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SANDWICH THEOREMS FOR HIGHER-ORDER DERIVATIVES OF p-VALENT FUNCTIONS DEFINED BY CERTAIN LINEAR OPERATOR

Mohamed K. Aouf and Tamer M. Seoudy

Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results for higher-order derivatives of p-valent functions involving certain linear operator. Some of our results improve and generalize previously known results.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \ (a \in \mathbb{C}). \tag{1.1}$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}(p)$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}). \tag{1.2}$$

which are p-valent in U. If $f,\ g \in H(U)$, we say that f is subordinate to g or f is superordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$, such that $f(z) = g(\omega(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [6], [9] and [10]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{1.3}$$

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then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.3) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z); z\right), \tag{1.4}$$

then $p(z)$ is a solution of the differential superordination (1.4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.4) is called the best subordinant. Using the results of Miller and Mocanu [10], Bulboaca [5] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [6]. Ali et al. [1], have used the results of Bulboaca [5] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}(1)$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [15] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}(1)$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [14] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}(1)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They [14] also obtained results for functions defined by using Carlson-Shaffer operator [7], Ruscheweyh derivative [12] and Sălăgean operator [13].

Upon differentiating both sides of (1.1) j-times with respect and to z, we have

$$f^{(j)}(z) = \delta(p;j)\, z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k;j)\, a_k z^{k-j}, \tag{1.5}$$

where

$$\delta(p;j) = \frac{p!}{(p-j)!} \quad (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.6}$$

For a function $f \in \mathcal{A}(p)$, we define the linear operator $D_p^n : \mathcal{A}(p) \to \mathcal{A}(p)$ by:

$$D_p^0 f^{(j)}(z) = f^{(j)}(z),$$

$$D_p^1 f^{(j)}(z) = D\left(f^{(j)}(z)\right) = \delta(p;j)\, z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k;j) \left(\frac{k-j}{p-j}\right) a_k z^{k-j},$$

$$D_p^2 f^{(j)}(z) = D\left(D_p^1 f^{(j)}(z)\right) = \delta(p;j)\, z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k;j) \left(\frac{k-j}{p-j}\right)^2 a_k z^{k-j},$$

and (in general)

$$D_p^n f^{(j)}(z) = D(D_p^{n-1} f^{(j)}(z)) = \delta(p;j)\, z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k;j) \left(\frac{k-j}{p-j}\right)^n a_k z^{k-j} \quad (p > j; p, n \in \mathbb{N}; j \in \mathbb{N}_0; z \in U). \tag{1.7}$$

From (1.7), we can easily deduce that

$$z\left(D_p^n f^{(j)}(z)\right)' = (p-j)\, D_p^{n+1} f^{(j)}(z) \quad (p > j; p \in \mathbb{N}; n, j \in \mathbb{N}_0; z \in U). \tag{1.8}$$

The operator $D_p^n f^{(j)}(z)$ $(p > j, p \in \mathbb{N}, n, j \in \mathbb{N}_0)$ was introduced and studied by Aouf [2, 3] where

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0).$$

We note that

(i) the differential operator $D_p^n f^{(0)}(z) = D_p^n f(z)$ was introduced by Kamali and Orhan [8] and Aouf and Mostafa [4];

(ii) the differential operator $D_1^n f^{(0)}(z) = D^n f(z)$ was introduced by Sălăgean [13].

In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator $D_p^n f^{(j)}(z)$.

2. Definitions and preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 1 ([10]). Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma 1 ([10]). *Let $q(z)$ be univalent in U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) = zq^{'}(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z). \tag{2.1}$$

Suppose that

(i) *$\psi(z)$ is starlike univalent in U,*

(ii) $\Re\left\{\frac{zh^{'}(z)}{\psi(z)}\right\} > 0$ *for $z \in U$.*

If $p(z)$ is analytic with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp^{'}(z)\varphi(p(z)) \prec \theta(q(z)) + zq^{'}(z)\varphi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1, Shanmugam et al. [14] obtained the following lemma.

