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DISTURBANCE ATTENUATION FOR A CLASS OF DISCRETE-TIME SWITCHED SYSTEMS WITH EXPONENTIAL UNCERTAINTY

Changlin Li, Fei Long, and Guohui Ren

Abstract. The disturbance attenuation problem for a class of discrete-time switched linear systems with exponential uncertainties via switched state feedback and switched dynamic output feedback is investigated, respectively. By using Taylor series approximation and convex polytope technique, exponentially uncertain discrete-time switched linear system is transformed into an equivalent switched polytopic model with additive norm bounded uncertainty. For such equivalent switched model, one designs its switching strategy and associated state feedback controllers and dynamic output feedback controllers so that whole switched model is asymptotical stabilization with H-infinity disturbance attenuation based on switched Lyapunov function and LMI approach. Finally, two numerical examples are presented to illustrate our results.

1. Introduction

As we know, many practical systems involve a mixture of continuous and discrete dynamics. Systems in which these two kinds of dynamics coexist and interact are usually called to hybrid systems. The history of hybrid system research can be traced back at least to the 1950's with the study of engineering systems. However, hybrid systems began to attract people's attention in the early 1990's, mainly because of the vast development and implementation of digital micro controllers and embedded devices. Hybrid system models can describe systems in a wide range of applications, including robotics, automotive electronics, manufacturing, automated highway systems, air traffic management systems, integrated circuit design, hybrid dynamic automobile, multimedia, and so on.

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Switched linear system belongs to a special class of hybrid system, which comprises a collection of subsystems described by linear dynamics together with a switching law that specifies the switching between the subsystems. The importance of switched linear systems scheme stems from the following facts:

1) Switched linear system can represent a wide class of practical systems;

2) Based on switching control approach, the two-level system structure provides an effective multiple-controller switching path;

3) Many tools and approaches for linear systems are applicable or extendable to cope with switched linear systems.

Hetel, et al. [4, 5] show that sampled model of linear system is derived and discrete time control methods are applied in order to design a computer based controller for the following switched linear system

$$\dot{x}(t) = M_\sigma x(t) + N_\sigma u(t).$$

Under the case that sampling and actuation are periodic and synchronous with the periodicity, the sampled model is given by

$$x(k+1) = A_\sigma(\rho(k))x(k) + B_\sigma(\rho(k))u(k),$$

where $A_\sigma(\rho(k)) = e^{M_\sigma \rho(k)}, B_\sigma(\rho(k)) = (\int_0^{\rho(k)} e^{M_\sigma s} ds)N_\sigma$; $\rho(k)$ is sampling periodicity; k denotes sampling step.

It is well known that, in many control problems, the sampling periodicity of system is often affected by some delays (delays between the sensor and the digital control, computing delays in the controller, communication delays between the controller and the actuator, and so on). Furthermore, these delays are often unknown, time-varying and bounded [17]. As a result, exponential uncertainty is encountered. Because of the practical background of exponential uncertainty, the control synthesis problem of switched linear system subject to exponential uncertainties is a very important and challenging one. Generally speaking, exponential uncertainty can be represented as $e^{M\rho}$ or $\int_0^\rho e^{M\tau} d\tau$ that depends on an unknown, possibly time-varying and bounded parameter ρ. In the literature [21], exponential uncertainty is treated by assuming estimable delay uncertainties. Andrea Balluchi, et al. [1] treats the uncertain exponential terms as bounded uncertainties. L. Hetel, et al. [5] studies the state feedback stabilization problem for a class of discrete-time switched linear systems with exponential uncertainties under the arbitrary switching rule, and then the obtained results are extended to cope with network controlled systems.

The H_∞ control problem for some uncertain switched linear systems was also investigated, such as bounded uncertainties [6, 15, 16, 18], linear fractional uncertainties [9, 11], polytopic uncertainties [20], actuator saturation [14, 19], modeling uncertainties [8, 10], and so on. Motivated by the reference [4, 5], F. Long, et al. [12, 13] have discussed H_∞ state feedback and dynamic output feedback control problem for a class of exponentially uncertain continuous-time switched linear system by using Taylor series approximation and convex polytope technique. In this manuscript, using similar idea of results [12, 13], one is intended to investigate the disturbance attenuation problem for discrete-time switched linear systems with exponential uncertainties via switched state feedback and dynamic output feedback. It is assumed that the switching strategy used in this paper is picked in such a way that there are finite switches in finite steps.

One's goal is, for discrete-time switched linear systems subject to exponential uncertainty, to design a switching $\sigma(k)$ strategy and associated state feedback controllers and dynamic output feedback controllers such that the resulting closed-loop system is asymptotical stabilization with a prescribed H_∞ disturbance attenuation level for all admissible uncertainties. Firstly, one shows that exponential uncertain switched system is transformed into an equivalent polytopic model with an additive norm bounded uncertainty based on Taylor series approximation and convex polytope technique. And then, by taking advantage of switched Lyapunov function and LMI approach, the robust H_∞ disturbance attenuation property of such equivalent switched model is investigated via switched state feedback and switched dynamic output feedback, respectively.

The remainder of this paper is organized as follows: The problem statement and some preliminaries are described in Section 2, while in Section 3 the asymptotical stabilization with H_∞ disturbance attenuation for exponentially uncertain discrete-time switched linear system is investigated via switched state feedback. In Section 4, one discusses the asymptotical stabilization with H_∞ disturbance attenuation for a class of discrete-time switched linear systems subject to exponential uncertainties via switched dynamic output feedback. Two numerical examples are presented in Section 5 to illustrate one's results. Finally, some conclusions are drawn in Section 6.

Notations: The symmetric terms in a symmetric matrix are denoted by $*$, $X^\perp$ denote any matrix whose columns form bases of the null space of X and $\lambda_{\max}(P)$ denote the maximum eigenvalues of matrix P.

2. Problem statement and preliminaries

In this paper, based on switched Lyapunov function techniques and LMI approach, one investigates the asymptotical stabilization with H_∞ disturbance attenuation for the exponentially uncertain discrete-time switched linear systems (1) and (2) via switched state feedback and switched dynamic output feedback, respectively.

$$\begin{cases} x(k+1) = A_\sigma(\rho(k))x(k) + B_\sigma(\rho(k))u(k) + B_{1\sigma}\omega(k) \\ \quad\; z(k) = C_{1\sigma}x(k) + D_\sigma u(k) \end{cases} \tag{1}$$

and

$$(2)\qquad \begin{cases} x(k+1) = A_\sigma(\rho(k))x(k) + B_\sigma(\rho(k))u(k) + B_{1\sigma}\omega(k) \\ z(k) = C_{1\sigma}x(k) + D_\sigma u(k) \\ y(k) = C_{2\sigma}x(k), \end{cases}$$

where $x \in \mathbb{R}^n$ is the system state; $u \in \mathbb{R}^m$ is the control input; $\omega \in \mathbb{R}^r$ is the exogenous disturbance input that satisfies $\sum_{k=0}^{\infty} \omega^T(k)\omega(k) < \infty$; $z \in \mathbb{R}^q$ is the controlled output variable. $y \in \mathbb{R}^p$ is the measured output. The switching signal $\sigma(\cdot) : \mathbb{N} \bigcup \{0\} \longrightarrow \overline{\mathbb{N}} = \{1, 2, \ldots, N\}, N < \infty$ denotes the piecewise constant switching rule. $\{M_i \in \mathbb{R}^{n\times n}, i \in \overline{\mathbb{N}}\}$ and $\{N_i \in \mathbb{R}^{n\times m}, i \in \overline{\mathbb{N}}\}$ are two families of matrices. $A_i(\rho(k)) = e^{M_i \rho(k)}$, $B_i(\rho(k)) = (\int_0^{\rho(k)} e^{M_i s} ds)N_i$, the uncertain parameter $\rho(k)$ is positive, time varying, bounded and $0 < \underline{\rho} < \rho(k) < \overline{\rho}$ (where $\underline{\rho}$ and $\overline{\rho}$ are two known constants). B_{1i}, C_{1i}, C_{2i} and $D_i (i \in \overline{\mathbb{N}})$ are constant matrices with appropriate dimensions. Uncertain switched systems such as (1) and (2) may be used to represent digital models for network controlled systems or for systems affected by sampling jitter. Each pair $\{(M_i, N_i, C_{1i}, D_i, B_{1i}), i \in \overline{\mathbb{N}}\}$ and $\{(M_i, N_i, C_{1i}, D_i, C_{2i}, B_{1i}), i \in \overline{\mathbb{N}}\}$ describes a discrete-time time model representing different regimes of system behavior. Here, σ will be considered a piecewise constant function that may change its value at the sampling period. The linear version of an uncertain system with exponential uncertainty can be obtained by considering only one model ($N = 1$).

