REFINEMENT RINGS, EXCHANGE PROPERTY AND COMPARABILITY

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Abstract. We introduce the notion of refinement rings, and generalize some comparability properties of regular rings to the setting of refinement rings.

Introduction

Recall that a monoid $(\mathbb{M}, +, 0)$ is a nonempty set $\mathbb{M}$ with a binary operation $+$ which is associative and which has an identity 0 such that $a = a + 0 = 0 + a$ for all $a \in \mathbb{M}$. Let R be an associative ring with identity. Let $FP(R)$ denote the category of finitely generated projective right R-modules. Let $V(R) = \{[A] |\ [A]$ is the class of R-modules isomorphic to $A,\ A \in FP(R)\}$. Define $[A] + [B] = [A \oplus B]$ for $[A],\ [B] \in V(R)$. Then $V(R)$ is a commutative monoid. Recall that a monoid $\mathbb{M}$ has refinement if for all $a_1,\ a_2,\ b_1,\ b_2 \in M$ with $a_1 + a_2 = b_1 + b_2$, there exist $c_{11},\ c_{12},\ c_{21},\ c_{22} \in M$ such that

$$a_1 = c_{11} + c_{12}, \quad a_2 = c_{21} + c_{22},$$

$$b_1 = c_{11} + c_{21}, \quad b_2 = c_{12} + c_{22}.$$

Note that for a positive integer s, a monoid $\mathbb{M}$ is said to satisfy s-comparability if for any $a,\ b \in \mathbb{M}$, either a is a summand of sb, or b is a summand of sa. In [8, p. 275], a von Neumann regular ring R is said to satisfy s-comparability in case for any $x,\ y \in R$, either xR is isomorphic to a summand of $s(yR)$, or yR is isomorphic to a summand of $s(xR)$. For the regular ring R, R satisfies s-comparability if and only if $V(R)$ satisfies s-comparability. A monoid $\mathbb{M}$ is said to be separative if for any $a,\ b \in \mathbb{M}$, $a + a = a + b = b + b$ implies that $a = b$. On the other hand, R is said to be separative if the following condition holds for all $A,\ B \in FP(R)$:

$$A \oplus A \cong A \oplus B \cong B \oplus B \Rightarrow A \cong B.$$

It is obvious that R is separative if and only if $V(R)$ is separative. It was proved that a conical ($x + y = 0$ implies that $x = y = 0$ for any $x, y \in \mathbb{M}$) refinement monoid $\mathbb{M}$ satisfying s-comparability is separative [14, Theorem 1.9]. For the ring theory, we know that any exchange ring (see the following definition) satisfying s-comparability is separative [14, Theorem 2.2].

For any right R-module M, Crawley and Jónsson [7] called M to have the exchange property if for every right R-module A and any two decompositions of A,

$$A = M' \oplus N = \oplus_{i \in I} A_i,$$

where $M' \cong M$, there are submodules $A_i' \subseteq A_i$ such that $A = M' \oplus (\oplus_{i \in I} A_i')$. It follows from the modular law that A_i' must be a direct summand of A_i for all i. Warfield [15] called a ring R an exchange ring if R has the exchange property as a right R-module. He proved that this definition is left-right symmetric [15, Corollary 2]. Many classes of rings belong to this class of rings, for instance, local rings, von Neumann regular rings, semiperfect rings and strongly π-regular rings, etc.

The notion of almost comparability for regular rings was first introduced by Ara and Goodearl [2], for giving an alternative proof of the outstanding O'Meara's Theorem: directly finite simple regular rings with weak comparability are unit-regular. It was proved that for a regular ring R, R satisfies almost comparability if and only if every finitely generated projective R-module satisfies almost comparability [12, Theorem 1.9], and that almost comparability is Morita invariant [12, Theorem 1.11]. O'Meara first introduced the notion of weak comparability, and proved that simple directly finite regular rings with weak comparability must be unit-regular [8, Open Problem 3]. Many authors studied regular rings with weak comparability [6, 10, 11]. For the regular ring R with weak comparability, it was proved [11, Theorem 1.6] that $A \oplus C \prec B \oplus C$ implies $A \prec B$ for any finitely generated projective R-modules A, B and C with $B \neq 0$, and was proved [11, Theorem 1.8] that $nA \prec nB$ implies $A \prec B$ for any positive integer n and any finitely generated projective R-modules A and B. In this note we define the notion of refinement rings, and give some generalizations of comparability from regular rings to refinement rings, especially exchange rings.

