curves and a finite number of isolated points, and Δ is g-stable.

In general, let S be a K3 surface, and g an automorphism of S of order n. Let T_S be its transcendental lattice. Note that g induces actions g^* on $\mathrm{Pic}(S) \otimes \mathbb{C}$ and on $T_S \otimes \mathbb{C}$. Since $g^n = \mathrm{id}$, these actions are diagonalizable and every eigenvalue of g^* is an nth root of unity, say ζ_n^i for some $0 \le i < n$. Since g^* is well-defined on $\mathrm{Pic}(S)$ and T_S, the number of eigenvalues ζ_n^i of $g^*|_{\mathrm{Pic}(S)\otimes\mathbb{C}}$ and $g^*|_{T_S\otimes\mathbb{C}}$ equals to that of the conjugate eigenvalues $\bar{\zeta}_n^i$, respectively. By noting that $\dim H^2(S, \mathbb{C}) = 22$, we have the following lemma:

Lemma 9 ([6, Lemma 2.0])**.** *With the notations above, let t_0 and r_0 be the rank of the invariant lattices $(\mathrm{Pic}(S))^{g^*}$ and $(T_S)^{g^*}$, respectively. Let I_s denote the identity matrix of size s.*

1) *If $n = 2k+1$ is odd, then $\rho(S) = t_0 + 2\sum_{i=1}^{k} t_i$ and*

$$g^*|_{\mathrm{Pic}(S)\otimes\mathbb{C}} = \mathrm{diag}(I_{t_0}, \zeta_n I_{t_1}, \bar\zeta_n I_{t_1}, \zeta_n^2 I_{t_2}, \bar\zeta_n^2 I_{t_2}, \dots, \zeta_n^k I_{t_k}, \bar\zeta_n^k I_{t_k}),$$
$$g^*|_{T_S\otimes\mathbb{C}} = \mathrm{diag}(I_{r_0}, \zeta_n I_{r_1}, \bar\zeta_n I_{r_1}, \zeta_n^2 I_{r_2}, \bar\zeta_n^2 I_{r_2}, \dots, \zeta_n^k I_{r_k}, \bar\zeta_n^k I_{r_k}),$$

and $t_0 + r_0 + 2\sum_{i=1}^{k} t_i + 2\sum_{i=1}^{k} r_i = 22$.

2) *If $n = 2k$ is even, then $\rho(S) = t_0 + 2\sum_{i=1}^{k-1} t_i + t_k$ and*

$$g^*|_{\mathrm{Pic}(S)\otimes\mathbb{C}} = \mathrm{diag}(I_{t_0}, \zeta_n I_{t_1}, \bar\zeta_n I_{t_1}, \zeta_n^2 I_{t_2}, \bar\zeta_n^2 I_{t_2}, \dots, \zeta_n^{k-1} I_{t_{k-1}}, \bar\zeta_n^{k-1} I_{t_{k-1}}, -I_{t_k}),$$
$$g^*|_{T_S\otimes\mathbb{C}} = \mathrm{diag}(I_{r_0}, \zeta_n I_{r_1}, \bar\zeta_n I_{r_1}, \zeta_n^2 I_{r_2}, \bar\zeta_n^2 I_{r_2}, \dots, \zeta_n^{k-1} I_{r_{k-1}}, \bar\zeta_n^{k-1} I_{r_{k-1}}, -I_{r_k}),$$

and $t_0 + r_0 + 2\sum_{i=1}^{k-1} t_i + 2\sum_{i=1}^{k} r_i + t_k + r_k = 22$.

4.1. Classification when $I = 3$

Let (S, g) be a pair of smooth K3 surface S and an automorphism g of S. We assume that $g^*\omega_S = \zeta_3\omega_S$ for a nowhere vanishing holomorphic 2-form ω_S on S.

Let P be an isolated g-fixed point on S. Then g^* can be written as $\mathrm{diag}(\zeta_3^a, \zeta_3^b)$ for some $a, b \in \{1, 2\}$ with $a + b \equiv 1 \pmod 3$ under some appropriate local coordinates around P because $g^*\omega_S = \zeta_3\omega_S$. We see that $a = b = 2$ and the action is $\mathrm{diag}(\zeta_3^2, \zeta_3^2)$. If C is a g-fixed irreducible curve and $Q \in C$, then it also follows from $g^*\omega_S = \zeta_3\omega_S$ that g^* can be written as $\mathrm{diag}(1, \zeta_3)$ under some appropriate local coordinates around Q. In particular, the g-fixed curves are smooth and mutually disjoint.

We need to use the following lemma in the classification for $I = 3$.

Lemma 10 ("Three Go" Lemma, [6, Lemma 2.2])**.** *Let (S, g) be a pair of smooth K3 surface S and an automorphism g of S. Assume that $g^3 = \mathrm{id}$ and $g^*\omega_S = \zeta_3\omega_S$.*

1) *Let $C_1 - C_2 - C_3$ be a linear chain of g-stable smooth rational curves. Then exactly one of C_i is g-fixed.*
2) *Let C be a g-stable but not g-fixed smooth rational curve. Then there is a unique g-fixed curve D such that $C \cdot D = 1$.*
3) *Let M and N be the number of smooth rational curves and the number of isolated points in S^g, respectively. Then $M - N = 3$.*

Suppose $I(Z) = 3$. Then the associated pair (S, g) satisfies the conditions in Lemma 10. We first determine a possible list of the Dynkin's types of Δ.

Proposition 11. *With the notations as in Main Theorem, suppose $I(Z) = 3$. Then $(S, g) \simeq (S_3, g_3)$, the Shioda-Inose's pair of discriminant 3. Moreover, Δ is of one of the following 13 types:*

I. A_{18};

II. D_{18};

III. $A_{3m} \oplus A_{3n}, \quad m+n=6$;
IV. $D_{3m} \oplus A_{3n}, \quad m+n=6$;
V. $D_{3m} \oplus D_{3n}, \quad m+n=6$;
VI. $D_{3m+1} \oplus A_{3n-1}, \quad m+n=6$;
VII. $A_{3m} \oplus A_{3n} \oplus A_{3r}, \quad m+n+r=6$;
VIII. $D_6 \oplus D_6 \oplus D_6$;
IX. $A_{3m} \oplus D_{3n} \oplus D_{3r}, \quad m+n+r=6$;
X. $A_{3m} \oplus A_{3n} \oplus D_{3r}, \quad m+n+r=6$;
XI. $D_{3m+1} \oplus A_{3n} \oplus A_{3r-1}, \quad m+n+r=6$;
XII. $D_{3m+1} \oplus D_{3n+1} \oplus A_{3r-2}, \quad m+n+r=6$;
XIII. $D_{3m+1} \oplus D_{3n} \oplus A_{3r-1}, \quad m+n+r=6$.

Proof. Let Δ_i be a connected component of Δ.

Step 1: Δ_i is g-stable.
If Δ_i is not g-stable, then its image in Z would be a Du Val singular point since $I(Z)=3$ is a prime. However, we have assumed that Z has no Du Val singularities.

Step 2: $\Delta_i = A_n$ or D_n.
Suppose there is a $\Delta_i = E_n$ for some n. Let C be the center of Δ_i, and C_1, C_2, C_3 the rational curves in Δ_i which intersect C. Suppose C_1 is the twig of length one. By the uniqueness of C and C_1, they are g-stable. If C is not g-fixed, then $\Delta_i = E_6$ and g switches the other two twigs, which contradicts $g^3 = \mathrm{id}$. If C is g-fixed, then each irreducible curve in Δ_i is g-stable. Let $C_2 - C_2'$ be a twig of Δ_i. Then C_2' is not g-fixed and it does not intersect with any g-fixed curve, which contradicts Lemma 10.