The objective of this paper is, for any given $\gamma > 0$, to find switched state feedback controller (switched output feedback controller) so that the discrete-time switched linear systems (1) (the discrete-time switched linear system (2)) satisfies:

a) With zero disturbance input condition $\omega \equiv 0$, it is asymptotically stable for all admissible uncertainties.

b) With zero-initial condition $x(0) = 0$,

$$\sum_{k=0}^{l} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{l} \omega^T(k)\omega(k)$$

for all nonzero ω satisfying $\sum_{k=0}^{\infty} \omega^T(k)\omega(k) < \infty$ and all admissible uncertainties.

Now, one introduces some support lemmas and the concept of switching sequence, which can be use in the later.

Lemma 2.1 ([5]). *Consider the uncertain polynomial parameter dependent n-order matrix $L(\rho) = L_0 + L_1\rho + L_2\rho^2 + \cdots + L_h\rho^h$ with the uncertain parameter ρ is positive, bounded and $0 < \underline{\rho} < \rho < \overline{\rho}$ (where $\underline{\rho}$ and $\overline{\rho}$ are two known constants). Then one can find a convex polytope with $h+1$ vertices that envelopes $L(\rho)$, i.e., there exist parameters $\mu_j(\rho), j = 1, 2, \ldots, h+1$ satisfying $\sum_{j=1}^{h+1} \mu_j(\rho) = 1, \mu_j(\rho) > 0$ such that $L(\rho) = \sum_{j=1}^{h+1} \mu_j(\rho)U_j$, where*

$U_j, j = 1, 2, \ldots, h+1$ *represent the polytope vertices given as follows.*

$$\begin{cases} U_1 = L_0 + L_1\underline{\rho} + L_2\underline{\rho}^2 + \cdots + L_h\underline{\rho}^h \\ U_2 = L_0 + L_1\overline{\rho} + L_2\underline{\rho}^2 + \cdots + L_h\underline{\rho}^h \\ U_3 = L_0 + L_1\overline{\rho} + L_2\overline{\rho}^2 + \cdots + L_h\underline{\rho}^h \\ \cdots \\ U_{h+1} = L_0 + L_1\overline{\rho} + L_2\overline{\rho}^2 + \cdots + L_h\overline{\rho}^h. \end{cases}$$

The relation between the uncertain parameter $\rho(k)$ and $\mu_j(\rho)$ is given by

$$\mu_j(\rho) = \begin{cases} 1 - \frac{\rho - \underline{\rho}}{\overline{\rho} - \underline{\rho}}, & j = 1 \\ \frac{\rho^{j-1} - \underline{\rho}^{j-1}}{\overline{\rho}^{j-1} - \underline{\rho}^{j-1}} - \frac{\rho^j - \underline{\rho}^j}{\overline{\rho}^j - \underline{\rho}^j}, & j = 2, 3, \ldots, h \\ \frac{\rho^h - \underline{\rho}^h}{\overline{\rho}^h - \underline{\rho}^h}, & j = h+1. \end{cases}$$

Lemma 2.2 ([2]). *Given a symmetric matrix $\Psi \in \mathbb{R}^{n\times n}$ and two matrices Γ, Ξ of column dimension m, consider the problem of finding some matrix Θ of compatible dimensions such that $\Psi + \Gamma^T\Theta^T\Xi + \Xi^T\Theta\Gamma < 0$. Denote by W_Γ,W_Ξ any matrices whose columns form bases of the null spaces of Γ and Ξ, respectively. Then the above matrix inequality is solvable for Θ if and only if $W_\Gamma^T\Psi W_\Gamma < 0$ and $W_\Xi^T\Psi W_\Xi < 0$.*

Lemma 2.3 ([3]). *Given symmetrically positive definite matrices $X, Y \in \mathbb{R}^{n\times n}$, if there exist matrices $X_2 \in \mathbb{R}^{n\times n_k}$ (where n_k is a positive integer) and symmetrical matrices $X_3 \in \mathbb{R}^{n_k\times n_k}$ such that*

$$\begin{pmatrix} X & X_2 \\ X_2^T & X_3 \end{pmatrix} > 0,$$

then

$$\begin{pmatrix} X & X_2 \\ X_2^T & X_3 \end{pmatrix}^{-1} = \begin{pmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{pmatrix}$$

if and only if

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0,$$

where

$$\operatorname{rank}\begin{pmatrix} X & I \\ I & Y \end{pmatrix} \leq n + n_k, Y_2 \in \mathbb{R}^{n\times n_k}, Y_3 = Y_3^T \in \mathbb{R}^{n_k\times n_k}.$$

Definition (switching sequence). The sequence $\{(k_m, r_m) | r_m = 1, 2, \ldots, N; m = 1, 2, \ldots\}$ is said to be switching sequence, if

(i) $\sigma(k_m) \neq \sigma(k_m - 1)$,

(ii) $\sigma(k_m) = \sigma(k_m + 1) = \cdots = \sigma(k_m + t_m) = r_m, t_m \geq 1$.

Remark 2.4. The switching sequence $\{(k_m, r_m)|r_m = 1, 2, \ldots, N; m = 1, 2, \ldots\}$ denotes that the r_m-th subsystem of discrete-time switched systems is active at the k_m step and the constant t_m is dwell steps in the r_m-th subsystem of this discrete-time switched systems.

3. Robust H_∞ control via switched state feedback

In this section, the robust H_∞ control problem of system (1) is investigated via switched state feedback. Before the design of switched state feedback controller, one firstly shows how the system (1) can be expressed as a switched polytopic system with additive norm bounded uncertainty. According to the Lemma 2.1 and the properties of exponential matrix:

$$e^{Mx} = \sum_{j=0}^{\infty} \frac{M^j}{j!} x_j, \int_0^x e^{Ms} ds = \sum_{j=1}^{\infty} \frac{M^{j-1}}{j!} x^j.$$

The following lemma is obvious.

Lemma 3.1 ([5]). *The discrete-time switched linear system* (1) *subject to exponential uncertainties can be expressed as*:

$$\left\{ \begin{aligned} x(k+1) &= (A_\sigma^h(\rho(k)) + \Delta A_\sigma^h(\rho(k)))x(k) \\ &\quad + (B_\sigma^h(\rho(k)) + \Delta B_\sigma^h(\rho(k)))u(k) + B_{1\sigma}\omega(k), \\ z(k) &= C_{1\sigma}x(k) + D_\sigma u(k), \end{aligned} \right. \tag{3}$$

where

$$A_\sigma^h(\rho(k)) = \sum_{j=1}^{k+1} \mu_j(\rho(k)) U_{\sigma j}^{Ah}, \tag{4}$$

$$B_\sigma^h(\rho(k)) = (\sum_{j=1}^{k+1} \mu_j(\rho(k)) U_{\sigma j}^{Bh}) N_\sigma, \tag{5}$$

$$\sum_{j=1}^{h+1} \mu_j(\rho) = 1, \mu_j(\rho) > 0, \ \ j = 1, 2, \ldots, h+1, \tag{6}$$

$$\left\{ \begin{aligned} U_{\sigma 1}^{Ah} &= I + M_\sigma \underline{\rho} + \frac{M_\sigma^2}{2!}\underline{\rho}^2 + \cdots + \frac{M_\sigma^h}{h!}\underline{\rho}^h \\ U_{\sigma 2}^{Ah} &= I + M_\sigma \overline{\rho} + \frac{M_\sigma^2}{2!}\underline{\rho}^2 + \cdots + \frac{M_\sigma^h}{h!}\underline{\rho}^h \\ U_{\sigma 3}^{Ah} &= I + M_\sigma \overline{\rho} + \frac{M_\sigma^2}{2!}\overline{\rho}^2 + \cdots + \frac{M_\sigma^h}{h!}\underline{\rho}^h \\ &\cdots \\ U_{\sigma,h+1}^{Ah} &= I + M_\sigma \overline{\rho} + \frac{M_\sigma^2}{2!}\overline{\rho}^2 + \cdots + \frac{M_\sigma^h}{h!}\overline{\rho}^h, \end{aligned} \right. \tag{7}$$

