THE COMPETITION NUMBERS OF HAMMING GRAPHS WITH DIAMETER AT MOST THREE

Boram Park and Yoshio Sano

ABSTRACT. The competition graph of a digraph D is a graph which has the same vertex set as D and has an edge between x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of a graph G is defined to be the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and it has been one of important research problems in the study of competition graphs. In this paper, we compute the competition numbers of Hamming graphs with diameter at most three.

1. Introduction and main results

The notion of a competition graph was introduced by Cohen [2] in connection with a problem in ecology (see also [3]). The *competition graph $C(D)$* of a digraph D is the (simple undirected) graph which has the same vertex set as D and has an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Roberts [16] defined the *competition number $k(G)$* of a graph G to be the smallest number k such that G together with k isolated vertices is the competition graph of an acyclic digraph. Opsut [12] showed that the computation of the competition number of a graph is an NP-hard problem. In the study of competition graphs, it has been one of important research problems to compute the competition numbers for various graph classes (see [1], [4], [5], [6], [7], [9], [11], [16], [18] for graphs whose competition numbers are known). For

some special graph families, we have explicit formulae for computing competition numbers. For example, if G is a chordal graph without isolated vertices, then $k(G) = 1$, and if G is a nontrivial triangle-free connected graph, then $k(G) = |E(G)| - |V(G)| + 2$ (see [16]).

Recently, it has been concerned to find the competition numbers of interesting graph families used in many areas of mathematics and computer science (see [8], [10], [13], [14], [15], [17]). Hamming graphs are known as an interesting graph family in connection with error-correcting codes, association schemes, and several branches of mathematics. For a positive integer q, we denote the q-set $\{1, 2, \ldots, q\}$ by $[q]$. Also we denote the set of n-tuples over $[q]$ by $[q]^n$. For positive integers n and q, the *Hamming graph* $H(n, q)$ is the graph which has the vertex set $[q]^n$ and in which two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are adjacent if $d_H(x, y) = 1$, where $d_H : [q]^n \times [q]^n \to \mathbb{Z}$ is the *Hamming distance* defined by

$$d_H(x, y) := |\{i \in [n] \mid x_i \neq y_i\}|.$$

Note that the diameter of the Hamming graph $H(n, q)$ is equal to n if $q \geq 2$. Since the Hamming graph $H(n, q)$ is an $n(q-1)$-regular graph with q^n vertices, it follows that the number of edges of the Hamming graph $H(n, q)$ is equal to $\frac{1}{2}n(q-1)q^n$.

In this paper, we study the competition numbers of Hamming graphs. If $q = 1$, then $H(n, 1)$ is K_1 and so the following holds:

Proposition 1. *For $n \geq 1$, we have $k(H(n, 1)) = 0$.*

If $q = 2$, then since $H(n, 2)$ triangle-free and connected, we have

$$
\begin{aligned}
k(H(n, 2)) &= |E(H(n, 2))| - |V(H(n, 2))| + 2 \\
&= n2^{n-1} - 2^n + 2 \\
&= (n - 2)2^{n-1} + 2.
\end{aligned}
$$

Proposition 2. *For $n \geq 1$, we have $k(H(n, 2)) = (n - 2)2^{n-1} + 2$.*

If $n = 1$, then $H(1, q)$ is the complete graph K_q with q vertices and so the following holds:

Proposition 3. *For $q \geq 2$, we have $k(H(1, q)) = 1$.*

However, in general, it is not easy to compute $k(H(n, q))$. In this paper, we give the exact values of $k(H(2, q))$ and $k(H(3, q))$. Our main results are the following:

Theorem 4. *For $q \geq 2$, we have $k(H(2, q)) = 2$.*

Theorem 5. *For $q \geq 3$, we have $k(H(3, q)) = 6$.*

We use the following notation and terminology in this paper. For a digraph D, a sequence $v_1, v_2, \ldots, v_n$ of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i > j$. It is well-known that a digraph D is acyclic

if and only if there exists an acyclic ordering of D. For a digraph D and a vertex v of D, we define the *out-neighborhood* $N_D^+(v)$ of v in D to be the set $\{w \in V(D) \mid (v, w) \in A(D)\}$, and the *in-neighborhood* $N_D^-(v)$ of v in D to be the set $\{w \in V(D) \mid (w, v) \in A(D)\}$. A vertex in the out-neighborhood $N_D^+(v)$ of a vertex v in a digraph D is called a *prey* of v in D. For a graph G and a vertex v of G, we define the *open neighborhood* $N_G(v)$ of v in G to be the set $\{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N_G[v]$ of v in G to be the set $N_G(v) \cup \{v\}$. We denote the subgraph of G induced by $N_G(v)$ (resp. $N_G[v]$) by the same symbol $N_G(v)$ (resp. $N_G[v]$).

For a clique S of a graph G and an edge e of G, we say e *is covered by* S if both of the endpoints of e are contained in S. An *edge clique cover* of a graph G is a family of cliques of G such that each edge of G is covered by some clique in the family. The *edge clique cover number* $\theta_E(G)$ of a graph G is the minimum size of an edge clique cover of G. An edge clique cover of G is called a *minimum edge clique cover* of G if its size is equal to $\theta_E(G)$. A *vertex clique cover* of a graph G is a family of cliques of G such that each vertex of G is contained in some clique in the family. The smallest size of a vertex clique cover of G is called the *vertex clique cover number*, and is denoted by $\theta_V(G)$.

