Bull. Korean Math. Soc. **48** (2011), No. 3, pp. 647–653
DOI 10.4134/BKMS.2011.48.3.647

A NOTE ON HYPONORMAL TOEPLITZ OPERATORS

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Abstract. In this note we are concerned with the hyponormality of Toeplitz operators T_ϕ with polynomial symbols $\phi = \bar{g} + f$ $(f, g \in H^\infty(\mathbb{T}))$ when g divides f.

1. Introduction

A bounded linear operator A on a complex Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ is positive semidefinite. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$ for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. Recall that given $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol ϕ is the operator T_ϕ on $H^2(\mathbb{T})$ defined by

$$T_\phi f = P(\phi \cdot f) \quad \text{for } f \in H^2(\mathbb{T})$$

and P denotes the projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Let $H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$. The hyponormality of Toeplitz operators has been studied by many authors (cf. [1-7, 9-16, 18]). In 1988, C. Cowen [3] characterized the hyponormality of a Toeplitz operator T_ϕ on $H^2(\mathbb{T})$ by properties of the symbol $\phi \in L^\infty(\mathbb{T})$. K. Zhu [18] reformulated Cowen's criterion and then showed that the hyponormality of T_ϕ with polynomial symbols ϕ can be decided by a method based on the classical interpolation theorem of I. Schur [17]. Also D. Farenick and W. Y. Lee [6] characterized the hyponormality of T_ϕ in terms of the Fourier coefficients of the trigonometric polynomial ϕ in the cases that the outer coefficients of ϕ have the same modulus. In [12], it was shown that the hyponormality of T_ϕ with polynomial symbols of the form $\phi(z) = \sum_{n=-m}^{N} a_n z^n$ can be determined by the zeros of the analytic polynomial $z^m \phi$. In this note we consider the hyponormality of Toeplitz operators T_ϕ with polynomial symbols $\phi = \bar{g} + f$ $(f, g \in H^\infty(\mathbb{T}))$ when g divides f.

Received October 27, 2009; Revised January 13, 2010.
2010 *Mathematics Subject Classification.* Primary 47B20, 47B35.
Key words and phrases. Toeplitz operators, hyponormal.
This research is financially supported by Changwon National University in 2009-2010.

2. Main results

We begin with:

Lemma 1 (Cowen's Theorem, [3, 16]). *Let* $\phi \in L^\infty(\mathbb{T})$. *If* $\mathcal{E}(\phi) := \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \le 1$ *and* $\phi - k\overline{\phi} \in H^\infty(\mathbb{T})\}$, *then* T_ϕ *is hyponormal if and only if* $\mathcal{E}(\phi) \neq \emptyset$.

On the other hand, in 1993, T. Nakazi and K. Takahashi characterized the hyponormality of a Toeplitz operator in the cases that its self-commutator is of finite rank.

Lemma 2 (Nakazi-Takahashi Theorem, [16]). *A Toeplitz operator* T_ϕ *is hyponormal and the rank of the selfcommutator* $[T_\phi^*, T_\phi]$ *is finite* (*e.g.,* ϕ *is a trigonometric polynomial*) *if and only if there exists a finite Blaschke product* $k \in \mathcal{E}(\phi)$ *such that* $\deg(k) = \operatorname{rank}[T_\phi^*, T_\phi]$.

We record here results on the hyponormality of Toeplitz operators with polynomial symbols, which have been recently developed in the literature. The statement (vii) appears to be new.