Step 3. Every irreducible curve in Δ_i is g-stable.
i) Let $\Delta_i = A_n$. Write the irreducible curves in Δ_i as a chain $C_1 - C_2 - \cdots - C_n$. For $n > 1$, if C_1 is not g-stable, we must have $g(C_1) = C_n$ and $g(C_n) = g(C_1)$, and this contradicts $g^3 = \mathrm{id}$.
ii) Let $\Delta_i = D_n$. Then by the uniqueness its center C is g-stable. Let C_1 and C_2 be twigs of length one, and C_3 the curve of another twig which intersects C.

Suppose $n > 4$. Then every irreducible component in the longest twig shall be g-stable. If C_1 is not g-stable, then $g(C_1) = C_2$ and $g(C_2) = C_1$, which contradicts $g^3 = \mathrm{id}$. Thus, every irreducible curve in Δ_i is g-stable. Suppose $n = 4$. If C_1 is not g-stable, we must have $g(C_1) = C_2$, $g(C_2) = C_3$ and $g(C_3) = g(C_1)$. In particular, C is not g-fixed, and it does not intersect with any g-fixed curve. This contradicts Lemma 10. Therefore, C_1 is g-stable. We see similarly as in the case $n > 4$ that C_2 and C_3 are both g-stable.

Step 4. The g-fixed curves of Δ_i are described as follows.

We use "f" to denote g-fixed curves, and "s" to denote g-stable but not g-fixed curves in Δ_i. k is the number of g-fixed curves in Δ_i.

i) Suppose $\Delta_i = A_n$.

a) $n = 3k - 2$:

$$f - s - s - f - s - \cdots - s - s - f$$

b) $n = 3k - 1$:

$$f - s - s - f - s - \cdots - s - f - s$$

c) $n = 3k$:

$$s - f - s - s - f - \cdots - s - f - s$$

ii) Suppose $\Delta_i = D_n$.

a) $n = 3k$:

$$\begin{array}{c} \phantom{s - {}} s \\ \phantom{s - {}} | \\ s - f - s - s - f - \cdots - s - s - f \end{array}$$

b) $n = 3k + 1$:

$$\begin{array}{c} \phantom{s - {}} s \\ \phantom{s - {}} | \\ s - f - s - s - f - \cdots - s - f - s \end{array}$$

The case $\Delta_i = A_n$ follows from Lemma 10. Suppose $\Delta_i = D_n$. Then by Step 3, the center C is g-fixed. So in the longest twig $C_3 - C_4 - \cdots - C_{n-1}$ of Δ_i, by induction, C_{3j+2} are g-fixed and others are not. If $n = 3k + 2$ for some k, then C_{n-2} and C_{n-1} are not g-fixed, and C_{n-1} does not intersect with any g-fixed curve, a contradiction to Lemma 10. Therefore, $n \not\equiv 2 \pmod 3$.

Step 5. $(S, g) \simeq (S_3, g_3)$.

Let M be the number of isolated g-fixed points and N the number of g-fixed curves in Δ. We can decompose

$$\Delta = \bigoplus_{i=1}^{a} D_{3\ell_i+1} \oplus \bigoplus_{i=1}^{b} D_{3m_i} \oplus \bigoplus_{i=1}^{c} A_{3p_i} \oplus \bigoplus_{i=1}^{d} A_{3q_i-1} \oplus \bigoplus_{i=1}^{e} A_{3r_i-2}.$$

Then

$$\begin{aligned} N &= \sum_{i=1}^{a} \ell_i + \sum_{i=1}^{b} m_i + \sum_{i=1}^{c} p_i + \sum_{i=1}^{d} q_i + \sum_{i=1}^{e} r_i, \\ M &\geq \sum_{i=1}^{a} (\ell_i + 2) + \sum_{i=1}^{b} (m_i + 1) + \sum_{i=1}^{c} (p_i + 1) + \sum_{i=1}^{d} q_i + \sum_{i=1}^{e} (r_i - 1) \\ &= N + (2a + b + c - e). \end{aligned}$$

Thus, by Lemma 10, $3 = M - N \geq 2a + b + c - e$. Recall that

$$\begin{aligned}\operatorname{rank}\Delta = 18 &= \sum_{i=1}^{a}(3\ell_i + 1) + \sum_{i=1}^{b} 3m_i + \sum_{i=1}^{c} 3p_i + \sum_{i=1}^{d}(3q_i - 1) + \sum_{i=1}^{e}(3r_i - 2) \\ &= 3N + a - d - 2e.\end{aligned}$$

Or equivalently, $N = 6 + \frac{-a+d+2e}{3}$. If $N \leq 5$, then $a \geq d + 2e + 3$, and we would have

$$3 \geq 2a + b + c - e \geq 2(d + 2e + 3) + b + c - e = b + c + 2d + 3e + 6 \geq 6.$$

Therefore, $N \geq 6$; and hence by Proposition 5, $N = 6$ and $M = 9$. Furthermore, we have $(S, g) \simeq (S_3, g_3)$.

Step 6. Determine the Dynkin's type of Δ.

Solving the system

$$d + 2e = a \quad \text{and} \quad 2a + b + c - e \leq 3,$$

we have 13 nonnegative integer solutions. So there are 13 types of Δ as listed in Proposition 11. □

To be more precise, we list all the 48 possible types of Δ in Table 1 in Section 5. Note that in Steps 3 and 4, we proved that each irreducible curve in Δ g-stable, and the action of g on Δ is uniquely determined, which is also included in Table 1. The case $I = 3$ for Main Theorem (5) is proved.

If Δ can be obtained from the 24 g-stable rational curves in S_3 (Figure 1) which contains the 6 g-fixed curves and satisfies the condition in the proof of Proposition 11 Step 4, let $S_3 \to \bar{S}$ be the contraction of Δ, then the automorphism g_3 on S_3 induces an automorphism on $\bar{S}$. We see that $Z = \bar{S}/\langle g_3 \rangle$ is a required log Enriques surface of type Δ. By verification, 40 cases are realizable. The detailed list is given in Table 1(A). Thus, we have completed the proof of Main Theorem (3).

Unfortunately, the remaining 8 cases are not realizable by the 24 curves on S_3, which are given in Table 1(B). We are unable to determine their realizability.

4.2. Classification when $I = 2$

Let (S, g) be a pair of a smooth K3 surface S and an automorphism g of S. We assume that $g^*\omega_S = -\omega_S$ for a nowhere vanishing holomorphic 2-form ω_S on S.

If $P \in S$ is an isolated g-fixed point, then g^* can be written as $\operatorname{diag}(-1, -1)$ under some appropriate local coordinates around P. However, this contradicts the assumption that $g^*\omega_S = -\omega_S$. So S has no isolated g-fixed point. Let C be a g-fixed irreducible curve and let $Q \in C$. Then g^* can be written as $\operatorname{diag}(1, -1)$ under some appropriate local coordinates around Q. So the g-fixed curves are smooth and mutually disjoint.

We need to use the following lemma in the classification.