We denote a path with n vertices by P_n, a cycle with n vertices by C_n, a graph with n vertices and no edges by I_n, and a complete multipartite graph by $K_{n_1,\ldots,n_m}$.

2. Proofs of Theorems 4 and 5

2.1. Cliques in a Hamming graph

Let $\pi_j : [q]^n \to [q]^{n-1}$ be a map defined by

$$(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

For $j \in [n]$ and $p \in [q]^{n-1}$, let

$$(1) \qquad S_j(p) := \pi_j^{-1}(p) = \{x \in [q]^n \mid \pi_j(x) = p\}.$$

Then $S_j(p)$ is a clique of $H(n, q)$ with size q. Let

$$(2) \qquad \mathcal{F}(n, q) := \{S_j(p) \mid j \in [n], p \in [q]^{n-1}\}.$$

Then $\mathcal{F}(n, q)$ is the family of maximal cliques of $H(n, q)$.

Lemma 6. *Let $n \geq 2$ and $q \geq 2$, and let K be a clique of $H(n, q)$ with size at least 2. Then there exists a unique maximal clique S of $H(n, q)$ containing K.*

Proof. Since $\mathcal{F}(n, q)$ is the family of maximal cliques of $H(n, q)$, it is sufficient to show that there is a unique maximal clique in $\mathcal{F}(n, q)$ containing K.

Take a vertex $x = (x_1, x_2, \ldots, x_n) \in K$. Now we will show that there exists a unique integer j such that $\pi_j(x) = \pi_j(y)$ for all vertices $y \in K \setminus \{x\}$. Take a vertex $y = (y_1, y_2, \ldots, y_n) \in K \setminus \{x\}$. Since K is a clique, x and y are adjacent. Then there is a unique integer $j \in [n]$ such that $x_j \neq y_j$ and $\pi_j(x) = \pi_j(y)$. Suppose that there is a vertex $z = (z_1, z_2, \ldots, z_n) \in K \setminus \{x, y\}$ such that

$\pi_j(z) \neq \pi_j(x)$. Since x and z are adjacent, there is $j_1 \in [n]$ with $j_1 \neq j$ such that $\pi_{j_1}(x) = \pi_{j_1}(z)$, and thus $x_j = z_j$. Since y and z are adjacent, there is $j_2 \in [n]$ with $j_2 \neq j$ such that $\pi_{j_2}(y) = \pi_{j_2}(z)$, and thus $y_j = z_j$. Thus we have $x_j = z_j = y_j$, which contradicts to the fact that $x_j \neq y_j$. Therefore, $\pi_j(z) = \pi_j(x)$.

It implies that j is the unique integer such that $\pi_j(x) = \pi_j(y)$ for all $y \in K$. Hence K is contained in $S_j(\pi_j(x)) \in \mathcal{F}(n,q)$. From the uniqueness of $j \in [n]$ and the fact that $\pi_j(x)$ does not depend on the choice of $x \in K$, it follows that $S_j(\pi_j(x))$ is the unique maximal clique containing K. $\square$

Lemma 7. *The following hold*:

    (a) *The family $\mathcal{F}(n,q)$ defined by (2) is an edge clique cover of $H(n,q)$.*
    (b) *The edge clique cover number of $H(n,q)$ is equal to nq^{n-1}.*
    (c) *Any minimum edge clique cover of $H(n,q)$ consists of edge disjoint maximum cliques.*

Proof. Since each edge is contained in a maximal clique and $\mathcal{F}(n.q)$ is the family of maximal cliques in $H(n,q)$, it follows that $\mathcal{F}(n,q)$ is an edge clique cover of $H(n,q)$.

Let $\mathcal{E}$ be a minimum edge clique cover of $H(n,q)$, that is, $\theta_E(H(n,q)) = |\mathcal{E}|$. Since $\mathcal{F}(n,q)$ is an edge clique cover with $|\mathcal{F}(n,q)| = nq^{n-1}$, we have $|\mathcal{E}| \leq nq^{n-1}$. Now we will show that $|\mathcal{E}| \geq nq^{n-1}$. For a clique S, let $E(S) := \binom{S}{2}$. Since $\mathcal{E}$ is an edge clique cover of $H(n,q)$, it holds that

$$(3) \qquad\qquad |E(H(n,q))| \ \leq \ \sum_{S \in \mathcal{E}} |E(S)|,$$

and the equality holds if and only if none of two distinct cliques in $\mathcal{E}$ have a common edge. Since the maximum size of a clique of $H(n,q)$ is equal to q, we have $|E(S)| \leq \binom{q}{2}$ for each $S \in \mathcal{E}$. Therefore,

$$(4) \qquad\qquad \sum_{S \in \mathcal{E}} |E(S)| \ \leq \ \binom{q}{2} \times |\mathcal{E}|,$$

and the equality holds if and only if any element of $\mathcal{E}$ is a maximum clique in $H(n,q)$. Since $|E(H(n,q))| = \frac{1}{2}n(q-1)q^n = \binom{q}{2} \times nq^{n-1}$, it follows from (3) and (4) that $nq^{n-1} \leq |\mathcal{E}|$, or $nq^{n-1} = |\mathcal{E}|$.