$$
(8) \qquad \begin{cases}
U_{\sigma 1}^{Bh} = \overline{\rho} I + \dfrac{M_\sigma}{2!}\underline{\rho}^2 + \dfrac{M_\sigma^2}{3!}\underline{\rho}^3 + \cdots + \dfrac{M_\sigma^h}{(h+1)!}\underline{\rho}^{h+1} \\
U_{\sigma 2}^{Bh} = \overline{\rho} I + \dfrac{M_\sigma}{2!}\overline{\rho}^2 + \dfrac{M_\sigma^2}{3!}\underline{\rho}^3 + \cdots + \dfrac{M_\sigma^h}{(h+1)!}\underline{\rho}^{h+1} \\
U_{\sigma 3}^{Bh} = \overline{\rho} I + \dfrac{M_\sigma}{2!}\overline{\rho}^2 + \dfrac{M_\sigma^2}{3!}\overline{\rho}^3 + \cdots + \dfrac{M_\sigma^h}{(h+1)!}\underline{\rho}^{h+1} \\
\quad \cdots \\
U_{\sigma, h+1}^{Bh} = \overline{\rho} I + \dfrac{M_\sigma}{2!}\overline{\rho}^2 + \dfrac{M_\sigma^2}{3!}\overline{\rho}^3 + \cdots + \dfrac{M_\sigma^h}{(h+1)!}\overline{\rho}^{h+1}.
\end{cases}
$$

The remainders of the Taylor series approximation $\Delta A_\sigma^h(\rho(k))$ *and* $\Delta B_\sigma^h(\rho(k))$ *are given as follows.*

$$
(9) \qquad \begin{cases}
\Delta A_\sigma^h(\rho(k)) = e^{M_\sigma \rho(k)} - \displaystyle\sum_{j=0}^{h} \frac{M^j}{j!}\rho^j(k), \\
\Delta B_\sigma^h(\rho(k)) = (\displaystyle\int_0^{\rho(k)} e^{M_\sigma s} ds - \sum_{j=1}^{h+1} \frac{M^{j-1}}{j!}\rho^j(k))N_\sigma.
\end{cases}
$$

The relation between the uncertain parameter $\rho(k)$ *and the coordinates* $\mu_j(\rho)$ *is given by Lemma 2.1.*

Remark 3.2. The describing and proof of Lemma 3.1 was come from the same idea of literature [5] and [12, 13].

Notice that the uncertain items $\Delta A_\sigma^h(\rho(k))$ and $\Delta B_\sigma^h(\rho(k))$ are bounded while $0 < \underline{\rho} < \rho(k) < \overline{\rho}$. Therefore one can write

$$
(10) \qquad \|\Delta A_\sigma^h(\rho(k))\| \le \gamma_A, \|\Delta B_\sigma^h(\rho(k))\| \le \gamma_B,
$$

where

$$
(11) \qquad \begin{cases}
\gamma_A = \displaystyle\max_{1 \le i \le N} \sup_{\underline{\rho} \le \rho(k) \le \overline{\rho}} \|e^{M_i \rho(k)} - \sum_{j=0}^{h} \frac{M^j}{j!}\rho^j(k)\|, \\
\gamma_B = \displaystyle\max_{1 \le i \le N} \sup_{\underline{\rho} \le \rho(k) \le \overline{\rho}} \|(\int_0^{\rho(k)} e^{M_i s} ds - \sum_{j=1}^{h+1} \frac{M^{j-1}}{j!}\rho^j(k))N_i\|.
\end{cases}
$$

By the above analysis, switched dynamic model (3) is actually a switched polytopic system subject to additive norm bounded uncertainty. As Lemma 3.1 has showed, such a switched model can be used to represent the discrete-time switched linear system (1). Therefore, stabilizing switched dynamic model (3) is equivalent to doing the system (1).

Our goal is in this section, for any given $\gamma > 0$, to design a switched state feedback controllers $u(k) = K_{\sigma(k)}^h x(k)$ such that the resulting closed-loop system of system (3) satisfies:

a) With zero disturbance input condition $\omega \equiv 0$, it is asymptotically stable for all admissible uncertainties.

b) With zero-initial condition $x(0) = 0$,

$$\sum_{k=0}^{l} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{l} \omega^T(k)\omega(k)$$

for all nonzero ω satisfying $\sum_{k=0}^{\infty} \omega^T(k)\omega(k) < \infty$ and all admissible uncertainties.

For disturbance attenuation performance of system (3), we have the following result.

Theorem 3.3. *Given any constant $\gamma > 0$ the discrete-time switched linear system (3) is asymptotically stabilization with H_∞ disturbance attenuation γ via switched state feedback if there exist symmetrically positive definite matrices X_i and matrices Y_i with $i \in \overline{\mathbb{N}}$ such that the following linear matrix inequality is satisfied for any $i = 1, 2, \ldots, N$.*

$$\begin{pmatrix} -X_i & * & * & * & * & * \\ 0 & 3\eta_i B_{1i}^T B_{1i} - \gamma^2 I & * & * & * & * \\ \begin{matrix}\sqrt{3}(A_i^h(\rho)X_i \\ +B_i^h(\rho)Y_i)\end{matrix} & 0 & -X_i & * & * & * \\ \sqrt{6\eta_i}\gamma_A X_i & 0 & 0 & -I & * & * \\ \sqrt{6\eta_i}\gamma_B Y_i & 0 & 0 & 0 & -I & * \\ C_{1i}X_i + D_i Y_i & 0 & 0 & 0 & 0 & -I \end{pmatrix} < 0, \tag{12}$$

where η_i is a constant and satisfies $\eta_i \geq \lambda_{\max}(X_i^{-1})$.

In this case, the state feedback controller gain and switching strategy are taken as $K_i^h = Y_i X_i^{-1}$ and

$$\sigma(k) = \arg\min_{1\le i\le N} \left\{ \begin{array}{l} x^T(k)\{3(A_i^h(\rho) + B_i^h(\rho)K_i^h)^T X_i^{-1}(A_i^h(\rho) \\ +\, B_i^h(\rho)K_i^h) + 6\eta_i\gamma_A^2 I + 6\eta_i\gamma_B^2 (K_i^h)^T K_i^h \\ +\, (C_{1i}D_iK_i^h)^T (C_{1i}D_iK_i^h)\}x(k) \\ <\, \min\limits_{1\le i\le N} \{x^T(k)(X_i^{-1})x(k)\} \end{array} \right\}. \tag{13}$$

Proof. Setting $X_i = P_i^{-1}, Y_i = K_i^h P_i^{-1}$, then the matrix inequality (12) is equivalent with the following matrix inequality.

$$\begin{pmatrix} -P_i^{-1} & * & * & * & * & * \\ 0 & 3\eta_i B_{1i}^T B_{1i} - \gamma^2 I & * & * & * & * \\ \begin{matrix}\sqrt{3}(A_i^h(\rho) \\ +B_i^h(\rho)K_i^h)P_i^{-1}\end{matrix} & 0 & -P_i^{-1} & * & * & * \\ \sqrt{6\eta_i}\gamma_A P_i^{-1} & 0 & 0 & -I & * & * \\ \sqrt{6\eta_i}\gamma_B K_i^h P_i^{-1} & 0 & 0 & 0 & -I & * \\ (C_{1i} + D_i K_i^h)P_i^{-1} & 0 & 0 & 0 & 0 & -I \end{pmatrix} < 0. \tag{14}$$

Pro-multiplying and post-multiplying the matrix $\mathrm{diag}(P_i, I, I, I, I, I)$ in left-side of matrix inequality (14), the matrix inequality implies the following matrix inequality.

$$\left(\begin{array}{cccccc} -P_i & * & * & * & * & * \\ 0 & 3\eta_i B_{1i}^T B_{1i} - \gamma^2 I & * & * & * & * \\ \sqrt{3}A_{Ci}(\rho) & 0 & -P_i^{-1} & * & * & * \\ \sqrt{6\eta_i}\gamma_A I & 0 & 0 & -I & * & * \\ \sqrt{6\eta_i}\gamma_B K_i^h & 0 & 0 & 0 & -I & * \\ C_{Ci} & 0 & 0 & 0 & 0 & -I \end{array}\right) < 0, \tag{15}$$

where $A_{Ci}(\rho(k)) = A_i^h(\rho(k)) + B_i^h(\rho(k))K_i^h, C_{Ci} = C_{1i} + D_i K_i^h$.