Lemma 12 ("Two Go" Lemma, [6, Lemma 3.2])**.** *Let (S,g) be a pair of smooth K3 surface and an automorphism g of S. Assume that $g^2 = \mathrm{id}$ and $g^*\omega_S = -\omega_S$.*

1) *If $C_1 - C_2$ is a linear chain of g-stable smooth rational curves, then exactly one of C_i is g-fixed.*
2) *If C_1 and C_2 are g-stable but not g-fixed smooth rational curves, then $C_1 \cdot C_2$ is even.*
3) *If C is a g-stable but not g-fixed smooth rational curve, then C has exactly 2 g-fixed points.*

Suppose $I(Z) = 2$. Then the associated pair satisfies the conditions in Lemma 12. We can now determine the possible Dynkin's types of (S,g).

Proposition 13. *With the notations as in Main Theorem. Suppose $I = 2$. Then $(S,g) \simeq (S_2, g_2)$, the Shioda-Inose's pair of discriminant 4. Moreover, Δ is of the type $A_{2m-1} \oplus A_{2n-1}$, where $m+n = 10$.*

Proof. Since $I = 2$ is a prime, each connected component Δ_i of Δ must be g-stable because Z is assumed to have no Du Val singular points.

Step 1. $\Delta_i = A_n$.

Suppose $\Delta_i = D_n$ or E_n. Let C be the center of Δ_i. Then C meets exactly 3 smooth rational curves in Δ_i, say C_1, C_2, C_3. By uniqueness, C is g-stable, and $g(\{C_1, C_2, C_3\}) = \{C_1, C_2, C_3\}$.

If every C_j is g-stable, then C has at least 3 g-fixed points, and it is g-fixed. Hence, C_j are not g-fixed. On the other hand, each C_j contains two g-fixed points, and one of them is not in C. There would be another g-fixed curve C_j' in Δ_i which intersects C_j, $j = 1,2,3$, a contradiction. Suppose C_1 is not g-stable, say $g(C_1) = C_2$. Then $g(C_2) = C_1$ and C is not g-fixed. Since C_3 is g-stable, by Lemma 12 it is also g-fixed. However, one of the two g-fixed points on C is not contained in C_3, so C should intersect with another g-fixed curve in Δ_i, a contradiction again.

Therefore, we can express $\Delta_i = A_n$ as a linear chain of smooth rational curves: $C_1 - C_2 - \cdots - C_n$.

Step 2. Each C_j is g-stable.

Suppose $g(C_1) \neq C_1$. Then $g(C_1) = C_n$, and consequently $g(C_j) = C_{n-j}$ for all j. There are two cases.

i) If $m = 2k$, let $\{P\} = C_k \cap C_{k+1}$, then P would be an isolated g-fixed point, absurd.

ii) If $m = 2k-1$, then C_k is g-stable, and there would be a g-fixed curve which intersects C_k. But Δ_i contains no g-fixed curve, a contradiction.

Therefore, $g(C_1) = C_1$ and it follows that each C_j is g-stable.

Step 3. $\Delta_i = A_{2m-1}$.

Note that each g-stable but not g-fixed curve must intersect g-fixed curves at two points. So C_1 is g-fixed and C_2 is not. Consequently, each C_{2j-1} is

g-fixed and C_{2j} is not. With the same reason, C_n must be g-fixed. So n is odd. Therefore, $\Delta_i = A_n$ has the form

$$f - s - f - s - f - \cdots - f - s - f$$

where "f" denotes the g-fixed curves and "s" denotes the g-stable but not g-fixed curves in Δ_i.

Step 4. Determine the Dynkin type of Δ.

Decompose $\Delta = \bigoplus_{i=1}^{r} A_{2n_i - 1}$. Recall that every smooth rational g-fixed curve in S is contained in Δ. Let N be the number of smooth rational g-fixed curves in S. Then $N = \sum_{i=1}^{r} n_i$ and

$$18 = \operatorname{rank} \Delta = \sum_{i=1}^{r} (2n_i - 1) = 2N - r.$$

So we have

$$N = \frac{18 + r}{2} > 9.$$

Then $N \geq 10$. It follows from Proposition 8 that $N = 10$ and $(S, g) \simeq (S_2, g_2)$. Moreover, $r = 2$. This completes the proof. □

We have the following configurations for Δ:

$$A_1 \oplus A_{17}, \quad A_3 \oplus A_{15}, \quad A_5 \oplus A_{13}, \quad A_7 \oplus A_{11}, \quad A_9 \oplus A_9.$$

Similarly as in the case when $I = 3$, if $S_2^g \subseteq \Delta$ and the divisor Δ can be obtained from the 24 smooth rational curves in S_2 (Figure 2) which satisfies the conditions in the proof of Proposition 13 Step 3, let $S_2 \to \bar{S}$ be the contraction of Δ, then the automorphism g_2 on S_2 induces an automorphism on $\bar{S}$, and $Z := \bar{S}/\langle g_2 \rangle$ is a required log Enriques surface of Dynkin's type Δ.

We can easily verify that these 5 cases are all realizable (cf. Table 2). We have proved Main Theorem (2). By noting the results in Steps 2 and 3 in the proof of Proposition 13, Main Theorem (5) for case $I = 2$ is also proved.

4.3. Classification when $I = 4$

Let (S, g) be a pair of a smooth K3 surface S and an automorphism g of S. Assume that $g^4 = \mathrm{id}$ and $g^*\omega_S = i\omega_S$ for a nowhere vanishing holomorphic 2-form on S, where $i = \sqrt{-1}$. Let P be an isolated g-fixed point. Then g^* can be written as $\mathrm{diag}(-1, -i)$ near P with appropriate coordinates. Let C be a g-fixed irreducible curve and Q a point in C. Then g^* can be written as $\mathrm{diag}(1, i)$ near Q with appropriate coordinates.

Similarly as in the case $I = 2$ (Lemma 12) or $I = 3$ (Lemma 10), we can state and prove the following lemma.

Lemma 14 ("Four Go" Lemma). *Let (S, g) be a pair of smooth K3 surface S and an automorphism g of S. Assume that $g^4 = \mathrm{id}$ and $g^*\omega_S = i\omega_S$.*

1) *Let $C_1 - C_2 - C_3 - C_4$ be a chain of g-stable smooth rational curves. Then exactly one of C_j is g-fixed, and exactly one of C_k is g^2-fixed but not g-fixed. Moreover, $\{j,k\} = \{1,3\}$ or $\{2,4\}$.*
2) *Let C be a g-stable but not h-fixed smooth rational curve on S. Then there exists a unique g-fixed curve D_1 and a unique g^2-fixed but not g-fixed curve D_2 such that $C \cdot D_1 = C \cdot D_2 = 1$.*
3) *Let M and N be the number of smooth rational curves and the number of isolated points in S^g, respectively. Then $M - 2N = 4$.*

Proof. 1) Applying Lemma 12 to $h := g^2$, we may assume that C_1, C_3 are h-fixed and C_2, C_4 are not. Note that $\{P\} = C_1 \cap C_2$ and $\{Q\} = C_2 \cap C_3$ are g-fixed. The action of g on the tangent space $T_{C_2,P}$ of C_2 at P is the multiplicative of i or $-i$, and the action of g on $T_{C_2,Q}$ is the multiplicative of $-i$ or i, respectively. For the first case, C_1 is g-fixed and C_3 not; and conversely for the second case.

2) Let P and Q be the g-fixed points on C. Then the actions of g on $T_{C,P}$ and $T_{C,Q}$ are the multiplication of i and $-i$, respectively. So there is a unique g-fixed curve passing through P and a unique h-fixed but not g-fixed curve passing through Q.