Moreover, since two equalities of (3) and (4) hold, we can conclude that any minimum edge clique cover of $H(n,q)$ consists of edge disjoint maximum cliques. $\square$

Corollary 8. *The family $\mathcal{F}(n,q)$ defined by (2) is a minimum edge clique cover of $H(n,q)$.*

Proof. It follows from the fact that $|\mathcal{F}(n,q)| = nq^{n-1}$ and Lemma 7. $\square$

2.2. Proof of Theorem 4

In this subsection, we give a proof of Theorem 4.

Lemma 9. *Let $n \geq 2$ and $q \geq 2$. For any vertex x of $H(n, q)$, we have $\theta_V(N_{H(n,q)}(x)) = n$.*

Proof. Take any vertex $x \in [q]^n$ of $H(n, q)$. Then the vertex x is adjacent to a vertex y such that $\pi_j(x) = \pi_j(y)$ for some $j \in [n]$. We can easily check from the definition of $H(n, q)$ that, for any $j \in [n]$, the set $S_j(\pi_j(x)) := \{y \in [q]^n \mid \pi_j(x) = \pi_j(y)\}$ forms a clique of $H(n, q)$. Since $N_{H(n,q)}(x) = \cup_{j \in [n]} S_j(\pi_j(x)) \setminus \{x\}$, the family $\{S_j(\pi_j(x)) \mid j \in [n]\}$ is a vertex clique cover of $N_{H(n,q)}(x)$ and so $\theta_V(N_{H(n,q)}(x)) \leq n$.

Moreover, note that $S_j(\pi_j(x)) \cap S_{j'}(\pi_{j'}(x)) = \{x\}$ for $j, j' \in [n]$ where $j \neq j'$. Take $y_j \in S_j(\pi_j(x)) \setminus \{x\}$ for each $j \in [n]$. Then $y_1, y_2, \ldots, y_n$ are n vertices of $N_{H(n,q)}(x)$ such that no two of them can be covered by a same clique and so $\theta_V(N_{H(n,q)}(x)) \geq n$. $\square$

Opsut showed the following lower bound for the competition number of a graph.

Theorem 10 ([12]). *For a graph G, it holds that $k(G) \geq \min\{\theta_V(N_G(v)) \mid v \in V(G)\}$.*

Corollary 11. *If $n \geq 2$ and $q \geq 2$, then $k(H(n, q)) \geq n$.*

Proof. It immediately follows from Lemma 9 and Theorem 10. $\square$

We define a total order $\prec$ on the set $[q]^n$ as follows. Take two distinct elements $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $[q]^n$. Then we define $x \prec y$ if there exists $j \in [n]$ such that $x_i = y_i$ for $i \leq j - 1$ and $x_j < y_j$. The *lexicographic ordering* of $[q]^n$ is the ordering $v_1, v_2, \ldots, v_{q^n}$ such that $v_1 \prec v_2 \prec \cdots \prec v_{q^n}$.

Proof of Theorem 4. By Corollary 11, it follows that $k(H(2, q)) \geq 2$. Now we show that $k(H(2, q)) \leq 2$. We define a digraph D as follows:

$$V(D) = V(H(2, q)) \cup \{z_1, z_2\},$$

$$A(D) = \left(\bigcup_{i=2}^{q} \{(x, (1, i-1)) \mid x \in S_1(i)\} \right) \cup \left(\bigcup_{i=2}^{q} \{(x, (i-1, q)) \mid x \in S_2(i)\} \right)$$
$$\cup \{(x, z_1) \mid x \in S_1(1)\} \cup \{(x, z_2) \mid x \in S_2(1)\},$$

where $S_j(i)$ with $j \in \{1, 2\}$ and $i \in [q]$ is the clique of $H(2, q)$ defined by (1). Since $\{N_D^-(v) \mid v \in V(D), |N_D^-(v)| \geq 2\} = \mathcal{F}(2, q)$, it is easy to check that $C(D) = H(2, q) \cup \{z_1, z_2\}$. In addition, the ordering obtained by adding z_1, z_2 on the head of the lexicographic ordering of $V(H(2, q))$ is an acyclic ordering of D. To see why, take an arc $(x, y) \in A(D)$. If $x \in S_1(1)$ or $x \in S_2(1)$, then y is either z_1 or z_2. If $x \in S_1(i)$ or $x \in S_2(i)$ for some $2 \leq i \in [q]$, then $x = (l, i)$

or $x = (i, l)$ for some $l \in [q]$. Since $y = (1, i - 1)$ or $y = (i - 1, q)$, we have $y \prec x$. Therefore D is acyclic. Hence we have $k(H(2, q)) \leq 2$. $\square$

2.3. Proof of Theorem 5

In this subsection, we give a proof of Theorem 5.

2.3.1. $k(H(3, q)) \geq 6$ $(q \geq 3)$

First, we improve the lower bound for the competition number of a Hamming graph $H(n, q)$ given in Corollary 11 in the case where $n \geq 3$ and $q \geq 3$.