Lemma 3. *Suppose that* ϕ *is a trigonometric polynomial of the form* $\phi(z) = \sum_{n=-m}^{N} a_n z^n$, *where* a_{-m} *and* a_N *are nonzero.*

- (i) *If* T_ϕ *is a hyponormal operator, then* $m \le N$ *and* $|a_{-m}| \le |a_N|$.
- (ii) *If* T_ϕ *is a hyponormal operator, then* $N - m \le \operatorname{rank}[T_\phi^*, T_\phi] \le N$.
- (iii) *The hyponormality of* T_ϕ *is independent of the particular values of* $a_0, a_1, \ldots, a_{N-m}$ *of* ϕ. *Moreover the rank of the selfcommutator* $[T_\phi^*, T_\phi]$ *is also independent of those coefficients.*
- (iv) *Write* $\phi = \overline{g} + f$ $(f, g \in H^\infty)$ *and put* $\widetilde{\phi} = \overline{g} + T_{\overline{z}^r} f$ $(r \le N - m)$. *Then* T_ϕ *is hyponormal if and only if* $T_{\tilde{\phi}}$ *is.*
- (v) *If* $|a_{-m}| = |a_N| \neq 0$, *then* T_ϕ *is hyponormal if and only if the following symmetric condition holds*:

$$\overline{a_N} a_{-j} = a_{-m} \overline{a_{N-m+j}} \quad (1 \le j \le m). \tag{3.1}$$

  In this case, the rank of $[T_\phi^*, T_\phi]$ *is* $N - m$ *and*

$$\mathcal{E}(\phi) = \{a_{-m} (\overline{a_N})^{-1} z^{N-m}\}.$$

- (vi) T_ϕ *is normal if and only if* $m = N$, $|a_{-m}| = |a_N|$, *and* (3.1) *holds with* $m = N$.
- (vii) *Write* $\phi := \bar{g} + f$, *where* f *and* g *are in* $H^\infty(\mathbb{T})$ *and put* $\tilde{\phi} := \alpha\, \bar{g} + f$ $(|\alpha| \le 1)$. *If* T_ϕ *is hyponormal, then so is* $T_{\tilde{\phi}}$.

Proof. The assertions (i) – (vi) were shown from [4, 6, 7, 10, 11, 12, 13, 16]. For the assertion (vii), suppose that there exists a function $k \in H^\infty(\mathbb{T})$ such that $\phi - k\,\bar{\phi} \in H^\infty(\mathbb{T})$ and $||k||_\infty \le 1$. Thus $\bar{g} - k\bar{f} \in H^\infty(\mathbb{T})$. Since $|\alpha| \le 1$ it follows that if we let $\tilde{k} = \alpha\, k$, then $\alpha\bar{g} - \tilde{k}\bar{f} = \alpha(\bar{g} - k\bar{f}) \in H^\infty(\mathbb{T})$ and $||\tilde{k}||_\infty = |\alpha|\, ||k||_\infty \le 1$. Therefore by Lemma 1, $T_{\tilde{\phi}}$ is hyponormal. □

Suppose $\phi = \bar{g} + f$, where $f = \sum_{n=1}^{N} a_n z^n$ and $g = \sum_{n=1}^{N} b_n z^n$. If T_ϕ is normal, then g divides f: indeed, by Lemma 3 (v),(vi), $g = e^{i\theta} \sum_{n=1}^{N} a_n z^n$ for some $\theta \in [0, 2\pi)$, so that g divides f. But if T_ϕ is hyponormal, then g need not divide f. For example, consider

$$g(z) = (z + \frac{1}{2})^2 \quad \text{and} \quad f(z) = 3(z+1)^2.$$

Using an argument of P. Fan [5, Theorem 1] – for every trigonometric polynomial ϕ of the form $\phi(z) = \sum_{n=-2}^{2} a_n z^n$,

$$T_\phi \text{ is hyponormal} \iff \left|\det \begin{pmatrix} a_{-1} & a_{-2} \\ a_1 & a_2 \end{pmatrix}\right| \le |a_2|^2 - |a_{-2}|^2, \tag{3.2}$$

a straightforward calculation shows that T_ϕ is hyponormal. How is the converse? That is, if g divides f, does it follow that T_ϕ is hyponormal? However we cannot also expect the hyponormality of T_ϕ when g divides f: for example, if $\phi = \overline{(z+1)^2} + (z+1)^3$, then by Lemma 3 (v), T_ϕ is not hyponormal.