By means of the schur complement formula, the matrix inequality (15) is equivalent to the following inequality.

$$\left(\begin{array}{cc} \begin{array}{c} 3A_{Ci}^T P_i A_{Ci} + 6\eta_i\gamma_A^2 I \\ +6\eta_i\gamma_B^2 (K_i^h)^T K_i^h + C_{Ci}^T C_{Ci} - P_i \end{array} & 0 \\ 0 & 3\eta_i B_{1i}^T B_{1i} - \gamma^2 I \end{array}\right) < 0. \tag{16}$$

Consider the notations:

$$\alpha_i(k) = \begin{cases} 1 & (k \in Q_i), \\ 0 & (k \overline{\in} Q_i), \end{cases}$$

where $Q_i = \{k \in \overline{\mathbb{N}}|$ the i-th subsystem is active at the k-th step$\}$.

The closed-loop dynamic of system (3) with state feedback controller $u(k) = K_{\sigma(k)}^h x(k)$ is given by:

$$\begin{cases} x(k+1) = (A_{C\sigma}(\rho(k)) + \Delta A_{C\sigma}(\rho(k)))x(k) + B_{1\sigma}\omega(k), \\ \quad z(k) = C_{C\sigma}x(k), \end{cases} \tag{17}$$

where $\Delta A_{C\sigma(k)}(\rho(k)) = \Delta A_{\sigma(k)}^h(\rho(k)) + \Delta B_{\sigma(k)}^h(\rho(k))K_{\sigma(k)}^h$.

Consider the following switched parameter dependent Lyapunov-like function

$$V(x(k)) = x^T(k)(\sum_{i=1}^{N} \alpha_i(k)P_i)x(k). \tag{18}$$

Then for any $k \in Q_{r_m} = \{k_m + 1, k_m + 2, \ldots, k_m + t_m\}$, the difference of (18) along with the trajectory of system (17) is given by

$$\begin{aligned} \Delta V &= x^T(k)[A_{Cr_m}^T P_{r_m} A_{Cr_m} + 2A_{Cr_m}^T P_{r_m} \Delta A_{Cr_m} \\ &\quad + \Delta A_{Cr_m}^T P_{r_m} \Delta A_{Cr_m} - P_{r_m}]x(k) + 2x^T(k)A_{Cr_m}^T P_{r_m} B_{1r_m}\omega \\ &\quad + 2x^T(k)\Delta A_{Cr_m}^T P_{r_m} B_{1r_m}\omega + \omega^T B_{1r_m}^T P_{r_m} B_{1r_m}\omega \\ &\le x^T(k)[3A_{Cr_m}^T P_{r_m} A_{Cr_m} + 3\Delta A_{Cr_m}^T P_{r_m} \Delta A_{Cr_m} - P_{r_m}]x(k) \\ &\quad + 3\omega^T B_{1r_m}^T P_{r_m} B_{1r_m}\omega \\ &= x^T(k)[3A_{Cr_m}^T P_{r_m} A_{Cr_m} + 3\Delta A_{Cr_m}^T P_{r_m} \Delta A_{Cr_m} + C_{Cr_m}^T C_{Cr_m} - P_{r_m}]x(k) \end{aligned}$$

$$
\begin{aligned}
&+3\omega^T B_{1r_m}^T P_{r_m} B_{1r_m}\omega - z^T(k)z(k)\\
&= x^T(k)[3A_{Cr_m}^T P_{r_m} A_{Cr_m} + 3\Delta A_{Cr_m}^T P_{r_m}\Delta A_{Cr_m} + C_{Cr_m}^T C_{Cr_m} - P_{r_m}]x(k)\\
&\quad + \omega^T(3B_{1r_m}^T P_{r_m} B_{1r_m} - \gamma^2 I)\omega + \gamma^2\omega^T\omega - z^T(k)z(k).
\end{aligned}
$$

By means of (10) and

$$
\begin{aligned}
&\Delta A_{Cr_m}^T P_{r_m}\Delta A_{Cr_m}\\
&= (\Delta A_{r_m}^h(\rho) + \Delta B_{r_m}^h(\rho)K_{r_m}^h)^T P_{r_m}(\Delta A_{r_m}^h(\rho) + \Delta B_{r_m}^h(\rho)K_{r_m}^h)\\
&\le 2(\Delta A_{r_m}^h(\rho))^T P_{r_m}\Delta A_{r_m}^h(\rho) + 2(\Delta B_{r_m}^h(\rho)K_{r_m}^h)^T P_{r_m}(\Delta B_{r_m}^h(\rho)K_{r_m}^h)\\
&\le 2\eta_{r_m}\gamma_A^2 I + 2\eta_{r_m}\gamma_B^2(K_{r_m}^h)^T K_{r_m}^h.
\end{aligned}
$$

The following inequality is obvious.

$$
\begin{aligned}
(19)\quad & z^T(k)z(k) - \gamma^2\omega^T\omega + \Delta V\\
&\le x^T(k)[3A_{Cr_m}^T P_{r_m} A_{Cr_m} + 3\Delta A_{Cr_m}^T P_{r_m}\Delta A_{Cr_m}\\
&\quad + C_{Cr_m}^T C_{Cr_m} - P_{r_m}]x(k) + \omega^T(3B_{1r_m}^T P_{r_m} B_{1r_m} - \gamma^2 I)\omega\\
&\le x^T(k)[3A_{Cr_m}^T P_{r_m} A_{Cr_m} + 6\eta_{r_m}\gamma_A^2 I + 6\eta_{r_m}\gamma_B^2(K_{r_m}^h)^T K_{r_m}^h\\
&\quad + C_{Cr_m}^T C_{Cr_m} - P_{r_m}]x(k) + \omega^T(3\eta_{r_m} B_{1r_m}^T B_{1r_m} - \gamma^2 I)\omega\\
&= \begin{pmatrix} x(k)\\ \omega \end{pmatrix}^T
\begin{pmatrix}
\begin{array}{c} 3A_{Cr_m}^T P_{r_m} A_{Cr_m}\\ +6\eta_{r_m}\gamma_A^2 I\\ +6\eta_{r_m}\gamma_B^2(K_{r_m}^h)^T K_{r_m}^h\\ +C_{Cr_m}^T C_{Cr_m} - P_{r_m}\end{array} & 0\\
0 & \begin{array}{c} 3\eta_{r_m} B_{1r_m}^T B_{1r_m}\\ -\gamma^2 I\end{array}
\end{pmatrix}
\begin{pmatrix} x(k)\\ \omega \end{pmatrix}.
\end{aligned}
$$

Assume $x(0) = 0$ and introduce the performance

$$
J = \sum_{k=0}^{l}(z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k)).
$$

Let $\{(k_m, r_m)|r_m \in \overline{\mathbb{N}}; m = 1, 2, \ldots; 0 = k_1 < k_2 < \cdots < k_s < l\}$ be switching sequence that is generated by the switching strategy (13) in the set $\{1, 2, \ldots, l\}$. Noting that $x(k_1) = x(0) = 0$, then for every satisfying $\sum_{k=0}^{\infty}\omega^T(k)\omega(k) < \infty$,

$$
\begin{aligned}
J &= \sum_{k=0}^{l}(z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(x(k))) - \sum_{k=0}^{l}\Delta V(x(k))\\
&= \sum_{k=0}^{l}(z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(x(k))) - V(x(l))\\
&\le \sum_{k=0}^{l}(z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(x(k))).
\end{aligned}
$$

By virtue of the matrix inequality (16) and (19), it follows that $J < 0$. That is to say $\sum_{k=0}^{l} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{l} \omega^T(k)\omega(k)$.

Now, one proves that the discrete-time switched linear system (3) with $\omega \equiv 0$ is asymptotically stable.

Let $\{(k_m, r_m)|r_m \in \overline{\mathbb{N}}; m = 1, 2, \ldots; 0 = k_1 < k_2 < \cdots < k_s < \cdots\}$ be a switching sequence that is generated by the switching strategy (13) in the set of natural numbers.