Lemma 2 ([14]). *Let $q(z)$ be univalent in U with $q(0) = 1$. Let $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{C}^*$, further assume that*

$$\Re\left\{1 + \frac{zq^{''}(z)}{q^{'}(z)}\right\} > \max\left\{0, -\Re\left(\frac{\alpha}{\gamma}\right)\right\}. \tag{2.3}$$

If $p(z)$ is analytic in U, and

$$\alpha p(z) + \gamma zp^{'}(z) \prec \alpha q(z) + \gamma zq^{'}(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3 ([5]). *Let $q(z)$ be convex univalent in U and ϑ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\Re\left\{\frac{\vartheta^{'}(q(z))}{\phi(q(z))}\right\} > 0$ *for $z \in U$,*

(ii) *$\Psi(z) = zq^{'}(z)\phi(q(z))$ is starlike univalent in U.*

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp^{'}(z)\phi(p(z))$ is univalent in U and

$$\vartheta(q(z)) + zq^{'}(z)\phi(q(z)) \prec \vartheta(p(z)) + zp^{'}(z)\phi(p(z)), \tag{2.4}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Taking $\vartheta(w) = \alpha w$ and $\phi(w) = \gamma$ in Lemma 3, Shanmugam et al. [14] obtained the following lemma.

Lemma 4 ([14]). *Let $q(z)$ be convex univalent in U, $q(0) = 1$. Let $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{C}^*$ and $\Re\left(\frac{\alpha}{\gamma}\right) > 0$. If $p(z) \in H[q(0), 1] \cap Q$, $\alpha p(z) + \gamma zp^{'}(z)$ is univalent in U and*

$$\alpha q(z) + \gamma zq^{'}(z) \prec \alpha p(z) + \gamma zp^{'}(z),$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Sandwich results

Unless otherwise mentioned, we assume throughout this paper that $p > j$; $p \in \mathbb{N}$ and $n, j \in \mathbb{N}_0$.

Theorem 1. *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further, assume that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}. \tag{3.1}$$

If $f \in \mathcal{A}(p)$ satisfy the following subordination condition:

(3.2)
$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z) = \frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \qquad (z \in U). \tag{3.3}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, differentiating (3.3) logarithmically with respect to z and using the identity (1.8) in the resulting equation, we have

$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\} = p(z) + \gamma z p'(z),$$

that is,

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z).$$

Therefore, Theorem 1 now follows by applying Lemma 2. □

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let $\gamma \in \mathbb{C}^*$ and*

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}.$$

If $f \in \mathcal{A}(p)$ satisfy the following subordination condition:

$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\} \prec \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Taking $p = 1$ and $j = 0$ in Theorem 1, we obtain the following subordination result for Sălăgean operator which improves the result of Shanmugam et al. [14, Theorem 5.1] and also obtained by Nechita [11, Corollary 7].

Corollary 2. *Let* $q(z)$ *be univalent in* U *with* $q(0) = 1$*, and* $\gamma \in \mathbb{C}^*$*. Further assume that* (3.1) *holds. If* $f \in \mathcal{A}(1)$ *satisfies the following subordination condition*:

$$\frac{D^n f(z)}{D^{n+1} f(z)} + \gamma \left\{ 1 - \frac{D^n f(z) D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2} \right\} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{D^n f(z)}{D^{n+1} f(z)} \prec q(z)$$

and $q(z)$ *is the best dominant.*

Remark 1. Taking $n = j = 0$ and $p = 1$ in Theorem 1, we obtain the subordination result of Shanmugam et al. [14, Theorem 3.1].

Now, by appealing to Lemma 4 it can be easily prove the following theorem.

Theorem 2. *Let* $q(z)$ *be convex univalent in* U *with* $q(0) = 1$*. Let* $\gamma \in \mathbb{C}$ *with* $\Re(\bar{\gamma}) > 0$*. If* $f \in \mathcal{A}(p)$ *such that* $\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \in H[q(0), 1] \cap Q$,

$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma (p-j) \left\{ 1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2} \right\}$$

is univalent in U*, and the following superordination condition*

$$q(z) + \gamma z q'(z) \prec \frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma (p-j) \left\{ 1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2} \right\}$$

holds, then

$$q(z) \prec \frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)}$$

and $q(z)$ *is the best subordinant.*

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we have the following corollary.

Corollary 3. *Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma}) > 0$. If $f \in \mathcal{A}(p)$ such that $\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \in H[q(0),1] \cap Q$,*

$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$\frac{1+Az}{1+Bz} + \gamma\frac{(A-B)z}{(1+Bz)^2} \prec \frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\}$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)}$$

and $q(z)$ is the best subordinant.

Taking $p = 1$ and $j = 0$ in Theorem 2, we obtain the following superordination result for Sălăgean operator which improves the result of Shanmugam et al. [14, Theorem 5.2] and also obtained by Nechita [11, Corollary 12].