We now consider the hyponormality of T_ϕ with $\phi = \bar{g} + f$ (f and g are analytic polynomials) when g divides f. If ψ is in $H^\infty(\mathbb{T})$, write $\mathcal{Z}(\psi)$ for the set of all zeros of ψ.

Theorem 4. *Suppose $\phi = \bar{g} + f$ with $f, g \in H^\infty(\mathbb{T})$ and $\psi := \frac{f}{g}$ has a factorization $\psi = up$, where u is an inner function and p is an analytic polynomial. If*

(i) $\mathcal{Z}(\psi) \subseteq \mathbb{D}$;

(ii) $\operatorname{ess\,inf} |\psi| \ge 1$,

then T_ϕ is hyponormal.

Proof. By the condition (i), p is of the form

$$p(z) = c \prod_{j=1}^{n} (z - \zeta_j) \quad \text{with } |\zeta_j| < 1 \quad (j = 1, \dots, n).$$

Then we have

$$\frac{1}{\bar{p}} = \frac{1}{\bar{c} \prod_{j=1}^{n} (\bar{z} - \bar{\zeta}_j)} = \frac{z^n}{\bar{c} \prod_{j=1}^{n} (1 - \bar{\zeta}_j z)},$$

which is in $H^\infty(\mathbb{T})$. Put $k := \frac{u}{\bar{p}}$. Then evidently, $k \in H^\infty(\mathbb{T})$. By the condition (ii),

$$||k||_\infty = \text{ess sup} \left|\frac{u}{\bar{p}}\right| = \text{ess sup} \frac{1}{|p|} = \frac{1}{\operatorname{ess\,inf} |p|} = \frac{1}{\operatorname{ess\,inf} |\psi|} \le 1.$$

Since $\bar{f} = \bar{g}\bar{u}\bar{p}$, it follows

$$\phi - k\,\bar{\phi} = (\bar{g} + f) - \frac{u}{\bar{p}}(g + \bar{g}\bar{u}\bar{p}) = f - \frac{u}{\bar{p}} g = f - kg \in H^\infty(\mathbb{T}).$$

Therefore by Cowen's theorem T_ϕ is hyponormal. □

The conditions (i) and (ii) in Theorem 4 need not be necessary for T_ϕ to be hyponormal. To see this consider the trigonometric polynomial $\phi = \bar{g} + f$, where

$$g(z) = (z-1)^2 \quad \text{and} \quad f(z) = 2(z-1)^2(z-\frac{3}{2}).$$

Then $\phi(z) = z^{-2} - 2z^{-1} - 2 + 8z - 7z^2 + 2z^3$. Put $\tilde{\phi}(z) := z^{-2} - 2z^{-1} - 7z + 2z^2$. By Lemma 3 (iv), T_ϕ is hyponormal if and only if $T_{\tilde{\phi}}$ is. A straightforward calculation with (3.2) shows that $T_{\tilde{\phi}}$ is hyponormal and hence so is T_ϕ. But note that $\mathcal{Z}(\frac{f}{g}) \subseteq \mathbb{C} \setminus \mathbb{D}$. Also if $\phi(z) = z^{-1} + z(z-\frac{1}{2})$, then T_ϕ is hyponormal, whereas ess $\inf|\psi| = \frac{1}{2}$.

Corollary 5 ([1])**.** *Suppose $\phi = \bar{g} + f$ with f and g inner. If g divides f, then T_ϕ is hyponormal.*

Proof. Apply Theorem 4 with $p = 1$. □

Corollary 6. *Let $\phi = \bar{g} + f$ with $f, g \in H^\infty(\mathbb{T})$ and suppose*

$$f(z) = c\, g(z) \prod_{j=1}^{n} (z - \zeta_j)$$

with $|\zeta_j| < 1$ $(j = 1, \ldots, n)$. If $|c| \geq \frac{1}{\prod_{j=1}^{n}(1-|\zeta_j|)}$, then T_ϕ is hyponormal.