Throughout this note, R is an associative ring with identity and R-modules are unitary right R-modules.

1. Refinement rings with almost comparability

Notation 1.1. For two R-modules M and N, we use $M \lesssim_\oplus N$ (respectively $M \lesssim N$) to denote that M is isomorphic to a direct summand of N (respectively M is isomorphic to a submodule of N), and $M \prec_\oplus N$ (respectively $M \prec N$) to denote that M is isomorphic to a proper direct summand of N (respectively M is isomorphic to a proper submodule of N). Let M and N be finitely generated projective R-modules. We write $M \lesssim_a N$ to mean that for any

nonzero principal right ideal C of R, $M \lesssim_\oplus N \oplus C$, and $M \prec_a N$ to mean that for any nonzero principal right ideal C of R, $M \prec_\oplus N \oplus C$. Other basic notations can be found in [8].

It is well-known that the concepts of almost comparability and s-comparability were defined over regular rings originally. Since we consider more generally the case, we give the following definition.

Definition 1.2. A ring R is said to satisfy almost comparability, if for x, $y \in R$ such that xR, $yR \lesssim_\oplus R$, either $xR \lesssim_a yR$ or $yR \lesssim_a xR$. A finitely generated projective R-module P satisfies almost comparability, if for any direct summands A, B of P either $A \lesssim_a B$ or $B \lesssim_a A$. Similarly, we can define the strictly almost comparability of R if $xR \prec_a yR$ or $yR \prec_a xR$, and strictly almost comparability of P if $A \prec_a B$ or $B \prec_a A$. Clearly, almost comparability for modules is inherited by direct summands, and R satisfies almost comparability if and only if so does the R-module R_R. For a positive integer s, R is said to satisfy s-comparability if, for any right ideals A, B such that A, $B \lesssim_\oplus R$, either $A \lesssim_\oplus sB$ or $B \lesssim_\oplus sA$. It is clear that almost comparability implies 2-comparability. We say that the finitely generated projective R-modules satisfy s-comparability if for any two finitely generated projective R-modules A and B, either $A \lesssim_\oplus sB$ or $B \lesssim_\oplus sA$.

Definition 1.3. Let R be a ring. If $V(R)$ has refinement, i.e., for finitely generated projective right R-modules A_1, A_2, B_1, B_2, $A_1 \oplus A_2 \cong B_1 \oplus B_2$ implies that there exist R-modules C_{ij}, i, $j = 1$, 2 such that

$$A_i = C_{i1} \oplus C_{i2}, \ B_j \cong C_{1j} \oplus C_{2j}, \ i, \ j = 1, \ 2,$$

we call R a refinement ring.

It is convenient to record refinement with the following matrix:

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} B_1 \quad B_2 \\ \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \end{array} .$$

By [4, Proposition 1.1], the class of exchange rings is included in the class of the refinement rings. There are lots of refinement rings, not necessarily exchange rings, for instance $R = K[x_1, \ldots, x_n]$, where K is a field. In fact, by famous Quillen-Suslin Theorem, every finitely generated projective R-module is free. Thus R is a refinement ring. Since the idempotent elements of R are those of K, R is not exchange. We also argue that there exists a refinement ring with almost comparability which is not von Neumann regular. Let $A_n = \mathbb{M}(\mathbb{C})_{2^n}$ be the C^*-algebra of $2^n \times 2^n$ complex matrices. Define $h_n : A_n \to A_{n+1}$ by $h_n(a) = \text{diag}(a, a)$. Let $A = \lim_{n \to \infty}(A_n, h_n)$ be the completion of $A'/p^{-1}(0)$ which is called the inductive limit of the sequence (A_n, h_n), where $A' = \{\{a_n\} \in \prod_{n=1}^\infty A_n : a_{n+1} = h_n(a_n)$ for all sufficiently large $n\}$ and $p(a) = \lim_{n \to \infty} ||a_n||$ (see [13, p. 53]). A is an infinite dimensional simple unital AF-algebra (see