Lemma 12. *Let $n \geq 2$ and $q \geq 2$. Let D be an acyclic digraph such that $C(D) = H(n, q) \cup I_k$ with $I_k = \{z_1, z_2, \ldots, z_k\}$. Let $z_1, z_2, \ldots, z_k, v_1, v_2, \ldots, v_{q^n}$ be an acyclic ordering of D. Let $U_i := \{v_1, \ldots, v_i\}$ for $i \in \{1, \ldots, q^n\}$. Then*

$$|\{S \in \mathcal{F}(n, q) \mid S \cap U_i \neq \emptyset\}| \leq k + i - 1.$$

Proof. Let

$$
\begin{aligned}
\mathcal{N}(D) &:= \{N_D^-(x) \mid x \in V(D), |N_D^-(x)| \geq 2\}, \\
\mathcal{S}_i &:= \{N_D^-(x) \mid x \in U_{i-1} \cup I_k, |N_D^-(x)| \geq 2\}, \\
\mathcal{K}_i &:= \{K \in \mathcal{N}(D) \mid K \cap U_i \neq \emptyset\}.
\end{aligned}
$$

Since D is acyclic, it holds that $\mathcal{K}_i = \{K \in \mathcal{S}_i \mid K \cap U_i \neq \emptyset\}$. Since $|\mathcal{S}_i| \leq k + i - 1$, it follows that

$$(5) \qquad |\mathcal{K}_i| = |\{K \in \mathcal{S}_i \mid K \cap U_i \neq \emptyset\}| \leq |\mathcal{S}_i| \leq k + i - 1.$$

For each $K \in \mathcal{K}_i$, there exists a unique element in $\mathcal{F}(n, q)$ containing K by Lemma 6, we denote it by S_K. From (5), it remains to show that

$$(6) \qquad |\{S \in \mathcal{F}(n, q) \mid S \cap U_i \neq \emptyset\}| \leq |\mathcal{K}_i|.$$

Take $S \in \mathcal{F}(n, q)$ such that $S \cap U_i \neq \emptyset$. Then there exists a vertex x in $S \cap U_i$. Since $q \geq 2$, there exists a vertex $y \in S \setminus \{x\}$. Since $C(D) = H(n, q) \cup I_k$ and the vertices x and y are adjacent, there is a common prey u of x and y in D. Then $x \in N_D^-(u) \cap U_i$ and so $N_D^-(u) \in \mathcal{K}_i$. Since $N_D^-(u)$ contains x and y, $S_{N_D^-(u)}$ is a maximal clique containing x and y. Then both S and $S_{N_D^-(u)}$ are maximal cliques containing x and y. By Lemma 6, we have $S = S_{N_D^-(u)}$, which implies that $S \in \{S_K \mid K \in \mathcal{K}_i\}$. It follows that

$$\{S \in \mathcal{F}(n, q) \mid S \cap U_i \neq \emptyset\} \subseteq \{S_K \mid K \in \mathcal{K}_i\},$$

and together with $|\{S_K \mid K \in \mathcal{K}_i\}| \leq |\mathcal{K}_i|$, (6) holds. Hence, the lemma holds. $\square$

Lemma 13. *For $n \geq 3$ and $q \geq 3$, we have $k(H(n, q)) \geq 3n - 4$.*

Proof. Let k be the competition number of $H(n, q)$ and let D be an acyclic digraph such that $C(D) = H(n, q) \cup I_k$ with $I_k = \{z_1, z_2, \ldots, z_k\}$. Let $z_1, z_2,$

$\ldots, z_k, v_1, v_2, \ldots, v_{q^n}$ be an acyclic ordering of D. Let $U_3 := \{v_1, v_2, v_3\}$. By Lemma 12, it holds that

$$(7) \qquad |\{S \in \mathcal{F}(n, q) \mid S \cap U_3 \neq \emptyset\}| \leq k + 2.$$

In addition, it holds that $|\{S \in \mathcal{F}(n, q) \mid S \cap U_3 \neq \emptyset\}| \geq 3n - 2$ whose proof will be shown in next paragraph. Therefore, we have $3n - 2 \leq k + 2$, or $k \geq 3n - 4$.

Now it remains to show that $|\{S \in \mathcal{F}(n, q) \mid S \cap U_3 \neq \emptyset\}| \geq 3n - 2$. Consider the subgraph of $H(n, q)$ induced by U_3, say H. Then H is isomorphic to one of the following:

$$\text{(i) } K_3 \qquad \text{(ii) } P_3 \qquad \text{(iii) } P_2 \cup I_1 \qquad \text{(iv) } I_3.$$

Case (i) $H \cong K_3$: By Lemma 6, U_3 is contained in exactly one maximal clique. Without loss of generality, we may assume that U_3 is contained in $S_1((\underbrace{1, \ldots, 1}_{n-1}))$, and so we may also assume that

$$U_3 = \{(\underbrace{1, 1, \ldots, 1}_{n}), (2, \underbrace{1, \ldots, 1}_{n-1}), (3, \underbrace{1, \ldots, 1}_{n-1})\}.$$

Then the family $\{S \in \mathcal{F}(n, q) \mid S \cap U_3 \neq \emptyset\}$ consists of the following $3n - 2$ elements:

$$S_1((\underbrace{1, \ldots, 1}_{n-1})), \quad S_j((i, \underbrace{1, \ldots, 1}_{n-2})) \ (i \in \{1, 2, 3\}, \ j \in [n] \setminus \{1\}).$$

Case (ii) $H \cong P_3$: Without loss of generality, we may assume that

$$U_3 = \{(\underbrace{1, \ldots, 1}_{n}), (2, \underbrace{1, \ldots, 1}_{n-1}), (1, 2, \underbrace{1, \ldots, 1}_{n-2})\}.$$