For any $k \in \{k_m+1, k_m+2, \ldots, k_m+t_m\} = Q_{r_m}$, the difference of $V(x(k))$ along with the trajectory of the discrete-time switched linear system (17) with $\omega \equiv 0$ is given by

$$
\begin{aligned}
\Delta V &= [x^T(k)A_{Cr_m}^T + x^T(k)\Delta A_{Cr_m}^T]P_{r_m}[A_{Cr_m}x(k) + \Delta A_{Cr_m}x(k)] \\
&\quad - x^T(k)P_{r_m}x(k) \\
&\le x^T(k)[3A_{Cr_m}^T P_{r_m} A_{Cr_m} + 6\eta_{r_m}\gamma_A^2 I + 6\eta_{r_m}\gamma_B^2 (K_{r_m}^h)^T K_{r_m}^h \\
&\quad + C_{Cr_m}^T C_{Cr_m} - P_{r_m}]x(k) \\
&= \begin{pmatrix} x(k) \\ 0 \end{pmatrix}^T \begin{pmatrix} \begin{array}{c} 3A_{Cr_m}^T P_{r_m} A_{Cr_m} \\ +6\eta_{r_m}\gamma_A^2 I \\ +6\eta_{r_m}\gamma_B^2 (K_{r_m}^h)^T K_{r_m}^h \\ +C_{Cr_m}^T C_{Cr_m} - P_{r_m} \end{array} & 0 \\ 0 & \begin{array}{c} 3\eta_{r_m} B_{1r_m}^T B_{1r_m} \\ -\gamma^2 I \end{array} \end{pmatrix} \begin{pmatrix} x(k) \\ 0 \end{pmatrix}.
\end{aligned}
$$

Therefore, by virtue of the matrix inequality (16), for any $k \in \{k_m+1, k_m+2, \ldots, k_m+t_m\} = Q_{r_m}$ the difference of (18) alone with the trajectory of the discrete-time switched linear system (17) with $\omega \equiv 0$ is less than zero.

Without loss of generality, suppose that $\sigma(k_m+t_m) = k$ and $\sigma(k_m+t_m+1) = k+1$. Then by $P_i^{-1} = X_i$, $A_{Ci}(\rho) = A_i^h(\rho) + B_i^h K_i^h$ and $C_{Ci} = C_{1i} + D_i K_i^h (i \in \overline{\mathbb{N}})$, one has

$$
\begin{aligned}
&V(x(k+1)) - V(x(k)) \\
&= x^T(k+1)P_{r_{m+1}}x(k+1) - x^T(k)P_{r_m}x(k) \\
&= [x^T(k)A_{Cr_{m+1}}^T + x^T(k)\Delta A_{Cr_{m+1}}^T]P_{r_{m+1}}[A_{Cr_{m+1}}x(k) \\
&\quad + \Delta A_{Cr_{m+1}}x(k)] - x^T(k)P_{r_m}x(k) \\
&\le x^T(k)[3A_{Cr_{m+1}}^T X_{r_{m+1}}^{-1} A_{Cr_{m+1}} + 6\eta_{r_{m+1}}\gamma_A^2 I \\
&\quad + 6\eta_{r_m}\gamma_B^2 (K_{r_{m+1}}^h)^T K_{r_{m+1}}^h + C_{Cr_{m+1}}^T C_{Cr_{m+1}} - X_{r_m}^{-1}]x(k) \\
&< 0.
\end{aligned}
$$

Hence by Lyapounv stability theory, under the action of switching controller (13), the asymptotic stability of system (3) with $\omega \equiv 0$ follows immediately. This completes the proof. □

4. Robust H_∞ control via dynamic output feedback

In this section, the robust control problem of the discrete-time switched linear system (2) is investigated via switched dynamic output feedback. As same doing in Section 3, one firstly shows that the system (2) can be expressed as a switched polytopic system with additive norm bounded uncertainty.

By Lemma 3.1, the following lemma is obvious.

Lemma 4.1. *The discrete-time switched linear system* (2) *subject to exponential uncertainties can be expressed as*:

$$\begin{cases} x(k+1) = (A_\sigma^h(\rho(k)) + \Delta A_\sigma^h(\rho(k)))x(k) \\ \qquad\qquad + (B_\sigma^h(\rho(k)) + \Delta B_\sigma^h(\rho(k)))u(k) + B_{1\sigma}\omega(k) \\ z(k) = C_{1\sigma}x(k) + D_\sigma u(k) \\ y(k) = C_{2\sigma}x(k), \end{cases} \tag{20}$$

where $A_\sigma^h(\rho(k))$ and $B_\sigma^h(\rho(k))$ are described as (4)-(5) *and* (7)-(8); *the remainders of the Taylor series approximation $\Delta A_\sigma^h(\rho(k))$ and $\Delta B_\sigma^h(\rho(k))$ are given by the functions* (9)-(11). *The description of parameters $\mu_j(\rho)$ is given by* (6) *and Lemma 3.1.*

In view of the Lemma 4.1, switched dynamic model (20) is actually a switched polytopic system subject to norm bounded uncertainty. Furthermore, such a switched model can be used to represent the discrete-time switched linear system (2). Therefore, stabilizing switched dynamic model (20) is equivalent to doing the system (2).

One's goal is, for any given $\gamma > 0$, to find a switched dynamic output feedback controller such that the resulting closed-loop system of system (20) satisfies:

a) With zero disturbance input condition $\omega \equiv 0$, it is asymptotically stable for all admissible uncertainties.

b) With zero-initial condition $x(0) = 0$,

$$\sum_{k=0}^{l} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{l} \omega^T(k)\omega(k)$$

for all nonzero ω satisfying $\sum_{k=0}^{\infty} \omega^T(k)\omega(k) < \infty$ and all admissible uncertainties.

For system (20), one is interested in constructing the form of the switched dynamic output-feedback controller as follows:

$$\begin{cases} \xi(k+1) = \hat{A}_{\sigma(k)}\xi(k) + \hat{B}_{\sigma(k)}y(k) \\ u(k) = \hat{C}_{\sigma(k)}\xi(k) + \hat{D}_{\sigma(k)}y(k), \end{cases} \tag{21}$$

where $\xi \in \mathbb{R}^n$.

The resulting closed-loop system of system (20) with switched dynamic output feedback controller (21) is given by

$$\left\{\begin{aligned}\tilde{x}(k+1) &= (A_{C\sigma(k)} + \Delta A_{C\sigma(k)})\tilde{x}(k) + B_{C\sigma(k)}\omega(k)\\ z(k) &= C_{C\sigma(k)}\tilde{x}(k),\end{aligned}\right. \tag{22}$$

where

$$\begin{aligned}&\tilde{x}^T = (x^T, \xi^T), A_{C\sigma(k)} = A^0_{\sigma(k)} + B^0_{\sigma(k)}K_{\sigma(k)}C^0_{2\sigma(k)},\\ &\Delta A_{C\sigma(k)} = \Delta A^0_{\sigma(k)} + \Delta B^0_{\sigma(k)}K_{\sigma(k)}C^0_{2\sigma(k)},\\ &C_{C\sigma(k)} = C^0_{1\sigma(k)} + D^0_{\sigma(k)}K_{\sigma(k)}C^0_{2\sigma(k)}, D^0_{\sigma(k)} = (D_{\sigma(k)} \quad 0),\\ &C^0_{1\sigma(k)} = (C_{1\sigma(k)} \quad 0), B_{C\sigma(k)} = B^0_{1\sigma(k)} = (B_{1\sigma(k)} \quad 0)^T,\end{aligned}$$

$$A^0_{\sigma(k)}) = \begin{pmatrix} A^h_{\sigma(k)} & 0 \\ 0 & 0 \end{pmatrix}, \Delta A^0_{\sigma(k)} = \begin{pmatrix} \Delta A^h_{\sigma(k)} & 0 \\ 0 & 0 \end{pmatrix},$$

$$B^0_{\sigma(k)} = \begin{pmatrix} B^h_{\sigma(k)} & 0 \\ 0 & I \end{pmatrix}, \Delta B^0_{\sigma(k)} = \begin{pmatrix} \Delta B^h_{\sigma(k)} & 0 \\ 0 & 0 \end{pmatrix},$$

$$C^0_{2\sigma(k)}) = \begin{pmatrix} C_{2\sigma(k)} & 0 \\ 0 & I \end{pmatrix}, K_{\sigma(k)} = \begin{pmatrix} \hat{D}_{\sigma(k)} & \hat{C}_{\sigma(k)} \\ \hat{B}_{\sigma(k)} & \hat{A}_{\sigma(k)} \end{pmatrix}.$$

Next, one gives the sufficient conditions for existence of the switched dynamic output feedback controllers (21) such that the resulting closed-loop system (22) is stabilization with disturbance attenuation level γ.