[13, Example 1.10.8]). Thus, by [13, Corollary 3.2.2], A has real rank zero. Hence A is a refinement ring by [6, Lemma 2.3]. Moreover, by [13, Chapter 3, 3.3 and 3.4], we know that A satisfies almost comparability. But A is not von Neumann regular by [9, Theorem, p. 58].

Lemma 1.4. *Let R be a refinement ring, and A_i, B_j be finitely generated projective right R-modules for $1 \le i \le n$, $1 \le j \le m$, if $A_1 \oplus A_2 \oplus \cdots \oplus A_n \lesssim_\oplus B_1 \oplus B_2 \oplus \cdots \oplus B_m$, then*

(1) there exist decompositions $A_i = A_{i1} \oplus \cdots \oplus A_{im}$ for $i = 1, ..., n$ such that $A_{1j} \oplus \cdots \oplus A_{nj} \lesssim_\oplus B_j$ for $j = 1, \ldots, m$.

(2) there exist decompositions $B_j = B_{j1} \oplus \cdots \oplus B_{jn}$ for $j = 1, \ldots, m$ such that $A_i \cong B_{1i} \oplus \cdots \oplus B_{mi}$ for $i = 1, \ldots, n-1$, and $A_n \lesssim_\oplus B_{1n} \oplus \cdots \oplus B_{mn}$.

The lemma below was obtained by Ara and Goodearl over regular rings, see [2, Lemma 3.3].

Lemma 1.5. *Let A, B be finitely generated projective right modules over a refinement ring R. If $A \lesssim_\oplus nB$ for some positive integer n, then there is a decomposition $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ such that $A_1 \lesssim_\oplus A_2 \lesssim_\oplus \cdots \lesssim_\oplus A_n \lesssim_\oplus B$.*

Proof. We prove it by the induction. By Lemma 1.4, there is a decomposition $A = U \oplus W$, where $U \lesssim_\oplus (n-1)B$, and $W \lesssim_\oplus B$. By the induction, there is a chain $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{n-1}$ of B such that $U \cong C_1 \oplus C_2 \oplus \cdots \oplus C_{n-1}$. Let $D_1, \ldots, D_n$ be submodules of B such that $D_1 = C_1$, $C_i = C_{i-1} \oplus D_i$ for $i = 2, \ldots, n-1$ and $B = C_{n-1} \oplus D_n$. Then $B = D_1 \oplus D_2 \oplus \cdots \oplus D_n$. By Lemma 1.4, it follows from $W \lesssim_\oplus B$ that there are submodules $X_i \subseteq D_i$, $i = 1, \ldots, n$ such that $W \cong X_1 \oplus X_2 \oplus \cdots \oplus X_n$. Hence $A = U \oplus W \cong C_1 \oplus C_2 \oplus \cdots \oplus C_{n-1} \oplus X_1 \oplus X_2 \oplus \cdots \oplus X_n = X_1 \oplus (C_1 \oplus X_2) \oplus \cdots \oplus (C_{n-1} \oplus X_n)$, where $X_1 \subseteq (C_1 \oplus X_2) \subseteq \cdots \subseteq (C_{n-1} \oplus X_n) \subseteq B$. Set $A_1 = X_1$, $A_i = (C_{i-1} \oplus X_i)$ for $i = 2, \ldots, n$, as desired. $\square$

Now we argue that s-comparability of refinement rings is inherited by finitely generated projective right R-modules, which is similar to a result about regular rings [5, Proposition 2.1]. So the idea of the following proposition is due to Ara, O'Meara and Tyukavkin.