3) We can write

$$S^g = \{P_1\} \coprod \cdots \coprod \{P_M\} \coprod C_1 \coprod \cdots \coprod C_N,$$

where P_j are the isolated g-fixed points, and C_k are the smooth irreducible rational g-fixed curves of S. Consider the holomorphic Lefschetz number $L(g)$, which can be evaluated in two different ways.

Method 1. $L(g) = \displaystyle\sum_{i=0}^{2} (-1)^i \operatorname{tr}(g^*|_{H^i(S,\mathcal{O}_S)})$ (cf. [1, §3]).

We see that $H^0(S, \mathcal{O}_S) \simeq \mathbb{C}$, $H^1(S, \mathcal{O}_S) = 0$, and by Serre duality

$$H^2(S, \mathcal{O}_S) \simeq H^0(S, \mathcal{O}_S(K_S))^\vee = H^0(S, \mathcal{O}_S)^\vee.$$

Then $g^*|_{H^0(S,\mathcal{O}_S)} = \mathrm{id}$, $g^*|_{H^1(S,\mathcal{O}_S)} = 0$ and $g^*|_{H^2(S,\mathcal{O}_S)} = i^{-1} = -i$.

Method 2. $L(g) = \displaystyle\sum_{j=1}^{M} a(P_j) + \sum_{k=1}^{N} b(C_k)$.

$$a(P_j) := \frac{1}{\det(1 - g^*|_{T_{P_j}})},$$

$$b(C_k) := \frac{1 - \pi(C_k)}{1 - \lambda_k^{-1}} - \frac{\lambda_k^{-1}}{(1 - \lambda_k^{-1})^2}(C_k)^2,$$

where $\pi(C_k)$ is the genus and $(C_k)^2$ is the self-intersection number of C_k, and λ_k is the eigenvalue of g^* on the normal bundle of C_k (cf. [2, §4]).

Recall that $g^*|_{T_{P_j}} = \operatorname{diag}(-1, -i)$. Then

$$a(P_j) = \frac{1}{(1+1)(1+i)} = \frac{1-i}{4}.$$

Since $g^*|_{T_{Q_k}} = \operatorname{diag}(1, i)$ near $Q_k \in C_k$, $\lambda_k = i^{-1}$ is the eigenvalue of g^* on the normal bundle. So

$$b(C_k) = \frac{1-0}{1-i} - \frac{i}{(1-i)^2}(-2) = -\frac{1-i}{2}.$$

Therefore, $1 - i = \dfrac{M}{4}(1-i) - \dfrac{N}{2}(1-i)$; that is, $M - 2N = 4$. □

Now suppose $I(Z) = 4$. Then the associated pair (S, g) satisfies the conditions in Lemmas 9 and 14. Set $h := g^2$. First of all, we claim that:

Lemma 15. *With the notations as in Main Theorem and above, each connected component Δ_i of Δ is h-stable.*

Proof. Suppose Δ_i is not h-stable. Then Δ_i, $g(\Delta_i)$, $h(\Delta_i)$ and $g^3(\Delta_i)$ are distinct components in Δ, and they are contracted to Du Val singular points on $\bar{S}/\langle g \rangle$, a contradiction to our assumption. □

Therefore, applying Proposition 8 to (S, h) we have $(S, h) \simeq (S_2, g_2)$, the Shioda-Inose's pair of discriminant 4. From now on, we set $(S, h) = (S_2, g_2)$. Since is known that $(g_2^*)^2 = \mathrm{id}$ on $\operatorname{Pic}(S)$, we can write $g^*|_{\operatorname{Pic}(S)\otimes\mathbb{C}} = \operatorname{diag}(I_s, -I_t)$, where $s + t = \rho(S) = 20$. Let $x \in T_S$. Suppose $g^*x = \pm x$. Then

$$x \cdot \omega_S = g^*(x \cdot \omega_S) = g^*x \cdot g^*\omega_S = \pm x \cdot i\omega_S = \pm i(x \cdot \omega_S).$$

It follows that $x \cdot \omega_S = 0$. Then $x \in \operatorname{Pic}(S) \cap T_S = \{0\}$. So ± 1 are not eigenvalues of $g^*|_{T_S\otimes\mathbb{C}}$. By Lemma 9, we can thus write $g^*|_{T_S\otimes\mathbb{C}} = \operatorname{diag}(i, -i)$.

Proposition 16. *With the notations as in Main Theorem. Suppose $I = 4$. Let $h = g^2$. Then $(S, h) \simeq (S_2, g_2)$, the Shioda-Inose's pair of discriminant 4. Moreover, Δ is of the type $A_1 \oplus A_{17}$, $A_5 \oplus A_{13}$ or $A_9 \oplus A_9$.*

Proof. We only need to check the second assertion. Let M be the number of isolated g-fixed points and N the number of smooth irreducible g-fixed curves. By Lemma 14, we have $M - 2N = 4$.

Step 1. $N \leq 4$.

We apply the topological Lefschetz fixed point theorem (cf. [7, Lemma 1.6]),

$$\chi_{\mathrm{top}}(S^g) = \sum_{i=0}^{4} (-1)^i \operatorname{tr}(g^*|_{H^i(S,\mathbb{Q})}).$$

The left-hand side is $M + 2N = 4N + 4$, and the right-hand side is

$$2 + \operatorname{tr}(g^*|_{\operatorname{Pic}(S)\otimes\mathbb{C}}) + \operatorname{tr}(g^*|_{T_S\otimes\mathbb{C}}) = 2 + s - t.$$

where $g^*|_{\mathrm{Pic}(S)\otimes\mathbb{C}} = \mathrm{diag}(I_s, -I_t)$. Since $s + t = \rho(S) = 20$, we have

$$s = 11 + 2N \quad \text{and} \quad t = 9 - 2N.$$

It follows that $N \leq 4$.

Step 2. $\Delta = A_{2m-1} \oplus A_{2n-1}$, where $m + n = 10$.
This follows immediately from Proposition 13.

Step 3. $\Delta \neq A_3 \oplus A_{15}$ and $\Delta \neq A_7 \oplus A_{11}$. So Proposition 16 will follow.

i) Suppose $\Delta = A_3 \oplus A_{15}$. Denote $A_3 = C_1 - C_2 - C_3$ and $A_{15} = D_1 - D_2 - \cdots - D_{15}$. Then it follows from the proof of Proposition 13 that all C_i and D_j are h-stable, and from which

$$C_1, C_3, D_1, D_3, D_5, D_7, D_9, D_{11}, D_{13}, D_{15}$$

are h-fixed and others are not. Clearly each connected component is g-stable, and $\mathrm{Aut}(\Delta) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. Note that $g(C_1) = C_1$ or C_3. For each case C_2 is g-stable but not h-fixed. By Lemma 14, C_2 intersects with a unique g-fixed curve. Then C_1 or C_3 is g-stable, and therefore all C_i are g-stable. Similarly, by noting that D_8 is g-stable but not h-fixed, we see that all D_j are g-stable. By Lemma 14 again, $C_1, D_1, D_5, D_9, D_{13}$ must be g-fixed. But this contradicts $N \leq 4$.

ii) Suppose $\Delta = A_7 \oplus A_{11}$. Denote $A_7 = C_1 - C_2 - \cdots - C_7$ and $A_{11} = D_1 - D_2 - \cdots - D_{11}$. Then using the same argument as for $A_3 \oplus A_{15}$, we can show that C_i and D_j are g-stable for all i, j, and therefore C_1, C_5, D_1, D_5, D_9 are g-fixed. This contradicts $N \leq 4$ again. □

Proof of Main Theorem (4). It remains to show that $A_1 \oplus A_{17}$, $A_5 \oplus A_{13}$ and $A_9 \oplus A_9$ are realizable.