Lemma 4.2. *Given any constant $\gamma > 0$, the uncertain switched linear system* (20) *is said to be asymptotically stabilizable with H_∞ disturbance attenuation level γ via switched dynamic output feedback, if there exist symmetrically positive definite matrices X_{Ci} such that the following non-linear matrix inequality hold for all $i \in \overline{\mathbb{N}}$.*

$$\begin{pmatrix} -X_{Ci} & * & * & * & * & * \\ 0 & 3\eta_i B^T_{Ci}B_{Ci} - \gamma^2 I & * & * & * & * \\ \sqrt{3}A_{Ci} & 0 & -X^{-1}_{Ci} & * & * & * \\ \sqrt{6\eta_i}\gamma_A I_0 & 0 & 0 & -I & * & * \\ \sqrt{6\eta_i}\gamma_B R_{Ci} & 0 & 0 & 0 & -I & * \\ C_{Ci} & 0 & 0 & 0 & 0 & -I \end{pmatrix} < 0, \tag{23}$$

where $R_{Ci} = I_0 K_i C^0_{2i}, I_0 = [I \quad 0], I_C = I^T_0 I_0$, η_i is a constant and satisfies $\eta_i \geq \lambda_{\max}(X_{Ci})$.

In this case, the dynamic output feedback controllers gain matrix:

$$K_i = \begin{pmatrix} \hat{D}_i & \hat{C}_i \\ \hat{B}_i & \hat{A}_i \end{pmatrix}, i = 1, 2, \ldots, N.$$

The switching strategy $\sigma(k)$ is given by

$$\sigma(k) = \arg\min_{1\le i\le N}\left\{\begin{array}{l}\tilde{x}^T(k)[3A_{Ci}^TX_{Ci}A_{Ci}+6\eta_i\gamma_A^2I_C\\ +6\eta_i\gamma_B^2R_{Ci}^TR_{Ci}+C_{Ci}^TC_{Ci}]\tilde{x}(k)\\ <\min\limits_{1\le i\le N}\{\tilde{x}^T(k)(X_{Ci})\tilde{x}(k)\}\end{array}\right\}. \tag{24}$$

Proof. By means of the Schur Complement formula, the matrix inequality (23) is equivalent to the following matrix inequality.

$$\begin{pmatrix} \begin{array}{c}3A_{Ci}^TX_{Ci}A_{Ci}+6\eta_i\gamma_A^2I_C\\ +6\eta_i\gamma_B^2R_{Ci}^TR_{Ci}+C_{Ci}^TC_{Ci}-X_{Ci}\end{array} & 0\\ 0 & 3\eta_iB_{Ci}^TB_{Ci}-\gamma^2I\end{pmatrix}<0. \tag{25}$$

Consider the notations:

$$\alpha_i(k)=\begin{cases}1 & (k\in Q_i)\\ 0 & (k\overline{\in}Q_i),\end{cases}$$

where $Q_i=\{k\in\overline{\mathbb{N}}|$ the i-th subsystem is active at the k-th step$\}$.

Constructing switched parameter dependent Lyapunov-like function as follows.

$$V(\tilde{x}(k))=\tilde{x}^T(k)(\sum_{i=1}^N\alpha_i(k)P_i)\tilde{x}(k). \tag{26}$$

Then for any $k\in Q_{r_m}=\{k_m+1,k_m+2,\ldots,k_m+t_m\}$, the difference of (26) along with the trajectory of system (22) is given by

$$\begin{aligned}\Delta V &= \tilde{x}^T(k)[A_{Cr_m}^TP_{r_m}A_{Cr_m}+2A_{Cr_m}^TP_{r_m}\Delta A_{Cr_m}+\Delta A_{Cr_m}^TP_{r_m}\Delta A_{Cr_m}\\ &\quad -P_{r_m}]\tilde{x}(k)+2\tilde{x}^T(k)A_{Cr_m}^TP_{r_m}B_{Cr_m}\omega+2\tilde{x}^T(k)\Delta A_{Cr_m}^TP_{r_m}B_{Cr_m}\omega\\ &\quad +\omega^TB_{Cr_m}^TP_{r_m}B_{Cr_m}\omega\\ &\le \tilde{x}^T(k)[3A_{Cr_m}^TP_{r_m}A_{Cr_m}+3\Delta A_{Cr_m}^TP_{r_m}\Delta A_{Cr_m}-P_{r_m}]\tilde{x}(k)\\ &\quad +3\omega^TB_{Cr_m}^TP_{r_m}B_{Cr_m}\omega\\ &= \tilde{x}^T(k)[3A_{Cr_m}^TP_{r_m}A_{Cr_m}+3\Delta A_{Cr_m}^TP_{r_m}\Delta A_{Cr_m}+C_{Cr_m}^TC_{Cr_m}-P_{r_m}]\tilde{x}(k)\\ &\quad +3\omega^TB_{Cr_m}^TP_{r_m}B_{Cr_m}\omega-z^T(k)z(k)\\ &= \tilde{x}^T(k)[3A_{Cr_m}^TP_{r_m}A_{Cr_m}+3\Delta A_{Cr_m}^TP_{r_m}\Delta A_{Cr_m}+C_{Cr_m}^TC_{Cr_m}-P_{r_m}]\tilde{x}(k)\\ &\quad +\omega^T(3B_{Cr_m}^TP_{r_m}B_{Cr_m}-\gamma^2I)\omega+\gamma^2\omega^T\omega-z^T(k)z(k).\end{aligned}$$

Again in view of (10) and

$$\begin{aligned}&\Delta A_{Cr_m}^TP_{r_m}\Delta A_{Cr_m}\\ &=(\Delta A_{r_m}^0+\Delta B_{r_m}^0K_{r_m}C_{2r_m}^0)^TP_{r_m}(\Delta A_{r_m}^0+\Delta B_{r_m}^0K_{r_m}C_{2r_m}^0)\\ &\le 2(\Delta A_{r_m}^0)^TP_{r_m}\Delta A_{r_m}^0+2(\Delta B_{r_m}^0K_{r_m}C_{2r_m}^0)^TP_{r_m}(\Delta B_{r_m}^0K_{r_m}C_{2r_m}^0)\\ &\le 2\eta_{r_m}\gamma_A^2I_C+2\eta_{r_m}\gamma_B^2R_{Cr_m}^TR_{Cr_m}.\end{aligned}$$

One has

$$
\begin{aligned}
& z^T(k)z(k) - \gamma^2\omega^T\omega + \Delta V \\
\le\ & \tilde{x}^T(k)[3A^T_{Cr_m}P_{r_m}A_{Cr_m} + 3\Delta A^T_{Cr_m}P_{r_m}\Delta A_{Cr_m} + C^T_{Cr_m}C_{Cr_m} \\
& - P_{r_m}]\tilde{x}(k) + \omega^T(3B^T_{Cr_m}P_{r_m}B_{Cr_m} - \gamma^2 I)\omega \\
\le\ & \tilde{x}^T(k)[3A^T_{Cr_m}P_{r_m}A_{Cr_m} + 6\eta_{r_m}\gamma_A^2 I_C + 6\eta_{r_m}\gamma_B^2 R^T_{Cr_m}R_{Cr_m} \\
& + C^T_{Cr_m}C_{Cr_m} - P_{r_m}]\tilde{x}(k) + \omega^T(3\eta_{r_m}B^T_{Cr_m}B_{Cr_m} - \gamma^2 I)\omega \\
=\ & \begin{pmatrix} \tilde{x}(k) \\ \omega \end{pmatrix}^T \begin{pmatrix} \begin{matrix} 3A^T_{Cr_m}P_{r_m}A_{Cr_m} \\ +6\eta_{r_m}\gamma_A^2 I_C \\ +6\eta_{r_m}\gamma_B^2 R^T_{Cr_m}R_{Cr_m} \\ +C^T_{Cr_m}C_{Cr_m} - P_{r_m} \end{matrix} & 0 \\ 0 & \begin{matrix} 3\eta_{r_m}B^T_{Cr_m}B_{Cr_m} \\ -\gamma^2 I \end{matrix} \end{pmatrix} \begin{pmatrix} \tilde{x}(k) \\ \omega \end{pmatrix}.
\end{aligned}
$$

Assume $\tilde{x}(0) = 0$ and introduce the performance

$$J = \sum_{k=0}^{l}(z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k)).$$

Let $\{(k_m, r_m)|r_m \in \overline{\mathbb{N}}; m = 1, 2, \ldots; 0 = k_1 < k_2 < \cdots < k_s < l\}$ be a switching sequence that is generated by the switching strategy (24) in the set $\{1, 2, \ldots, l\}$. Noting that $\tilde{x}(k_1) = \tilde{x}(0) = 0$, then for every ω satisfying $\sum_{k=0}^{\infty}\omega^T(k)\omega(k) < \infty$,

$$
\begin{aligned}
J &= \sum_{k=0}^{l}(z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(\tilde{x}(k))) - \sum_{k=0}^{l}\Delta V(\tilde{x}(k)) \\
&= \sum_{k=0}^{l}(z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(\tilde{x}(k))) - V(\tilde{x}(l)) \\
&\le \sum_{k=0}^{l}(z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(\tilde{x}(k))).
\end{aligned}
$$

Consequently, setting $P_i = X_{Ci}$ and by virtue of the matrix inequality (23), it follows that $J < 0$. That is to say

$$\sum_{k=0}^{l} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{l}\omega^T(k)\omega(k).$$

Now, one proves that the discrete-time switched linear system (20) with $\omega \equiv 0$ is asymptotically stable.