Proof. If $\psi := \frac{f}{g}$, then $\mathcal{Z}(\psi) \subseteq \mathbb{D}$. Further by assumption,

$$\text{ess } \inf|\psi| = \text{ess } \inf|c| \prod_{j=1}^{n} |z - \zeta_j| \geq \text{ess } \inf \prod_{j=1}^{n} \left| \frac{z-\zeta_j}{1-|\zeta_j|} \right| \geq 1.$$

Therefore by Theorem 4, T_ϕ is hyponormal. □

For example if $\phi = \bar{g} + f$, where

$$g(z) = \prod_{j=1}^{n} (z - \zeta_j) \quad \text{and} \quad f(z) = \left(\frac{1}{1-\alpha} \right)^m (z-\alpha)^m \prod_{j=1}^{n} (z - \zeta_j) \quad (|\alpha| < 1),$$

then by Corollary 6, T_ϕ is hyponormal.

If $\phi = \bar{g} + f$ (f and g are analytic polynomials), if g divides f, and if the modulus of the leading coefficient of $\psi := \frac{f}{g}$ is 1, then we can easily check the hyponormality of T_ϕ.

Theorem 7. *Let $\phi = \bar{g} + f$, where f and g are analytic polynomials of degrees N and m $(m \geq 2)$, respectively. Suppose that g divides f and the modulus of the leading coefficient of $\psi := \frac{f}{g}$ is 1. Then T_ϕ is hyponormal if and only if $\hat{\psi}(n) = 0$ for $N - 2m + 1 \leq n \leq N - m - 1$, where $\hat{\psi}(n)$ is the n-th Fourier coefficient of ψ. Hence, in particular, if $N < 2m$, then T_ϕ is hyponormal if and only if $\psi(z) = e^{i\omega} z^{N-m}$ for some $\omega \in [0, 2\pi)$.*

Proof. By assumption, we may write $\psi(z) = e^{i\omega} \prod_{j=1}^{N-m} (z - \zeta_j)$ for some $\omega \in [0, 2\pi)$. If T_ϕ is hyponormal, then by Lemma 3(v), the finite Blaschke product $k \in \mathcal{E}(\phi)$ should be of the form $k(z) = e^{i\theta} z^{N-m}$ for some $\theta \in [0, 2\pi)$. Thus we have

$$\begin{aligned} T_\phi \text{ hyponormal} &\iff \phi - k\,\bar{\phi} \in H^\infty \quad \text{with } k(z) = e^{i\theta} z^{N-m} \\ &\iff \bar{g} - e^{i\theta}\, z^{N-m} \bar{f} \in H^\infty \\ &\iff \bar{g} - e^{i\theta}\, z^{N-m} \cdot \bar{g}\, e^{-i\omega} \prod_{j=1}^{N-m} (\bar{z} - \overline{\zeta_j}) \in H^\infty \\ &\iff \bar{g} \left(1 - e^{i(\theta-\omega)} \prod_{j=1}^{N-m} (1 - \overline{\zeta_j} z) \right) \in H^\infty \\ &\iff 1 - e^{i(\theta-\omega)} \prod_{j=1}^{N-m} (1 - \overline{\zeta_j} z) \in z^m\, H^\infty. \end{aligned}$$

Therefore if T_ϕ is hyponormal and $N < 2m$ then $e^{i(\theta-\omega)} \prod_{j=1}^{N-m} (1 - \overline{\zeta_j} z) = 1$, which implies $\zeta_j = 0$ for $1 \le j \le N - m$. Thus we have that $\psi(z) = e^{i\omega} z^{N-m}$. The converse immediately follows from applying Theorem 4 with $p = 1$. If instead $N \ge 2m$, write $\eta(z) := e^{i(\theta-\omega)} \prod_{j=1}^{N-m} (1 - \overline{\zeta_j} z)$. Then we have