Let g_4 be the automorphism of S_2 induced by the action $\mathrm{diag}(i, 1)$ on $E^2_{\zeta_4}$. Then $g_4^2 = g_2$ as in Definition 6. From the construction of the 24 rational curves in S_2 (Figure 2), we see that

I) 4 curves are g_4-fixed, say F_1, F_2 and G_1, G_3;

II) 6 curves are g_2-fixed but not g_4-fixed, say $F_2, G_2, H_{11}, H_{13}, H_{31}, H_{33}$;

III) $g_4(H_{22}) = H'_{22}$ and $g_4(H'_{22}) = H_{22}$;

IV) the remaining 12 curves are g_4-stable, but not g_2-fixed.

Let $g := g_4$ and $h := g^2$. Then Δ contains exactly 4 g-fixed curves (i.e., $N = 4$), and 6 h-fixed but not g-fixed curves. Consider the following three possible types of Δ.

i) $A_1 \oplus A_{17}$.

Since A_1 contains at most 1 g-fixed curve, A_{17} must contain at least 3 g-fixed curves. Then every curve in A_{17} is g-stable. Moreover, it contains 9 h-fixed curves. Noting that Δ has exactly 4 g-fixed curves, we see that C_3, C_7, C_{11}, C_{15} are the g-fixed curves and $C_1, C_5, C_9, C_{13}, C_{17}, A_1$ are the h-fixed but not g-fixed curves.

ii) $A_5 \oplus A_{13}$.

Since A_5 contains at most 2 g-fixed curves, A_{13} has a g-fixed curve. So every curve in A_{13} is g-stable. We write

$$A_5 = C_1 - C_2 - C_3 - C_4 - C_5,$$
$$A_{13} = D_1 - D_2 - D_3 - \cdots - D_{13}.$$

If C_1 is not g-stable, then only C_3 in A_5 is h-fixed. Note that it is not g-fixed. Then A_{13} shall contain 4 g-fixed curves: D_1, D_5, D_9, D_{13}. However, Δ would have only 5 h-fixed but not g-fixed curves $D_3, D_7, D_{11}, D_{15}, C_3$, a contradiction. Therefore, every curve in A_5 is g-stable. Then A_5 contains at least 1 g-fixed curve, and A_{13} contains at most 3 g-fixed curves. It follows that exactly 4 curves C_3, D_3, D_7, D_{11} in Δ are g-fixed.

iii) $A_9 \oplus A_9$.

We call the second A_9 as A_9'. If A_9 is not g-stable, then $g(A_9) = A_9'$ and $g(A_9') = A_9$. There would be no g-fixed curve in Δ, absurd. So both A_9 and A_9' are g-stable. Since A_9 contains at most 3 g-fixed curves, A_9' contains at least 1 g-fixed curve. Hence every curve in A_9' is g-stable. Similarly, every curve in A_9 is g-stable. On the other hand, A_9 should contain at least 2 g-fixed curves, so does A_9'. If we write

$$A_9 = C_1 - C_2 - C_3 - \cdots - C_9,$$
$$A_9' = D_1 - D_2 - D_3 - \cdots - D_9,$$

then exactly C_3, C_7, D_3 and D_7 are g-fixed.

Since we have determined the action of g on Δ and these Δ can be obtained from the 22 g-stable rational curves in S_2 (Figure 2), they are all realizable. The dual graphs are given in Table 2 (1), (3) and (5). □

Note that in the proof of above, we showed that for each of the every cases, every irreducible curve in Δ is g-stable.

4.4. Impossibility of $I = 6$

In order to complete the proof of Main Theorem, in this section we will explore the method used in [4, Proposition 2.12, Lemma 2.13] to prove the following.

Proposition 17. *With the notations in Main Theorem, $I \neq 6$.*

Proof. We assume that there is a log Enriques surface Z of rank 18 without Du Val singularities. Let (S, g) be the associated pair. Let P be an isolated g-fixed point. Then g^* can be written as either

i) $\mathrm{diag}(\zeta_6^2, \zeta_6^5)$, or
ii) $\mathrm{diag}(\zeta_6^3, \zeta_6^4)$

with appropriate coordinates around P.

Step 1. There are even number of isolated g-fixed points of the second type.

Suppose $g^* = \mathrm{diag}(\zeta_6^2, \zeta_6^5)$ near P. Then $(g^2)^* = (\zeta_6^4, \zeta_6^4)$ near P. It follows that P is an isolated g^2-fixed point. Suppose $g^* = \mathrm{diag}(\zeta_6^3, \zeta_6^4)$ near P. Then $(g^2)^* = \mathrm{diag}(1, \zeta_6^2)$, and there exists a unique smooth rational g^2-fixed curve C passing through P. Since S^{g^2} is smooth, C is g-stable but not g-fixed. Let Q be the other g-fixed point on C. Since Q is not an isolated g^2-fixed point, it is also an isolated g-fixed point of the second type. Therefore, the g-fixed points of the second type come in pairs. There are even number of such points.

Step 2. The number of isolated g-fixed points of the first type equals that of the second type.

Let P be an isolated g-fixed point. Since $S^g \subseteq S^{g^3}$, a disjoint union of smooth rational curves, there is a unique g^3-fixed curve C passing through P. Hence, C is g-stable but not g-fixed, and it contains exactly 2 g-fixed points. Note that if P is of the first type $\mathrm{diag}(\zeta_6^2, \zeta_6^5)$, then $g^*|_{T_{C,P}} = \zeta_6^2$; if P is of the second type $\mathrm{diag}(\zeta_6^3, \zeta_6^4)$, then $g^*|_{T_{C,P}} = \zeta_6^4$. So the other isolated g-fixed point on C is of different type of P. Therefore, there is a one-to-one correspondence between the set of g-fixed points of the first type and that of the second type. Step 2 is proved.

Now we can set $P_1, \dots, P_{2\ell}$ and $Q_1, \dots, Q_{2\ell}$ to be the isolated S^g-fixed points of type $\mathrm{diag}(\zeta_6^2, \zeta_6^5)$ and of type $\mathrm{diag}(\zeta_6^3, \zeta_6^4)$, respectively. Suppose there are c rational smooth g-fixed curves, say $C_1, \dots, C_c$. We claim that

Step 3. $\ell = c + 1$.

Similarly as in the proof of Lemma 14, we use the holomorphic Lefschetz fixed point formula

$$L(g) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(g^*|_{H^i(S,\mathcal{O}_S)}) = \sum_{i=1}^{2\ell} a(P_i) + \sum_{i=1}^{2\ell} a(Q_i) + \sum_{i=1}^{c} b(C_i).$$

We can compute that

$$\sum_{i=0}^{2} (-1)^i \operatorname{tr}(g^*|_{H^i(S,\mathcal{O}_S)}) = 1 + 0 + \frac{1}{\zeta_6} = \frac{3 - i\sqrt{3}}{2},$$

$$a(P_i) = \frac{1}{\det(1 - g^*|_{T_{P_i}})} = \frac{1}{(1 - \zeta_6^2)(1 - \zeta_6^5)} = \frac{3 - i\sqrt{3}}{6},$$

$$a(Q_i) = \frac{1}{\det(1 - g^*|_{T_{Q_i}})} = \frac{1}{(1 - \zeta_6^3)(1 - \zeta_6^4)} = \frac{3 - i\sqrt{3}}{12},$$

$$b(C_i) = \frac{1 - \pi(C_i)}{1 - \zeta_6} - \frac{\zeta_6 C_i^2}{(1 - \zeta_6)^2} = -\frac{3 - i\sqrt{3}}{2}.$$

Therefore, $\ell = c + 1$.