Proposition 1.6. *Let R be a refinement ring satisfying s-comparability. Then the finitely generated projective right R-modules also satisfy s-comparability.*

Proof. Let A, B be finitely generated projective right R-modules. There is a positive number n such that A, $B \lesssim_\oplus nR$. We can prove the assertion by induction on n. If $n = 1$, it is true since R satisfies s-comparability. Assume that the assertion holds for $n-1$ and suppose A, $B \lesssim_\oplus nR$. Since R is a refinement ring, by Lemma 1.4, we can write that $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ where A_i, $B_i \lesssim_\oplus (n-1)R$ for $i = 1$, 2. By the induction, we have $A_1 \lesssim_\oplus sB_1$ or $B_1 \lesssim_\oplus sA_1$, and $A_2 \lesssim_\oplus sB_2$ or $B_2 \lesssim_\oplus sA_2$. If $A_1 \lesssim_\oplus sB_1$ and $A_2 \lesssim_\oplus sB_2$, or $B_1 \lesssim_\oplus sA_1$ and $B_2 \lesssim_\oplus sA_2$, the assertion is obviously true. Now assume

that $A_1 \lesssim_\oplus sB_1$ and $B_2 \lesssim_\oplus sA_2$. By Lemma 1.5, there are a direct summand V of A_1 such that $V \lesssim_\oplus B_1$ and $A_1 \lesssim_\oplus sV$ and a direct summand U of B_2 such that $U \lesssim_\oplus A_2$ and $B_2 \lesssim_\oplus sU$. So $B_1 \cong V \oplus C$ and $A_2 \cong U \oplus D$ where C and D are finitely generated projective right R-modules. Since C, $D \lesssim_\oplus (n-1)R$, by the induction, we get $C \lesssim_\oplus sD$, or $D \lesssim_\oplus sC$. If the former is true, then

$$B = B_1 \oplus B_2 \cong V \oplus C \oplus B_2$$
$$\lesssim_\oplus V \oplus sD \oplus sU$$
$$\cong V \oplus sA_2$$
$$\lesssim_\oplus sA_1 \oplus sA_2$$
$$= sA.$$

Similarly, if the latter is true, then $A \lesssim_\oplus sB$, as desired. $\qquad\square$

In order to use easily later, we give the following result which is well-known for regular rings [12, Lemma 1.2].

Proposition 1.7. *Let R be a refinement ring satisfying almost comparability.*
(1) If S is a simple projective R-module, then $S \lesssim_\oplus A$ for any nonzero finitely generated projective R-module A.
(2) All simple projective R-modules are isomorphic.

Proof. (1) Since R satisfies almost comparability, it has 2-comparability, hence $S \lesssim_\oplus 2A$, or $A \lesssim_\oplus 2S$, by Proposition 1.6. If $S \lesssim_\oplus 2A$, since R has refinement, S has decomposition $S = S_1 \oplus S_2$ such that $S_i \lesssim_\oplus A$, $i = 1$, 2 by Lemma 1.4. Since S is simple, $S_1 = 0$, or $S_2 = 0$. If $S_1 = 0$, then $S = S_2 \lesssim_\oplus A$. If $S_2 = 0$, then $S = S_1 \lesssim_\oplus A$. If $A \lesssim_\oplus 2S$, since R has refinement, A has decomposition $A = A_1 \oplus A_2$ such that $A_1 \lesssim_\oplus A_2 \lesssim_\oplus S$ by Lemma 1.5. Since S is simple, $A_1 = 0$, or $A_2 = 0$. If $A_2 = 0$, then $S = 0$. If $A_1 = 0$, then $A = A_2 \cong S$. So we have $S \lesssim_\oplus A$.

(2) Let A, B be two simple projective R-modules, by (1), we have that $A \lesssim_\oplus B$. So $A \cong B$ as B is simple. $\qquad\square$

Corollary 1.8. *Let R be an exchange ring satisfying almost comparability. The statements in Proposition 1.7 hold.*

In 2007, Kutami [12, Proposition 1.3] proved that over any regular ring R, almost comparability and strictly almost comparability of every finitely generated projective R-module are equivalent. Actually, they coincide with each other over any ring.