Then the family $\{S \in \mathcal{F}(n, q) \mid S \cap U_3 \neq \emptyset\}$ consists of the following $3n - 2$ elements:

$$S_j((\underbrace{1, \ldots, 1}_{n-1})) \ (j \in [n]), \quad S_j((2, \underbrace{1, \ldots, 1}_{n-2})) \ (j \in [n] \setminus \{1\}),$$
$$S_j((1, 2, \underbrace{1, \ldots, 1}_{n-3})) \ (j \in [n] \setminus \{2\}).$$

Case (iii) $H \cong P_2 \cup I_1$ or **(iv)** $H \cong I_3$: Let v be an isolated vertex of H. Since the n cliques in $\mathcal{F}(n, q)$ containing v do not contain the other vertices of U_3, it is sufficient to show that $\{S \in \mathcal{F}(n, q) \mid S \cap (U_3 \setminus \{v\}) \neq \emptyset\}$ has at least $2n - 2$ elements. Since, for each vertex $u \in U_3 \setminus \{v\}$, there are n cliques in $\mathcal{F}(n, q)$ containing u and there is at most one clique in $\mathcal{F}(n, q)$ containing the two vertices of $U_3 \setminus \{v\}$. Thus we can conclude that $\{S \in \mathcal{F}(n, q) \mid S \cap (U_3 \setminus \{v\}) \neq \emptyset\}$ has at least $2n - 1$ elements.

We complete the proof. $\qquad \square$

If $n = 3$, then the above lower bound gives $k(H(3, q)) \geq 5$. In this case, however, we can improve the bound as follows.

Lemma 14. *For $q \geq 3$, we have $k(H(3, q)) \geq 6$.*

Proof. By Lemma 13, we have $k(H(3, q)) \geq 5$. Suppose that $k(H(3, q)) = 5$. Then there exists an acyclic digraph D such that $C(D) = H(3, q) \cup I_5$ with $I_5 = \{z_1, z_2, \ldots, z_5\}$. Let $z_1, z_2, \ldots, z_5, v_1, v_2, \ldots, v_{q^3}$ be an acyclic ordering of D. Let $U_4 := \{v_1, v_2, v_3, v_4\}$. For convenience, let

$$\mathcal{A}_1 := \{S \in \mathcal{F}(3, q) \mid S \cap U_4 \neq \emptyset\}, \quad \mathcal{A}_2 := \{S \in \mathcal{F}(3, q) \mid v_5 \in S\}.$$

Now we consider the subgraph G of $H(3, q)$ induced by U_4. Any graph on 4 vertices is isomorphic to one of the following graphs:

$$\begin{array}{llll}
\text{(i) } K_4 & \text{(ii) } K_{1,1,2} & \text{(iii) } K_4 - E(P_3) & \text{(iv) } C_4 \\
\text{(v) } P_4 & \text{(vi) } K_{1,3} & \text{(vii) } K_3 \cup I_1 & \text{(viii) } K_2 \cup K_2 \\
\text{(ix) } P_3 \cup I_1 & \text{(x) } K_2 \cup I_2 & \text{(xi) } I_4.
\end{array}$$

Since $H(3, q)$ does not contain an induced subgraph isomorphic to $K_{1,1,2}$ by Lemma 6, G is one of the above graphs except (ii). For each cases, the number $|\mathcal{A}_1|$ is given as follows:

$$\begin{array}{llll}
\text{(i) } 9 & \text{(ii) } - & \text{(iii) } 9 & \text{(iv) } 8 \\
\text{(v) } 9 & \text{(vi) } 9 & \text{(vii) } 10 & \text{(viii) } 10 \\
\text{(ix) } 10 & \text{(x) } 11 & \text{(xi) } 12.
\end{array}$$

By Lemma 12, we have $|\mathcal{A}_1| \leq 8$. Therefore $G \cong C_4$ and so $|\mathcal{A}_1| = 8$. Since each vertex of $H(3, q)$ is contained in exactly 3 cliques in $\mathcal{F}(3, q)$, $|\mathcal{A}_2| = 3$. From the fact that

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \{S \in \mathcal{F}(3, q) \mid S \cap (U_4 \cup \{v_5\}) \neq \emptyset\},$$

it holds that $|\mathcal{A}_1 \cup \mathcal{A}_2| \leq 9$ by Lemma 12. Since $|\mathcal{A}_1| = 8$, $|\mathcal{A}_2| = 3$, and $|\mathcal{A}_1 \cup \mathcal{A}_2| \leq 9$, we have $|\mathcal{A}_1 \cap \mathcal{A}_2| = |\mathcal{A}_1| + |\mathcal{A}_2| - |\mathcal{A}_1 \cup \mathcal{A}_2| \geq 8 + 3 - 9 = 2$.