Let $\{(k_m, r_m)|r_m \in \overline{\mathbb{N}}; m = 1, 2, \ldots; 0 = k_1 < k_2 < \cdots < k_s < \cdots\}$ be a switching sequence that is generated by the switching strategy (24) in the set of natural numbers.

For any $k \in \{k_m+1, k_m+2, \ldots, k_m+t_m\} = Q_{r_m}$, the difference of $V(\tilde{x}(k))$ along with the trajectory of the discrete-time switched linear system (22) with $\omega \equiv 0$ is given by

$$\begin{aligned}
\Delta V &= [\tilde{x}^T(k)A^T_{Cr_m} + \tilde{x}^T(k)\Delta A^T_{Cr_m}]P_{r_m}[A_{Cr_m}\tilde{x}(k) + \Delta A_{Cr_m}\tilde{x}(k)] \\
&\quad - \tilde{x}^T(k)P_{r_m}\tilde{x}(k) \\
&\le \tilde{x}^T(k)[3A^T_{Cr_m}P_{r_m}A_{Cr_m} + 6\eta_{r_m}\gamma_A^2 I_C + 6\eta_{r_m}\gamma_B^2 R^T_{Cr_m}R_{Cr_m} \\
&\quad + C^T_{Cr_m}C_{Cr_m} - P_{r_m}]\tilde{x}(k) \\
&= \begin{pmatrix} \tilde{x}(k) \\ 0 \end{pmatrix}^T \begin{pmatrix} \begin{matrix} 3A^T_{Cr_m}P_{r_m}A_{Cr_m} \\ +6\eta_{r_m}\gamma_A^2 I_C \\ +6\eta_{r_m}\gamma_B^2 R^T_{Cr_m}R_{Cr_m} \\ +C^T_{Cr_m}C_{Cr_m} - P_{r_m} \end{matrix} & 0 \\ 0 & \begin{matrix} 3\eta_{r_m}B^T_{Cr_m}B_{Cr_m} \\ -\gamma^2 I \end{matrix} \end{pmatrix} \begin{pmatrix} \tilde{x}(k) \\ 0 \end{pmatrix}.
\end{aligned}$$

Therefore, by virtue of the matrix inequality (23), for any $k \in \{k_m+1, k_m+2, \ldots, k_m+t_m\} = Q_{r_m}$ the difference of (26) alone with the trajectory of the discrete-time switched linear system (22) with $\omega \equiv 0$ is less than zero.

Without loss of generally, suppose that $\sigma(k_m+t_m) = k$ and $\sigma(k_m+t_m+1) = k+1$. Then by Lyapunov-like functions (26), switching strategy (24) and $P_i = X_{Ci}, R_{Ci} = I_0 K_i C^0_{2i}, A_{Ci} = A^0_i + B^0_i K_i C^0_{2i}, C_{Ci} = C^0_{1i} + D^0_i K_i C^0_{2i} (i \in \overline{\mathbb{N}})$. One has

$$\begin{aligned}
&V(\tilde{x}(k+1)) - V(\tilde{x}(k)) \\
&= \tilde{x}^T(k+1)P_{r_{m+1}}\tilde{x}(k+1) - \tilde{x}^T(k)P_{r_m}\tilde{x}(k) \\
&= [\tilde{x}^T(k)A^T_{Cr_{m+1}} + \tilde{x}^T(k)\Delta A^T_{Cr_{m+1}}]P_{r_{m+1}}[A_{Cr_{m+1}}\tilde{x}(k) \\
&\quad + \Delta A_{Cr_{m+1}}\tilde{x}(k)] - \tilde{x}^T(k)P_{r_m}\tilde{x}(k) \\
&\le \tilde{x}^T(k)[3A^T_{Cr_{m+1}}X_{Cr_{m+1}}A_{Cr_{m+1}} + 6\eta_{r_{m+1}}\gamma_A^2 I_C \\
&\quad + 6\eta_{r_m}\gamma_B^2 R^T_{Cr_{m+1}}R_{Cr_{m+1}} + C^T_{Cr_{m+1}}C_{Cr_{m+1}} - X_{Cr_m}]\tilde{x}(k) \\
&< 0.
\end{aligned}$$

Hence, by Lyapounv stability theory, under the action of switching controller (21), the asymptotic stability of system (20) with $\omega \equiv 0$ follows immediately. This completes the proof. □

It should be pointed out, however, that the matrix inequality (23) is not linear matrix inequality. In the following part, our goal is to transform matrix inequality (23) into linear matrix inequalities.

The matrix inequality (23) is transformed into the following matrix inequality.

$$H_{X_{Ci}} + L^T k_i Q + Q^T k_i^T L < 0, \tag{27}$$

where

$$L = (0 \quad 0 \quad \sqrt{3}(B_i^0)^T \quad 0 \quad \sqrt{6\eta_i}\gamma_B I_0^T \quad (D_i^0)^T), Q = (C_{2i}^0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

and

$$H_{X_{Ci}} = \begin{pmatrix} -X_{Ci} & * & * & * & * & * \\ 0 & 3\eta_i (B_{1i}^0)^T B_{1i}^0 - \gamma^2 I & * & * & * & * \\ \sqrt{3}A_i^0 & 0 & -X_{Ci}^{-1} & * & * & * \\ \sqrt{6\eta_i}\gamma_A I_0 & 0 & 0 & -I & * & * \\ 0 & 0 & 0 & 0 & -I & * \\ C_{1i}^0 & 0 & 0 & 0 & 0 & -I \end{pmatrix}.$$

By means of Lemma 2.2, the matrix inequality (27) is equivalent to the following matrix inequalities.

$$L^{\perp T} H_{X_{Ci}} L^{\perp} < 0, Q^{\perp T} H_{X_{Ci}} Q^{\perp} < 0. \tag{28}$$

The above analysis shows that the matrices L and Q are the non-zero constant matrices. Hence, if there exist the matrices $H_{X_{Ci}} < 0$, then the matrix inequalities (28) hold for all $i \in \overline{\mathbb{N}}$. In view of Lemma 2.3, the matrix inequality $H_{X_{Ci}} < 0$ is transformed into the following inequalities.

$$\begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} \geq 0, \tag{29}$$

$$\begin{pmatrix} -X_i & * & * & * & * \\ 0 & 3\eta_i (B_{1i}^0)^T B_{1i}^0 - \gamma^2 I & * & * & * \\ \sqrt{3}A_i^0 & 0 & -Y_i & * & * \\ \sqrt{6\eta_i}\gamma_A I_0 & 0 & 0 & -I & * \\ C_{1i}^0 & 0 & 0 & 0 & -I \end{pmatrix} < 0. \tag{30}$$

According to the above analysis, one can obtain the following result for disturbance attenuation performance of system (20).

Theorem 4.3. *Given any constant $\gamma > 0$, the uncertain discrete-time switched linear system* (20) *is said to be asymptotically stabilization with H_∞ disturbance attenuation level γ via switched dynamic output feedback, if there exist symmetrically positive definite matrices X_i and matrices Y_i such that the above linear matrix inequalities* (29) *and* (30) *are satisfied for all $i \in \overline{\mathbb{N}}$, where the switching strategy $\sigma(k)$ is given by* (24)*.*

In this case, the dynamic output feedback controllers gain matrix K_i can be solved by using the following algorithm.