Step 4. Determine S^{g^2}.

If P is a g^2-fixed but not g-fixed point, then so is $g(P)$. Therefore, there are even number of g^2-fixed but not g-fixed points. If C is a rational smooth irreducible g^2-fixed curve which does not contain any g-fixed point, so is $g(C)$. Hence, there are even number of such curves.

Suppose the isolated g^2-fixed points are $P_1, \dots, P_{2c+2}, R_1, \dots, R_{2k}$, and the smooth rational g^2-fixed curves are $C_1, \dots, C_c, D_1, \dots, D_{c+1}, \dots, F_1, \dots, F_{2p}$, where R_i is not g-fixed, $Q_{2i-1}, Q_{2i} \in D_i$, and F_i does not contain at g-fixed point. Then apply Lemma 10 to (S, g^2), we obtain

$$(2c+2+2k) - (c+c+1+2p) = 3,$$

which implies $k = p+1$.

Step 5. Determine S^{g^3}.

We note g^3 is a non-symplectic involution on S, and so there is no isolated g^3-fixed point. If G is a g^3-fixed curve which does not contain any g-fixed point, then so are $g(G)$ and $g^2(G)$. Therefore, the smooth rational g^3-fixed curves are $C_1, \dots, C_c$, $E_1, \dots, E_{2c+2}$, $G_1, \dots, G_{3q}$, where $P_i, Q_i \in E_i$ and G_i does not contain any g-fixed point.

Step 6. $c + p + q \leq 2$.

Since $\operatorname{ord}(g) = 6$, we can write

$$g^*|_{H^2(S,\mathbb{Q})} = \operatorname{diag}(I_\alpha, -I_\beta, \zeta_6^2 I_\gamma, \bar{\zeta}_6^2 I_\gamma, \zeta_6 I_{1+\delta}, \bar{\zeta}_6 I_{1+\delta}),$$

where $\alpha, \beta, \gamma, \delta \geq 0$. Let $j = 1$ in the topological Lefschetz fixed point formula

$$\chi_{\mathrm{top}}(S^{g^j}) = \sum_{i=0}^{4} (-1)^i \operatorname{tr}((g^j)^*|_{H^i(S,\mathbb{Q})}).$$

We have

$$(2c+2) + (2c+2) + 2 \cdot c = 2 + \alpha - \beta - \gamma + (\delta + 1).$$

$(g^2)^*|_{H^2(S,\mathbb{Q})} = \operatorname{diag}(I_{\alpha+\beta}, \zeta_6^2 I_{\gamma+\delta+1}, \bar{\zeta}_6^2 I_{\gamma+\delta+1})$. Then for $j = 2$ we have

$$(2c+2) + (2p+2) + 2[c + (c+1) + 2p] = 2 + (\alpha+\beta) - (\gamma+\delta+1).$$

$(g^3)^*|_{H^2(S,\mathbb{Q})} = \operatorname{diag}(I_{\alpha+2\gamma}, -I_{\beta+2+2\delta})$. Then for $j = 3$ we have

$$2[c + (2c+2) + 3q] = 2 + (\alpha + 2\gamma) - (\beta + 2 + 2\delta).$$

We also note that

$$\alpha + \beta + 2\gamma + 2(1+\delta) = \dim H^2(S, \mathbb{Q}) = 22.$$

It can be solved that $\delta = -c - p - q + 2$. In particular, $c + p + q \leq 2$.

Step 7. Determine the possible types of Δ.

Let Δ_i be a connected component of Δ. Then Δ_i is either g^3-stable or g^2-stable, otherwise $g^k(\Delta_i)$, $k = 0, \dots, 5$, would be contracted to a single Du Val singular point in $\bar{S}/\langle g \rangle$, which should not exist by assumption.

Suppose Δ_i, $i = 1, \dots, m$, are the g^3-stable connected components of Δ. Since $(g^3)^*\omega_S = -\omega_S$, using the same argument as for $I = 2$, we see that $\Delta_i = A_{2m_i-1}$ for some m_i, which contains exactly m_i smooth rational g^3-fixed curves. On the other hand, by computation in Step 4, there are $c+(2c+c)+3q = 3(c+q)+2$ g-fixed curves. Therefore,

$$\sum_{i=1}^{n} \operatorname{rank} \Delta_i = \sum_{i=1}^{m} (2m_i - 1) = 6(c+q) + 4 - m.$$

Since $\ell = c + 1 > 0$, $S^g \neq \emptyset$. We see that $m \geq 1$.

Suppose Δ'_j, $j = 1, \dots, n$, are the g^2-stable but not g-stable connected components of Δ. Since $(g^2)^*\omega_S = \zeta_3\omega_S$, using the same argument as for $I = 3$, we see that each Δ'_j has Dynkin type A or D.

Since each Δ'_j contains at least one g^2-fixed curve and $F_1, \dots, F_{2p}$ are the only g^2-fixed curves in Δ'_j, we have $n \leq 2p$. On the other hand, from the proof of Proposition 11 Step 4, if $\operatorname{rank} \Delta'_j = \alpha_j$, then Δ_j contains at least $\lceil(\alpha_j - 1)/3\rceil$ smooth g^2-fixed curves. We have an estimation

$$2p \geq \sum_{j=1}^{n} \lceil(\alpha_j - 1)/3\rceil \geq \sum_{j=1}^{n} (\alpha_j - 1)/3.$$

That is,

$$\sum_{j=1}^{n} \operatorname{rank} \Delta'_j \leq 6p + n.$$

Note that Δ'_j is not g^3-stable, otherwise it would also be g-stable. So Δ'_j and $g^3(\Delta'_j)$ are disjoint connected components in Δ. In particular, n is even. It follows from $\operatorname{rank} \Delta = 18$ that

$$\begin{aligned}
18 &\leq 6(c+q) + 4 - m + 6p + n = 6(c+p+q) + 4 - m + n \\
&\leq 6 \cdot 2 + 4 - m + n = 16 - m + n \\
&\leq 16 - 1 + n = 15 + n \\
&\leq 15 + 2p.
\end{aligned}$$

Then $p \geq 2$ and it follows from $c + p + q \leq 2$ that $p = 2$ and $c = q = 0$. So Δ has no g-fixed curve. Since n is even, $n = 4$ and $m = 1$ or 2. We are left to show that these two cases are impossible.

Recall that Δ_i has the form A_{2m_i-1} and contains exactly m_i g^3-fixed curves, and the 2 irreducible g^3-fixed curves are contained in $\coprod_{i=1}^{m} \Delta_i$. We have $\sum_{i=1}^{m} m_i = 2$.

If $m = 1$, then $m_1 = 2$ and $\Delta_1 = A_3$. However, this would imply that $\sum_{j=1}^{4} \operatorname{rank} \Delta'_j = 15$, which needs to be even. If $m = 2$, then $m_1 = m_2 = 1$ and $\Delta_1 = \Delta_2 = A_1$. They are g^3-fixed. On the other hand, note that $\operatorname{ord}(g^2) = 3$. By considering the g^2-action on Δ, we see that Δ_1 and Δ_2 are also g^2-fixed. It follows that Δ_1 and Δ_2 g-fixed, which contradicts our computation that there is no g-fixed curve. $\square$

This completes the proof of Proposition 17 and also Main Theorem (1).