Take two distinct cliques $S, S' \in \mathcal{A}_1 \cap \mathcal{A}_2$. Then $S \cap U_4 \neq \emptyset$, $S' \cap U_4 \neq \emptyset$ and so take $x \in S \cap U_4$ and $y \in S' \cap U_4$. If $x = y$ or x and y are adjacent, then $S = S'$ by Lemma 6. Therefore x and y are not adjacent. Since $G \cong C_4$, without loss of generality, we may assume that

$$U_4 = \{(1, 1, 1), (1, 1, 2), (1, 2, 2), (1, 2, 1)\}, \quad x = (1, 1, 1), \quad y = (1, 2, 2).$$

Since x and v_5 are adjacent, one of the following holds:

(8)
$$\begin{aligned}
&\pi_1(x) = (1, 1) = \pi_1(v_5), \quad \pi_2(x) = (1, 1) = \pi_2(v_5), \\
&\pi_3(x) = (1, 1) = \pi_3(v_5).
\end{aligned}$$

Since y and v_5 are adjacent, one of the following holds:

(9)
$$\begin{aligned}
&\pi_1(y) = (2, 2) = \pi_1(v_5), \quad \pi_2(y) = (1, 2) = \pi_2(v_5), \\
&\pi_3(y) = (1, 2) = \pi_3(v_5).
\end{aligned}$$

However, it is impossible that v_5 satisfies both one of (8) and one of (9) since $v_5 \notin U_4$. We reach a contradiction. Hence we conclude $k(H(3, q)) \geq 6$. $\quad\square$

2.3.2. $k(H(3, q)) \leq 6 \ (q \geq 3)$

Next, we show the upper bound $k(H(3, q)) \leq 6$ for $q \geq 3$. To show the upper bound, we introduce a graph $K_{q_1} \square K_{q_2} \square K_{q_3}$ as an extension of $H(3, q)$.

For graphs G and H, the *Cartesian product* $G \square H$ of G and H is the graph which has the vertex set $V(G) \times V(H)$ and has an edge between two vertices (g, h) and (g', h') if and only if $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. Note that the Cartesian product of n complete graphs K_q of size q is the Hamming graph $H(n, q)$.

Let $q_1, q_2, q_3 \geq 2$ be integers and we consider the graph $K_{q_1} \square K_{q_2} \square K_{q_3}$. Define

$$\pi_1 : [q_1] \times [q_2] \times [q_3] \rightarrow [q_2] \times [q_3], \qquad (x_1, x_2, x_3) \mapsto (x_2, x_3),$$
$$\pi_2 : [q_1] \times [q_2] \times [q_3] \rightarrow [q_1] \times [q_3], \qquad (x_1, x_2, x_3) \mapsto (x_1, x_3),$$
$$\pi_3 : [q_1] \times [q_2] \times [q_3] \rightarrow [q_1] \times [q_2], \qquad (x_1, x_2, x_3) \mapsto (x_1, x_2).$$

For $p_1 \in [q_1]$, $p_2 \in [q_2]$, and $p_3 \in [q_3]$, let

$$
\begin{aligned}
S_1((p_2, p_3)) &:= \{x \in [q_1] \times [q_2] \times [q_3] \mid \pi_1(x) = (p_2, p_3)\}, \\
S_2((p_1, p_3)) &:= \{x \in [q_1] \times [q_2] \times [q_3] \mid \pi_2(x) = (p_1, p_3)\}, \\
S_3((p_1, p_2)) &:= \{x \in [q_1] \times [q_2] \times [q_3] \mid \pi_3(x) = (p_1, p_2)\}.
\end{aligned}
$$

Note that $S_1((p_2, p_3))$, $S_2((p_1, p_3))$, and $S_3((p_1, p_2))$ are maximal cliques of $K_{q_1} \square K_{q_2} \square K_{q_3}$. We denote the set of all maximal cliques $S_1((p_2, p_3)), S_2((p_1, p_3))$ and $S_3((p_1, p_2))$ by $\mathcal{F}_{(q_1, q_2, q_3)}$. Then $\mathcal{F}_{(q_1, q_2, q_3)}$ is an edge clique cover of $K_{q_1} \square K_{q_2} \square K_{q_3}$.

Lemma 15. *For $q_1, q_2, q_3 \geq 2$, there exists an acyclic digraph D such that $C(D) = (K_{q_1} \square K_{q_2} \square K_{q_3}) \cup I_6$ and*

$$\{N_D^-(v) \mid v \in V(D), |N_D^-(v)| \geq 2\} = \mathcal{F}_{(q_1, q_2, q_3)}.$$

Consequently, we have $k(K_{q_1} \square K_{q_2} \square K_{q_3}) \leq 6$.

Proof. For any digraph D, we define $\mathcal{N}(D) := \{N_D^-(v) \mid v \in V(D), |N_D^-(v)| \geq 2\}$. We prove the lemma by induction on $m = q_1 + q_2 + q_3$. Since $q_1, q_2, q_3 \geq 2$, we have $m \geq 6$. Suppose $m = 6$, i.e., $q_1 = q_2 = q_3 = 2$. Note that $K_2 \square K_2 \square K_2 = H(3, 2)$. Since $H(3, 2)$ is a triangle-free graph, there exists an acyclic digraph D such that $C(D) = H(3, 2) \cup I_6$ and that $N_D^-(v)$ is either the empty set or a maximum clique in $H(3, 2)$ for each vertex $v \in V(D)$ (see Figure 1 for an illustration of such a digraph). Thus the statement is true for $m = 6$.