Proposition 1.9. *Let R be a ring. For each finitely generated projective R-module P, the following are equivalent:*
(1) P has almost comparability;
(2) P has strictly almost comparability.

Proof. $(2) \Rightarrow (1)$ is clear.

$(1) \Rightarrow (2)$ For any direct summands A and B of P, by hypothesis, $A \lesssim_a B$ or $B \lesssim_a A$. If $A \lesssim_a B$, for any nonzero principal right ideal C of R, $A \lesssim_\oplus B \oplus C$. If there is a nonzero principal right ideal D such that $A \cong B \oplus D$, then for any nonzero principal right ideal E of R, we have $B \prec_\oplus A \prec_\oplus A \oplus E$. In this case, $B \prec_a A$. Otherwise, $A \prec_a B$. $\square$

Lemma 1.10. *Let R be a refinement ring satisfying almost comparability. And let A, B be nonzero finitely generated projective R-modules. Then there exists a nonzero principal right ideal X of R such that $X \lesssim_\oplus A$, and $X \lesssim_\oplus B$.*

Proof. At first, we argue that every finitely generated projective R-module has a nonzero cyclic submodule as a direct summand. Let P be a finitely generated projective right R-module. There is another finitely generated projective right R-module Q such that $P \oplus Q \cong R^n$ for some n. Since R is a refinement ring, there are right R-modules $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ such that $P \cong P_1 \oplus \cdots \oplus P_n$ and $Q \cong Q_1 \oplus \cdots \oplus Q_n$ and $P_i \oplus Q_i \cong R$ for all i. Thus P has a nonzero cyclic submodule isomorphic to, for example, P_1 as a direct summand. Now we can assume that both A and B are nonzero cyclic projective R-modules. Since R satisfies almost comparability, $A \lesssim_a B$, or $B \lesssim_a A$. Without loss of generality, we assume that $A \lesssim_a B$. Thus $A \lesssim_\oplus 2B$. Since R is a refinement ring, A has a decomposition $A = A_1 \oplus A_2$ such that $A_2 \lesssim_\oplus A_1 \lesssim_\oplus B$ by Lemma 1.5. Let X be a nonzero principal right ideal of R such that $X \cong A_1$. Then we have $X \lesssim_\oplus A$, and $X \lesssim_\oplus B$. $\square$

Corollary 1.11. *Let R be an exchange ring satisfying almost comparability, and let A and B be nonzero finitely generated projective right R-modules. Then there exists a nonzero principal right ideal X of R such that $X \lesssim_\oplus A$, and $X \lesssim_\oplus B$.*

Lemma 1.12 ([14, Theorem 1.9])**.** *Let R be a refinement ring satisfying s-comparability. Then R is separative.*

The following lemma for regular rings is first proved by Kutami [12, Lemma 1.7]. For the readers' convenience, we prove it here.

Lemma 1.13. *Let R be a refinement ring and assume that nR satisfies almost comparability for some positive integer n. Let A, B, C, D be finitely generated projective R-modules isomorphic to direct summands of nR. If $A \prec_a C$ and $B \prec_a D$, then $A \oplus B \prec_a C \oplus D$.*

Proof. We may assume that $A \neq 0$. Let X be any nonzero principal right ideal of R. Since $A \prec_a C$, we have $A \prec_\oplus C \oplus X$. Hence there is a nonzero finitely generated projective R-module Y such that $A \oplus Y \cong C \oplus X$. By Lemma 1.10, since R satisfies almost comparability, we get a nonzero principal right ideal Y^* of R such that $Y^* \lesssim_\oplus Y$, and $Y^* \lesssim_\oplus A$. Further, we have that $B \prec_\oplus D \oplus Y^*$ as $B \prec_a D$. Thus

$$A \oplus B \oplus Y^* \lesssim_\oplus A \oplus B \oplus Y \cong B \oplus C \oplus X \prec_\oplus C \oplus X \oplus D \oplus Y^*.$$