5. The list of Dynkin's types of Δ

TABLE 1. $I = 3$

"f" denotes the g-fixed curve and s denotes the g-stable but not g-fixed curve. We use the same labeling for curves as in Figure 1.

(A) Realizable Cases.

Case I: A_{18}: $s - f - s - s - f - s - s - f - s - s - f - s - s - f - s - s - f - s$
$E_{33} - G_3 - E_{13} - E'_{13} - F_1 - E'_{11} - E_{11} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2 - E_{22} - E'_{22} - F_2 - E'_{21}$.

Case II: D_{18}: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s - s - f - s - s - f$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21} - E'_{21} - F_2 - E'_{22} - E_{22} - G_2$

Case III: $A_{3m} \oplus A_{3n}$, where $m + n = 6$, $1 \leq m \leq n \leq 5$.

(1) $A_3 \oplus A_{15}$: $s - f - s$, $s - f - s - s - f - s - s - f - s - s - f - s - s - f - s$
$E'_{11} - F_1 - E'_{12}$
$E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21} - E'_{21} - F_2 - E'_{22} - E_{22} - G_2 - E_{32}$

(2) $A_6 \oplus A_{12}$: $s - f - s - s - f - s$, $s - f - s - s - f - s - s - f - s - s - f - s$
$E_{21} - G_1 - E_{11} - E'_{11} - F_1 - E'_{12}$
$E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{22} - E_{22} - G_2 - E_{32} - E'_{32} - F_3 - E'_{33}$

(3) $A_9 \oplus A_9$: $s - f - s - s - f - s - s - f - s$, $s - f - s - s - f - s - s - f - s$
$E'_{11} - F_1 - E'_{12} - E_{12} - G_2 - E_{22} - E'_{22} - F_2 - E'_{23}$
$E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

Case IV: $D_{3m} \oplus A_{3n}$, where $m + n = 6$.

(1) $D_6 \oplus A_{12}$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $s - f - s - s - f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$
$E'_{33} - F_3 - E'_{32} - E_{32} - G_2 - E_{22} - E'_{22} - F_2 - E'_{21} - E_{21} - G_1 - E_{31}$

(2) $D_9 \oplus A_9$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$, $s - f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2$
$E_{22} - G_2 - E_{32} - E'_{32} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

(3) $D_{12} \oplus A_6$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f$, $s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{22} - E_{22} - G_2$
$E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

(4) $D_{15} \oplus A_3$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s - s - f$, $s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3$
$E_{22} - G_2 - E_{32}$

Case V: $D_{3m} \oplus D_{3n}$, where $m + n = 6$, $2 \leq m \leq n \leq 4$.

(1) $D_6 \oplus D_{12}$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f$.

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$
$\begin{matrix} E'_{33} \\ E'_{32} \end{matrix} > F_3 - E'_{31} - E_{31} - G_1 - E_{21} - E'_{21} - F_2 - E'_{22} - E_{22} - G_2$

Case VI: $D_{3n+1} \oplus A_{3m-1}$, $m + n = 6$, $1 \leq m, n \leq 5$.

(1) $D_4 \oplus A_{14}$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $f - s - s - f - s - s - f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$
$G_3 - E_{23} - E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2 - E_{22}$

(2) $D_7 \oplus A_{11}$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s$, $f - s - s - f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23}$
$G_2 - E_{22} - E'_{22} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{33}$

(3) $D_{10} \oplus A_8$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s$, $f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{21}$
$G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2 - E_{22}$

(4) $D_{13} \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s$, $f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$
$G_2 - E_{22} - E'_{22} - F_2 - E'_{23}$

(5) $D_{16} \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s$, $f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32}$

Case VII: $A_{3m} \oplus A_{3n} \oplus A_{3r}$, $m + n + r = 6$, $1 \leq m \leq n \leq r \leq 4$.

(1) $A_3 \oplus A_3 \oplus A_{12}$: $s - f - s$, $s - f - s$, $s - f - s - s - f - s - s - f - s - s - f - s$
$E_{13} - G_3 - E_{23}$
$E'_{32} - F_3 - E'_{33}$
$E'_{11} - F_1 - E'_{12} - E_{12} - G_2 - E_{22} - E'_{22} - F_2 - E'_{21} - E_{21} - G_1 - E_{31}$

(2) $A_3 \oplus A_6 \oplus A_9$: $s - f - s$, $s - f - s - s - f - s$, $s - f - s - s - f - s - s - f - s$
$E_{13} - G_3 - E_{33}$
$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32}$
$E'_{11} - F_1 - E'_{12} - E_{12} - G_2 - E_{22} - E'_{22} - F_2 - E'_{23}$

(3) $A_6 \oplus A_6 \oplus A_6$: $s - f - s - s - f - s$, $s - f - s - s - f - s$, $s - f - s - s - f - s$
$E'_{11} - F_1 - E'_{12} - E_{12} - G_2 - E_{22}$
$E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{32}$
$E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31}$

Case VIII: $D_6 \oplus D_6 \oplus D_6$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$

$\begin{matrix} E'_{21} \\ E'_{23} \end{matrix} > F_2 - E'_{22} - E_{22} - G_2$

$\begin{matrix} E'_{32} \\ E'_{33} \end{matrix} > F_3 - E'_{31} - E_{31} - G_1$

Case X: $A_{3m} \oplus A_{3n} \oplus D_{3r}$, where $m + n + r = 6$, $m \leq n$.

(1) $A_3 \oplus A_3 \oplus D_{12}$: $s - f - s$, $s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f$

$E_{22} - G_2 - E_{32}$

$E'_{31} - F_3 - E'_{33}$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{21} - E_{21} - G_1$

(2) $A_3 \oplus A_6 \oplus D_9$: $s - f - s$, $s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$

$E_{22} - G_2 - E_{32}$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{33}$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2$

(3) $A_3 \oplus A_9 \oplus D_6$: $s - f - s$, $s - f - s - s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$.

$E_{22} - G_2 - E_{32}$

$E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{33}$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$

(4) $A_6 \oplus A_6 \oplus D_6$: $s - f - s - s - f - s$, $s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - f - s$

$E_{22} - G_2 - E_{32} - E'_{32} - F_3 - E'_{33}$

$E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31}$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$

Case XI: $D_{3m+1} \oplus A_{3n} \oplus A_{3r-1}$, where $m + n + r = 6$.