Corollary 4. *Let $q(z)$ be convex univalent in U with $q(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma}) > 0$. If $f \in \mathcal{A}(1)$ such that $\frac{D^n f(z)}{D^{n+1} f(z)} \in H[q(0),1] \cap Q$,*

$$\frac{D^n f(z)}{D^{n+1} f(z)} + \gamma\left\{1 - \frac{D^n f(z).D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z) + \gamma z q'(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)} + \gamma\left\{1 - \frac{D^n f(z).D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2}\right\}$$

holds, then

$$q(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)}$$

and $q(z)$ is the best subordinant.

Remark 2. Taking $j = n = 0$ and $p = 1$ in Theorem 2, we obtain the superordination result of Shanmugam et al. [14, Theorem 3.2].

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem for the linear operator $D_p^n f^{(j)}(z)$.

Theorem 3. *Let $q_1(z)$ be convex univalent in U with $q_1(0) = 1$, $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma}) > 0, q_2(z)$ be univalent in U with $q_2(0) = 1$, and satisfies (3.1). If $f \in \mathcal{A}(p)$ such that $\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \in H[q(0),1] \cap Q$,*

$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\}$$

is univalent in U*, and*

$$q_1(z) + \gamma z q_1'(z) \prec \frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\} \prec q_2(z) + \gamma z q_2'(z)$$

holds, then

$$q_1(z) \prec \frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Taking $q_i(z) = \frac{1+A_i z}{1+B_i z}$ $(i = 1, 2; -1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1)$ in Theorem 3, we obtain the following corollary.

Corollary 5. *Let* $\gamma \in \mathbb{C}$ *with* $\Re(\bar{\gamma}) > 0$*. If* $f \in \mathcal{A}(p)$ *such that* $\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \in H[q(0), 1] \cap Q$,

$$\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\}$$

is univalent in U*, and*

$$\begin{aligned} &\frac{1+A_1 z}{1+B_1 z} + \frac{\gamma}{\lambda} \frac{(A_1 - B_1) z}{(1+B_1 z)^2} \\ \prec &\frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} + \gamma(p-j)\left\{1 - \frac{D_p^n f^{(j)}(z) D_p^{n+2} f^{(j)}(z)}{\left[D_p^{n+1} f^{(j)}(z)\right]^2}\right\} \\ \prec &\frac{1+A_2 z}{1+B_2 z} + \frac{\gamma}{\lambda} \frac{(A_2 - B_2) z}{(1+B_2 z)^2} \end{aligned}$$

holds, then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{D_p^n f^{(j)}(z)}{D_p^{n+1} f^{(j)}(z)} \prec \frac{1+A_2 z}{1+B_2 z}$$

and $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are, respectively, the best subordinant and the best dominant.*

Taking $p = 1$ and $j = 0$ in Theorem 3, we obtain the following sandwich result for Sălăgean operator which improves the result of Shanmugam et al. [14, Theorem 5.3].

Corollary 6. *Let* $q_1(z)$ *be convex univalent in* U *with* $q_1(0) = 1$*,* $\gamma \in \mathbb{C}$ *with* $\Re(\bar{\gamma}) > 0, q_2(z)$ *be univalent in* U *with* $q_2(0) = 1$*, and satisfies* (3.1)*. If* $f \in \mathcal{A}(1)$ *such that* $\frac{D^n f(z)}{D^{n+1} f(z)} \in H[q(0), 1] \cap Q$,

$$\frac{D^n f(z)}{D^{n+1} f(z)} + \gamma\left\{1 - \frac{D^n f(z) . D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2}\right\}$$

is univalent in U*, and*

$$q_1(z) + \gamma z q_1'(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)} + \gamma \left\{ 1 - \frac{D^n f(z).D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2} \right\} \prec q_2(z) + \gamma z q_2'(z)$$

holds, then

$$q_1(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)} \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Remark 3. Taking $n = j = 0$ and $p = 1$ in Theorem 3, we obtain the sandwich result of Shanmugam et al. [14, Corollary 3.3].

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Mohamed K. Aouf
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
E-mail address: mkaouf127@yahoo.com

Tamer M. Seoudy
Department of Mathematics
Faculty of Science
Fayoum University
Fayoum 63514, Egypt
E-mail address: tmseoudy@gmail.com