Suppose that the statement is true for $m = q_1 + q_2 + q_3$ where $m \geq 6$. Consider a graph $K_{q_1} \square K_{q_2} \square K_{q_3}$ such that $m + 1 = q_1 + q_2 + q_3$. Since $q_1 + q_2 + q_3 > 6$, at least one of q_1, q_2, or q_3 is greater than 2. Without loss of generality, we may assume that $q_1 > 2$. Now we consider the graph $K_{q_1-1} \square K_{q_2} \square K_{q_3}$ which is a subgraph of $K_{q_1} \square K_{q_2} \square K_{q_3}$. Then by the induction hypothesis, there exists an acyclic digraph D_0 such that $C(D_0) = (K_{q_1-1} \square K_{q_2} \square K_{q_3}) \cup I_6$
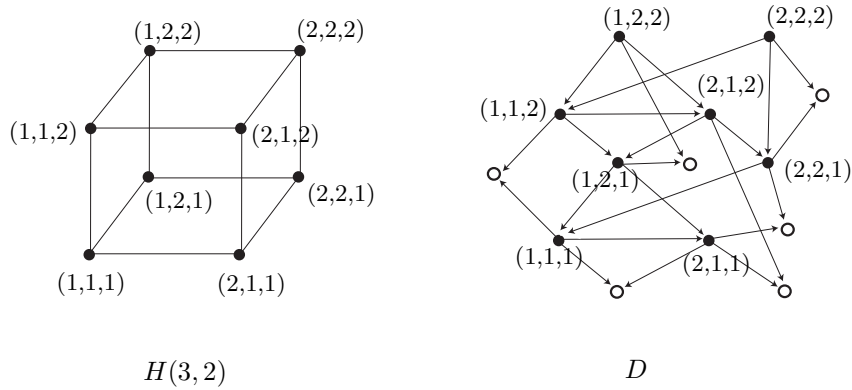
FIGURE 1. The Hamming graph $H(3,2)$ and an acyclic digraph D satisfying $C(D) = H(3,2) \cup I_6$

and

$$\mathcal{N}(D_0) = \mathcal{F}_{(q_1-1,q_2,q_3)}. \tag{10}$$

Let $v_1, v_2, \ldots, v_{q_1 q_2 q_3 - q_2 q_3 + 6}$ be an acyclic ordering of D_0. For convenience, let $w_1 := v_{q_1 q_2 q_3 - q_2 q_3 + 5}$ and $w_2 := v_{q_1 q_2 q_3 - q_2 q_3 + 6}$.

Let H^* be the subgraph of $K_{q_1} \square K_{q_2} \square K_{q_3}$ induced by

$$\begin{aligned} V^* &:= V(K_{q_1} \square K_{q_2} \square K_{q_3}) - V(K_{q_1-1} \square K_{q_2} \square K_{q_3}) \\ &= \{(q_1, p_2, p_3) \mid p_2 \in [q_2], p_3 \in [q_3]\}. \end{aligned}$$

Now we define a digraph D_1 as follows:

$$\begin{aligned} V(D_1) &= V^* \cup \{w_1, w_2\}, \\ A(D_1) &= \left(\bigcup_{i=2}^{q_3} \{(x, (q_1, 1, i-1)) \mid x \in S_2((q_1, i))\} \right) \\ &\quad \cup \left(\bigcup_{i=2}^{q_2} \{(x, (q_1, i-1, q)) \mid x \in S_3((q_1, i))\} \right) \\ &\quad \cup \{(x, w_1) \mid x \in S_2((q_1, 1))\} \cup \{(x, w_2) \mid x \in S_3((q_1, 1))\}. \end{aligned}$$

The ordering obtained by adding w_1, w_2 on the head of the lexicographic ordering of V^* is an acyclic ordering of D_1, and let $w_1, w_2, \ldots, w_{q_2 q_3 + 2}$ be the ordering. In addition,

$$\mathcal{N}(D_1) = \{S_2((q_1, i)) \mid i \in [q_3]\} \cup \{S_3((q_1, i')) \mid i' \in [q_2]\}. \tag{11}$$

Therefore D_1 is an acyclic digraph such that $C(D_1) = H^* \cup \{w_1, w_2\}$.

Note that, for $(p_2, p_3) \in [q_2] \times [q_3]$, the clique in $K_{q_1} \square K_{q_2} \square K_{q_3}$ obtained by deleting the vertex (q_1, p_2, p_3) from an element $S_1((p_2, p_3))$ of $K_{q_1} \square K_{q_2} \square K_{q_3}$

is a maximal clique of $K_{q_1-1}\square K_{q_2}\square K_{q_3}$. Then by (10), for each $(p_2,p_3) \in [q_2] \times [q_3]$, the set $\{v \in V(D_0) \mid N^-_{D_0}(v) = S_1((p_2,p_3)) \setminus \{(q_1,p_2,p_3)\}\}$ is not empty, and so we take an element $y_{(p_2,p_3)}$ of this set.