$$\begin{aligned} &T_\phi \text{ hyponormal} \\ \iff &\eta(z) = 1 + \sum_{j=m}^{N-m} a_j z^j \quad \text{for some } a_j\ (j = m, \dots, N-m) \\ \iff &z^{N-m} \overline{\eta(z)} = \sum_{j=0}^{N-2m} \overline{a_{N-m-j}} z^j + z^{N-m} \quad \text{for some } a_j\ (j = m, \dots, N-m) \\ \iff &e^{i(\omega-\theta)} \prod_{j=1}^{N-m} (z - \zeta_j) = \sum_{j=0}^{N-2m} \overline{a_{N-m-j}} z^j + z^{N-m} \\ &\text{for some } a_j\ (j = m, \dots, N-m) \\ \iff &\hat{\psi}(n) = 0 \quad \text{for } N - 2m + 1 \le n \le N - m - 1. \end{aligned}$$

This completes the proof. □

Since the hyponormality is translation-invariant, it follows from Lemma 3(ii) and Theorem 4 that the conclusion of Theorem 7 via its proof can be rewritten as: $z^m T_{\bar{z}^{N-m}} f = \frac{a_N}{a_{-m}} z^m g$, or equivalently, $T_{\bar{z}^{N-m}} f = \frac{a_N}{a_{-m}} g$. Therefore we can recapture Lemma 3(v): if $\phi(z) = \sum_{n=-m}^{N} a_n z^n$ with $|a_{-m}| = |a_N| \neq 0$, then

T_ϕ is hyponormal if and only if

$$a_N \begin{pmatrix} \overline{a_{-1}} \\ \overline{a_{-2}} \\ \vdots \\ \vdots \\ \overline{a_{-m}} \end{pmatrix} = \overline{a_{-m}} \begin{pmatrix} a_{N-m+1} \\ a_{N-m+2} \\ \vdots \\ \vdots \\ a_N \end{pmatrix}.$$

Example 8. If

$$\phi(z) = \prod_{j=1}^{m} \overline{(z-\alpha_j)} + \prod_{j=1}^{N} (z-\alpha_j) \quad (m < N < 2m,\ \alpha_N \neq 0),$$

then T_ϕ is not hyponormal.

Proof. This follows immediately from Theorem 7. □

Theorem 7 is not true in general if the leading coefficient of $\frac{f}{g}$ does not have modulus 1. Hyponormality for such a case is very complicated.

Corollary 9. *Let $\phi = \bar{g} + f$, where f and g are analytic polynomials of degrees N and m, respectively $(m \geq 2)$. Suppose that g divides f and the leading coefficient of $\psi \equiv \frac{f}{g}$ has modulus ≥ 1. If $\mathcal{Z}(\psi) \subseteq \mathbb{T}$ and $\hat{\psi}(n) = 0$ for $n = 1, \ldots, m-1$, then T_ϕ is hyponormal.*

Proof. Without loss of generality we may write $g(z) = \prod_{j=1}^{m}(z-\gamma_j)$. Define

$$k(z) := \frac{\psi(z)}{\psi(0)\overline{\psi(z)}}.$$

Since $\mathcal{Z}(\psi) \subset \mathbb{T}$, it follows that $|\psi(0)| = 1$. But since $\frac{\psi}{\bar{\psi}}$ is unimodular it follows that $k \in H^\infty$ and $||k||_\infty \leq 1$. Thus

$$\begin{aligned} \bar{g} - k\,\bar{f} &= \prod_{j=1}^{m} (\bar{z} - \overline{\gamma_j}) \left(1 - \frac{\psi(z)}{\psi(0)\overline{\psi(z)}} \overline{\psi(z)} \right) \\ &= \frac{1}{z^m} \prod_{j=1}^{m} (1 - \overline{\gamma_j} z) \left(1 - \frac{\psi(z)}{\psi(0)} \right) \\ &\in H^\infty \quad (\text{because } \hat{\psi}(n) = 0 \text{ for } n = 1, \ldots, m-1), \end{aligned}$$

which implies that $\phi - k\,\bar{\phi} \in H^\infty$, and hence T_ϕ is hyponormal. □

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