Since R is separative by Lemma 1.12, and $Y^* \lesssim_\oplus A$ and $Y^* \lesssim_\oplus Y \lesssim_\oplus C \oplus X$, we have that $A \oplus B \prec_\oplus C \oplus X \oplus D$ by [3, Lemma 2.1], i.e., $A \oplus B \prec_a C \oplus D$, as desired. □

Corollary 1.14. *Let R be an exchange ring satisfying almost comparability, and let A and B be nonzero finitely generated projective right R-modules. If $A \prec_a C$ and $B \prec_a D$, then $A \oplus B \prec_a C \oplus D$.*

Proof. By [3, Proposition 1.2], R is a refinement ring. From Lemma 1.13, the assertion is true. □

Theorem 1.15. *Let R be a refinement ring, and assume that nR satisfies almost comparability for some positive integer n. Let A, B, C, D be finitely generated projective R-modules isomorphic to direct summands of nR. If $A \prec_a C$ and $D \prec_a B$, then $A \oplus B \prec_a C \oplus D$ or $C \oplus D \prec_a A \oplus B$.*

Proof. (1) If $A = C = 0$ or $B = D = 0$, then the result obviously holds.

(2) If $A = D = 0$, either $B \prec_a C$ or $C \prec_a B$ by the hypothesis of almost comparability of nR_R, as desired. Similarly, the result also holds in case $B = C = 0$.

(3) Either $A \neq 0$ or $B \neq 0$, and either $C \neq 0$ or $D \neq 0$. We consider the following four cases:

Case 1. $B \prec_\oplus nR_R$ and $C \prec_\oplus nR_R$;

Case 2. $B \cong nR_R$ and $C \cong nR_R$;

Case 3. $B \prec_\oplus nR_R$ and $C \cong nR_R$;

Case 4. $B \cong nR_R$ and $C \prec_\oplus nR_R$.

Case 1. There exist nonzero finitely generated projective R-modules E and F such that $B \oplus E \cong nR_R$ and $C \oplus F \cong nR_R$. Since $A \prec_a C$ and $D \prec_a B$, by Lemma 1.10, we can take a nonzero principal right ideal X of R such that X is isomorphic to a direct summand of A, C, E, and F, and hence $A \prec_\oplus C \oplus X$ and $D \prec_\oplus B \oplus X$. So, there exist Y and Z such that $A \oplus Y \cong C \oplus X$ and $D \oplus Z \cong B \oplus X$. It follows from $X \lesssim_\oplus F$ and $X \lesssim_\oplus E$ that $C \oplus X \lesssim_\oplus nR_R$ and $B \oplus X \lesssim_\oplus nR_R$. Hence Y and Z are finitely generated projective R-modules which are isomorphic to direct summands of nR_R. By the hypothesis of almost comparability of nR_R, we have that either $Y \prec_a Z$ or $Z \prec_a Y$. Assume that $Y \prec_a Z$. Then, $C \oplus X \oplus D \cong A \oplus D \oplus Y \prec_a A \oplus D \oplus Z \cong A \oplus B \oplus X$. Note that $X \lesssim_\oplus A$ and $X \lesssim_\oplus C$, as R is separative by Lemma 1.12, then $C \oplus D \prec_a A \oplus B$ by [3, Lemma 2.1].

Case 2. We have that $B \cong C$. Since nR_R satisfies almost comparability, either $A \prec_a D$ or $D \prec_a A$. Therefore, either $A \oplus B \prec_a C \oplus D$ or $C \oplus D \prec_a A \oplus B$.