(1) $D_4 \oplus A_3 \oplus A_{11}$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $s - f - s$, $f - s - s - f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$E_{21} - G_1 - E_{31}$

$F_3 - E'_{32} - E_{32} - G_2 - E_{22} - E'_{22} - F_2 - E'_{23} - E_{23} - G_3 - E_{33}$

(2) $D_4 \oplus A_6 \oplus A_8$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $s - f - s - s - f - s$, $f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32}$

$G_2 - E_{22} - E'_{22} - F_2 - E'_{23} - E_{23} - G_3 - E_{33}$

(3) $D_4 \oplus A_9 \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $s - f - s - s - f - s - s - f - s$, $f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{33} - E_{33} - G_3 - E_{23}$

$F_2 - E'_{22} - E_{22} - G_2 - E_{32}$

(4) $D_4 \oplus A_{12} \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $s - f - s - s - f - s - s - f - s - s - f - s$, $f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2 - E_{22} - E'_{22} - F_2 - E'_{23}$

$G_3 - E_{33}$

(5) $D_7 \oplus A_3 \oplus A_8$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s,\ s - f - s,\ f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$E_{22} - G_2 - E_{32}$

$F_3 - E'_{31} - E_{31} - G_1 - E_{21} - E'_{21} - F_2 - E'_{23}$

(6) $D_7 \oplus A_6 \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s,\ s - f - s - s - f - s,\ f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$E'_{23} - F_2 - E'_{22} - E_{22} - G_2 - E_{32}$

$F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

(7) $D_7 \oplus A_9 \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s,\ s - f - s - s - f - s - s - f - s,\ f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32}$

$G_2 - E_{22}$

(8) $D_{10} \oplus A_3 \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s,\ s - f - s,\ f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31}$

$E_{22} - G_2 - E_{32}$

$G_1 - E_{21} - E'_{21} - F_2 - E'_{23}$

(9) $D_{10} \oplus A_6 \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s,\ s - f - s - s - f - s,\ f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31}$

$E'_{23} - F_2 - E'_{22} - E_{22} - G_2 - E_{32}$

$G_1 - E_{21}$

(10) $D_{13} \oplus A_3 \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s,\ s - f - s,\ f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

$E_{22} - G_2 - E_{32}$

$F_2 - E'_{23}$

Case XII: $D_{3m+1} \oplus D_{3n+1} \oplus A_{3r-2}$, where $m + n + r = 6$, $m \leq n$.

(2) $D_4 \oplus D_7 \oplus A_7$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s,\ f - s - s - f - s - s - f$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$\begin{matrix} E'_{21} \\ E'_{22} \end{matrix} > F_2 - E'_{23} - E_{23} - G_3 - E_{33}$

$G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2$

(5) $D_7 \oplus D_7 \oplus A_4$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s,\ f - s - s - f$

$\begin{matrix} E'_{12} \\ E'_{13} \end{matrix} > F_1 - E'_{11} - E_{11} - G_1 - E_{31}$

$\begin{matrix} E'_{21} \\ E'_{22} \end{matrix} > F_2 - E'_{23} - E_{23} - G_3 - E_{33}$

$G_2 - E_{32} - E'_{32} - F_3$

(6) $D_7 \oplus D_{10} \oplus A_1$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s$, f

$\begin{matrix} E'_{12} \\ E'_{13} \end{matrix} > F_1 - E'_{11} - E_{11} - G_1 - E_{31}$

$\begin{matrix} E'_{21} \\ E'_{22} \end{matrix} > F_2 - E'_{23} - E_{23} - G_3 - E_{33} - E'_{33} - F_3 - E'_{32}$

G_2

Case XIII: $D_{3n+1} \oplus D_{3m} \oplus A_{3r-1}$, where $m + n + r = 6$, $m \geq 2$.

(3) $D_4 \oplus D_{12} \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f$, $f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$\begin{matrix} E'_{21} \\ E'_{22} \end{matrix} > F_2 - E'_{23} - E_{23} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1$

$G_2 - E_{32}$

(4) $D_7 \oplus D_6 \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$\begin{matrix} E'_{22} \\ E'_{23} \end{matrix} > F_2 - E'_{21} - E_{21} - G_1$

$G_2 - E_{32} - E'_{32} - F_3 - E'_{31}$

(5) $D_7 \oplus D_9 \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$, $f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$\begin{matrix} E'_{22} \\ E'_{23} \end{matrix} > F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3$

$G_2 - E_{32}$

(6) $D_{10} \oplus D_6 \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31}$

$\begin{matrix} E'_{22} \\ E'_{23} \end{matrix} > F_2 - E'_{21} - E_{21} - G_1$

$G_2 - E_{32}$

(B) Indeterminate Cases

Case V: (2) $D_9 \oplus D_9$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$

Case IX: (1) $A_3 \oplus D_6 \oplus D_9$: $s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$

Case IX: (2) $A_6 \oplus D_6 \oplus D_6$: $s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$

Case XII: (1) $D_4 \oplus D_4 \oplus A_{10}$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s$, $f - s - s - f - s - s - f - s - s - f$

Case XII: (3) $D_4 \oplus D_{10} \oplus A_4$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s$, $f - s - s - f$

Case XII: (4) $D_4 \oplus D_{13} \oplus A_1$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s$, f

Case XIII:(1) $D_4 \oplus D_6 \oplus A_8$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $f - s - s - f - s - s - f - s$

Case XIII:(2) $D_4 \oplus D_9 \oplus A_5$: $\begin{smallmatrix} s \\ s \end{smallmatrix} > f - s$, $\begin{smallmatrix} s \\ s \end{smallmatrix} > f - s - s - f - s - s - f$, $f - s - s - f - s$.

TABLE 2. $I = 2, 4$

We use the same labeling as in Figure 2. For $I = 2$, "f" denotes the g-fixed curve and s denotes the g-stable but not g-fixed curve. For $I = 4$, define $h = g^2$; "f" denotes the g-fixed curve, "h" denotes the h-fixed but not g-fixed curve and "s" denotes the g-stable but not h-fixed curve.

(1) $A_1 \oplus A_{17}$:
$I = 2$: f, $f - s - f - s - f - s - f - s - f - s - f - s - f - s - f - s - f$
$I = 4$: h, $h - s - f - s - h - s - f - s - h - s - f - s - h - s - f - s - h$
H_{11}
$H_{13} - E'_{13} - F_1 - E_{12} - G_2 - E_{32} - F_3 - E'_{33} - H_{33} - G_3 - E'_{23} - F_2 - E'_{21} - G_1 - E_{31} - H_{31}$.

(2) $A_3 \oplus A_{15}$:
$I = 2$: $f - s - f$, $f - s - f - s - f - s - f - s - f - s - f - s - f - s - f$
$F_2 - E_{22} - G_2$
$H_{11} - E'_{11} - F_1 - E'_{13} - H_{13} - E_{13} - G_3 - E_{33} - H_{33} - E'_{33} - F_3 - E'_{31} - H_{31} - E_{31} - G_1$.

(3) $A_5 \oplus A_{13}$:
$I = 2$: $f - s - f - s - f$, $f - s - f - s - f - s - f - s - f - s - f - s - f$
$I = 4$: $h - s - f - s - h$, $h - s - f - s - h - s - f - s - h - s - f - s - h$
$H_{13} - E_{13} - G_3 - E_{33} - H_{33}$
$H_{11} - E'_{11} - F_1 - E_{12} - G_2 - E_{32} - F_3 - E'_{31} - H_{31} - E_{31} - G_1 - E'_2 - F_2$

(4) $A_7 \oplus A_{11}$:
$I = 2$: $f - s - f - s - f - s - f$, $f - s - f - s - f - s - f - s - f - s - f$
$H_{13} - E_{13} - G_3 - E_{33} - H_{33} - E'_{33} - F_3$
$H_{11} - E'_{11} - F_1 - E_{12} - G_2 - E_{22} - F_2 - E'_{21} - G_1 - E_{31} - H_{31}$

(5) $A_9 \oplus A_9$:
$I = 2$: $f - s - f - s - f - s - f - s - f$, $f - s - f - s - f - s - f - s - f$
$I = 4$: $h - s - f - s - h - s - f - s - h$, $h - s - f - s - h - s - f - s - h)$
$H_{11} - E'_{11} - F_1 - E_{12} - G_2 - E_{32} - F_3 - E'_{33} - H_{33}$
$H_{13} - E_{13} - G_3 - E'_{23} - F_2 - E'_{21} - G_1 - E_{31} - H_{31}$

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