Now we define a digraph D as follows:

$$\begin{aligned} V(D) &= V(K_{q_1}\square K_{q_2}\square K_{q_3}) \cup I_6, \\ A(D) &= A(D_0) \cup A(D_1) \cup \left\{((q_1,p_2,p_3), y_{(p_2,p_3)}) \mid (p_2,p_3) \in [q_2] \times [q_3]\right\}. \end{aligned}$$

Note that since $N^-_{D_0}(w_1) = N^-_{D_0}(w_2) = \emptyset$,

$$(12) \qquad \{N^-_D(w_1), N^-_D(w_2)\} = \{S_2((q_1,1)), S_3((q_1,1))\}.$$

By the definition of D and (10), (11), and (12), we can conclude that $\mathcal{N}(D) = \mathcal{F}_{(q_1,q_2,q_3)}$ and so $E(C(D)) = E(K_{q_1}\square K_{q_2}\square K_{q_3})$. Thus,

$$C(D) = (K_{q_1}\square K_{q_2}\square K_{q_3}) \cup I_6.$$

Then the ordering

$$(13) \qquad v_1, v_2, \ldots, v_{q_1q_2q_3-q_2q_3+6}, w_3, w_4, \ldots, w_{q_2q_3+2}$$

of the vertices of D is an acyclic ordering. To see this, take an arc $a = (x,y) \in A(D)$. If $a \in A(D_0) \cup A(D_1)$, then y appears before x in (13), since D_0 and D_1 are acyclic. If $a \notin A(D_0) \cup A(D_1)$, then $x \in \{w_3, w_4, \ldots, w_{q_2q_3+2}\}$ and $y \in \{v_1, v_2, \ldots, v_{q_1q_2q_3-q_2q_3+6}\}$, thus y appears before x in (13). Thus the digraph D is acyclic. Hence the lemma holds. $\square$

Proof of Theorem 5. By Lemma 14, we have $k(H(3,q)) \geq 6$ for $q \geq 3$. By Lemma 15, we have $k(H(3,q)) = k(K_q\square K_q\square K_q) \leq 6$ for $q \geq 3$. Hence Theorem 5 holds. $\square$

3. Concluding remarks

In this paper, we gave the exact values of the competition numbers of Hamming graphs with diameter 2 or 3.

We conclude this paper with leaving the following questions for further study:

- What is the competition number of a Hamming graph $H(4,q)$ with diameter 4 for $q \geq 3$?
- Give the exact values or a good bound for the competition numbers of Hamming graphs $H(n,q)$.

References

[1] H. H. Cho and S.-R. Kim, *The competition number of a graph having exactly one hole*, Discrete Math. **303** (2005), no. 1-3, 32–41.

[2] J. E. Cohen, *Interval Graphs and Food Webs: a finding and a problem*, Document 17696-PR, RAND Corporation, Santa Monica, CA, 1968.

[3] ———, *Food Webs and Niche Space*, Princeton University Press, Princeton, NJ, 1978.

[4] S.-R. Kim, *The competition number and its variants*, Quo vadis, graph theory?, 313–326, Ann. Discrete Math., 55, North-Holland, Amsterdam, 1993.

[5] ⎯⎯⎯, *On competition graphs and competition numbers*, Commun. Korean Math. Soc. **16** (2001), no. 1, 1–24.

[6] ⎯⎯⎯, *Graphs with one hole and competition number one*, J. Korean Math. Soc. **42** (2005), no. 6, 1251–1264.

[7] S.-R. Kim, J. Y. Lee, and Y. Sano, *The competition number of a graph whose holes do not overlap much*, Discrete Appl. Math. **158** (2010), no. 13, 1456–1460.

[8] S.-R. Kim, B. Park, and Y. Sano, *The competition numbers of Johnson graphs*, Discuss. Math. Graph Theory **30** (2010), 449–459.

[9] S.-R. Kim and F. S. Roberts, *Competition numbers of graphs with a small number of triangles*, Discrete Appl. Math. **78** (1997), no. 1-3, 153–162.

[10] S.-R. Kim and Y. Sano, *The competition numbers of complete tripartite graphs*, Discrete Appl. Math. **156** (2008), no. 18, 3522–3524.

[11] J. Y. Lee, S.-R. Kim, S.-J. Kim, and Y. Sano, *The competition number of a graph with exactly two holes*, Ars Combin. **95** (2010), 45–54.

[12] R. J. Opsut, *On the computation of the competition number of a graph*, SIAM J. Algebraic Discrete Methods **3** (1982), no. 4, 420–428.

[13] B. Park, S.-R. Kim, and Y. Sano, *On competition numbers of complete multipartite graphs with partite sets of equal size*, preprint **RIMS-1644** October 2008. (http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1644.pdf)

[14] ⎯⎯⎯, *The competition numbers of complete multipartite graphs and mutually orthogonal Latin squares*, Discrete Math. **309** (2009), no. 23-24, 6464–6469.

[15] B. Park and Y. Sano, *The competition numbers of ternary Hamming graphs*, Appl. Math. Lett. **24** (2011), 1608–1613.

[16] F. S. Roberts, *Food webs, competition graphs, and the boxicity of ecological phase space*, Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976), pp. 477–490. Lecture Notes in Math., Vol. 642, Springer, Berlin, 1978.

[17] Y. Sano, *The competition numbers of regular polyhedra*, Congr. Numer. **198** (2009), 211–219.

[18] Y. Wu and J. Lu, *Dimension-2 poset competition numbers and dimension-2 poset double competition numbers*, Discrete Appl. Math. **158** (2010), no. 6, 706–717.

BORAM PARK
DEPARTMENT OF MATHEMATICS EDUCATION
SEOUL NATIONAL UNIVERSITY
SEOUL 151-742, KOREA
E-mail address: kawa22@snu.ac.kr

YOSHIO SANO
NATIONAL INSTITUTE OF INFORMATICS
TOKYO 101-8430, JAPAN
E-mail address: sano@nii.ac.jp