Assume that

$$X_i = \begin{pmatrix} X_{i1} & X_{i2} \\ X_{i2}^T & X_{i3} \end{pmatrix}, Y_i = \begin{pmatrix} Y_{i1} & Y_{i2} \\ Y_{i2}^T & Y_{i3} \end{pmatrix},$$

where $X_{i1}, Y_{i1} \in \mathbb{R}^{n \times n}$, $X_{i2}, Y_{i2} \in \mathbb{R}^{n \times n_k}$.

Step 1: *To solve the matrices X_i and Y_i by using the conditions* (29) *and* (30);

Step 2: *One first solves* $\overline{X}_i \in \mathbb{R}^{n\times n_k}$ *via the matrix equality* $\overline{X}_i\overline{X}_i^T = X_{i1} - Y_{i1}^{-1}$, *and then construct the matrix*:

$$X_{Ci} = \begin{pmatrix} X_{i1} & \bar{X}_i \\ \bar{X}_i^T & I \end{pmatrix} > 0,$$

where $n_k = \mathrm{rank}(X_{i1} - Y_{i1}^{-1})$;

Step 3: *To solve* K_i *via the matrix inequality* (27).

Proof. The proof of Theorem 4.3 is obvious. □

5. Numerical example

The aim of the following two examples is used to illustrate that neither of the designed state feedback and dynamic output feedback sub-controllers stabilizes discrete-time switched linear system (1) and (2) with H_∞ disturbance attenuation level γ, respectively. But the discrete-time switched linear system (1) and (2) is stabilization with disturbance attenuation level γ via switched state feedback and switched dynamic output feedback, respectively.

Example 1. Consider the discrete-time switched linear system (1) with $N = 2$, $\sigma(k) : \mathbb{N}\bigcup\{0\} \to \{1, 2\}$ and

$$M_1 = \begin{pmatrix} 0 & 4 \\ -1 & -3 \end{pmatrix}, M_2 = \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}, N_1 = \begin{pmatrix} -10 & -3 \\ -4 & -1 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} -9 & -6 \\ -4 & -1 \end{pmatrix}, B_{11} = \begin{pmatrix} -0.1 & 0 \\ 0.1 & -0.1 \end{pmatrix}, B_{12} = \begin{pmatrix} 0.1 & 0.1 \\ 0 & -0.1 \end{pmatrix},$$

$$C_{11} = \begin{pmatrix} 0 & -2 \\ 2 & -4 \end{pmatrix}, C_{12} = \begin{pmatrix} -2 & 0.1 \\ -1 & 2 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -7 & 1 \\ 1 & 4 \end{pmatrix}, D_2 = \begin{pmatrix} -3 & 2.1 \\ -1.7 & 2.5 \end{pmatrix}.$$

Set $\underline{\rho} = 0.1, \overline{\rho} = 0.5, h = 3, \eta_1 = \eta_2 = 10$, the system (1) is sampled with $T = 0.1s$ and the disturbance attenuation level $\gamma = 0.9$, then $\gamma_A = 0.0624, \gamma_B = 0.0669$. By Theorem 3.3, one has

$$X_1 = \begin{pmatrix} 0.1930 & 0.1631 \\ 0.1631 & 1.2814 \end{pmatrix}, Y_1 = \begin{pmatrix} 0.0139 & -0.1715 \\ 0.1232 & 1.2417 \end{pmatrix};$$

$$X_2 = \begin{pmatrix} 1.4494 & -0.5334 \\ -0.5334 & 0.4916 \end{pmatrix}, Y_2 = \begin{pmatrix} -0.3083 & 0.2002 \\ 0.8920 & -0.3780 \end{pmatrix}$$

and by $K_i^h = Y_i X_i^{-1}$, the gain matrices of the state feedback sub-controller are given by

$$K_1^h = \begin{pmatrix} 0.2071 & -0.1602 \\ -0.2023 & 0.9947 \end{pmatrix}, K_2^h = \begin{pmatrix} -0.1046 & 0.2937 \\ 0.5535 & -0.1684 \end{pmatrix}.$$

Example 2. Consider the discrete-time switched linear system (2) with $N = 2$, $\sigma(k) : \mathbb{N}\bigcup\{0\} \to \{1, 2\}$ and

$$M_1 = \begin{pmatrix} -2.5 & 4.3 \\ -1 & 1.41 \end{pmatrix}, M_2 = \begin{pmatrix} 0.71 & 2.3 \\ -0.9 & -2.02 \end{pmatrix}, N_1 = \begin{pmatrix} 2 & -1 \\ 0 & -2 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 1.7 & -1.3 \\ -1 & -0.2 \end{pmatrix}, B_{11} = \begin{pmatrix} -0.01 & 0 \\ 0.01 & 0 \end{pmatrix}, B_{12} = \begin{pmatrix} -0.01 & 0 \\ -0.01 & 0 \end{pmatrix},$$

$$C_{11} = \begin{pmatrix} -3 & -2 \\ 2 & -4 \end{pmatrix}, C_{12} = \begin{pmatrix} -3 & -1.8 \\ 1.7 & -3.9 \end{pmatrix}, D_1 = \begin{pmatrix} -4 & 0 \\ 2 & 4 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} -1.6 & 2.8 \\ 1.9 & 3.2 \end{pmatrix}, C_{21} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}, C_{22} = \begin{pmatrix} 2.4 & 1.4 \\ -0.2 & 2 \end{pmatrix}.$$

Set $\underline{\rho} = 0.1, \overline{\rho} = 0.5, h = 3, \eta_1 = 100, \eta_2 = 40$, the system (2) is sampled with $T = 0.1s$ and the disturbance attenuation level $\gamma = 0.9$, then

$$X_{11} = \begin{pmatrix} 52.1889 & -0.4158 \\ -0.4158 & 50.5443 \end{pmatrix}, Y_{11} = \begin{pmatrix} 34.1746 & -4.3079 \\ -4.3079 & 32.4563 \end{pmatrix};$$

$$X_{21} = \begin{pmatrix} 23.0332 & -0.4016 \\ -0.4016 & 25.0708 \end{pmatrix}, Y_{21} = \begin{pmatrix} 7.9361 & -0.2951 \\ -0.2951 & 10.6943 \end{pmatrix}.$$

By Theorem 4.3, the system (2) with $N = 2$ satisfies robust H_∞ performance with the disturbance attenuation level $\gamma = 0.9$ via dynamic output feedback controllers and a switching strategy, where the dynamic output feedback controllers are given by:

$$\Gamma_1 : \begin{cases} \xi(k+1) = \begin{pmatrix} 0.0011 & 0.0048 \\ 0.0007 & 0.0030 \end{pmatrix} \xi(k) + \begin{pmatrix} 0.7336 & -1.5968 \\ 0.0269 & 0.1278 \end{pmatrix} y(k) \\ u(k) = \begin{pmatrix} -0.0001 & -0.0003 \\ 0.0001 & 0.0004 \end{pmatrix} \xi(k) + \begin{pmatrix} -0.3126 & 0.1878 \\ 0.2813 & -0.9691 \end{pmatrix} y(k) \end{cases}$$

$$\Gamma_2 : \begin{cases} \xi(k+1) = \begin{pmatrix} -0.0024 & 0.0015 \\ 0.0014 & 0.0001 \end{pmatrix} \xi(k) + \begin{pmatrix} 0.32186 & -0.5985 \\ -1.0162 & -0.1632 \end{pmatrix} y(k) \\ u(k) = \begin{pmatrix} 0.0004 & -0.0001 \\ -0.0001 & 0.0001 \end{pmatrix} \xi(k) + \begin{pmatrix} -0.5135 & 0.6055 \\ 0.1472 & 0.3612 \end{pmatrix} y(k) \end{cases}$$

They are evident that neither of the designed controllers makes the associated subsystem asymptotically stable.

6. Conclusions

The robust H_∞ control problem has been studied via switched state feedback and switched dynamic output feedback for discrete-time switched linear systems with exponential uncertainties by using Taylor series approximation, convex polytope technique and LMI method. Sufficient conditions are, by solving linear matrix inequalities, presented to realize the H_∞ control design. How to design switched controllers to improve the performance of singular uncertain switched systems should be further studied in the future work.

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Changlin Li
College of Computer Science and Information
Guizhou University
Guiyang, 550025, Guizhou, P. R. China
E-mail address: clin817@yahoo.cn

Fei Long
Institute of Intelligent Information Processing
Guizhou University
Guiyang, 550025, Guizhou, P. R. China
E-mail address: feilong73@yahoo.com

Guohui Ren
College of Computer Science and Information
Guizhou University
Guiyang, 550025, Guizhou, P. R. China
E-mail address: guohuiren@163.com