Case 3. If $A \cong nR_R$, then $D \prec_a B$ implies $C \oplus D \prec_a A \oplus B$, as desired. Therefore, we can assume that $A \prec_\oplus nR_R$. There exist nonzero finitely generated projective R-modules X and C^* such that $C = X \oplus C^*$, where $X \cong A$. Since $D \prec_a B$ and $B \prec_\oplus nR_R$, by Lemma 1.10, there exists a nonzero principal right ideal X^* of R such that $X^* \lesssim_\oplus X$, $X^* \lesssim_\oplus A$, and $D \lesssim_\oplus B \oplus X^* \lesssim_\oplus nR_R$. Thus, we have a nonzero finitely generated projective R-module Z such that

$D \oplus Z \cong B \oplus X^*$. Set $Y = X^* \oplus C^*$, so $Y \lesssim_\oplus nR_R$ and $Z \lesssim_\oplus nR_R$. Since nR_R satisfies almost comparability, either $Y \prec_a Z$ or $Z \prec_a Y$. Assume that $Y \prec_a Z$. Then $C \oplus D \oplus X^* \cong X \oplus D \oplus C^* \oplus X^* \cong A \oplus D \oplus Y \prec_a A \oplus D \oplus Z \cong A \oplus B \oplus X^*$. Since R is separative, and $X^* \lesssim_\oplus C$ and $X^* \lesssim_\oplus A$, we see that $C \oplus D \prec_a A \oplus B$ by [3, Lemma 2.1].

Case 4. The same as the proof of the case 3. $\square$

Corollary 1.16. *Let R be an exchange ring, and assume that nR satisfies almost comparability for some positive integer n. Let A, B, C, D be finitely generated projective R-modules isomorphic to direct summands of nR. If $A \prec_a C$ and $D \prec_a B$, then $A \oplus B \prec_a C \oplus D$ or $C \oplus D \prec_a A \oplus B$.*

Theorem 1.17. *Let R be a refinement ring. Then the following conditions are equivalent*:

(1) *R satisfies almost comparability*;

(2) *Every finitely generated projective R-module satisfies almost comparability*;

(3) *nR_R satisfies almost comparability for all positive integers n*;

(4) *nR_R satisfies almost comparability for some positive integer n.*

Proof. $(3) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1)$ are clear.

$(1) \Rightarrow (3)$. Assume that R satisfies almost comparability. We shall prove (3) by using the induction on n. We assume that $(n-1)R_R$ satisfies almost comparability, and let V and W be any direct summands of nR_R. Then there exist decompositions $V = A \oplus B$ and $W = C \oplus D$ such that all A, B, C, and D are isomorphic to direct summands of $(n-1)R_R$. From Lemma 1.13 and Theorem 1.15, we see that either $V \prec_a W$ or $W \prec_a V$. Therefore, nR_R satisfies almost comparability. $\square$

Proposition 1.18. *Let M be a finitely generated projective R-module over a refinement ring R, and set $S = \mathrm{End}_R(M)$. Then M satisfies almost comparability if and only if so does S as an S-module.*

Proof. $\Rightarrow$ Let A, B be summands of S. By [1, Lemma 29.4], $A \otimes_S M$, $B \otimes_S M$ are summands of M_R. Since M satisfies almost comparability, $A \otimes_S M \lesssim_a B \otimes_S M$, or $B \otimes_s M \lesssim_a A \otimes_S M$. Assume that $A \otimes_S M \lesssim_a B \otimes_S M$. For any nonzero principal right ideal $C \lesssim_\oplus S$, we have $A \otimes_S M \lesssim_\oplus (B \otimes_S M) \oplus (C \otimes_S M)$. Thus there is a right R-module X such that $(A \otimes_S M) \oplus X \cong (B \otimes_S M) \oplus (C \otimes_S M)$. Hence $\mathrm{Hom}_R(M, (A \otimes_S M) \oplus X) \cong \mathrm{Hom}_R(M, (B \otimes_S M) \oplus (C \otimes_S M))$. That is, $\mathrm{Hom}_R(M, A \otimes_S M) \oplus \mathrm{Hom}_R(M, X) \cong \mathrm{Hom}_R(M, B \otimes_S M) \oplus \mathrm{Hom}_R(M, C \otimes_S M)$. Thus by using [1, Lemma 29.4] again, we have that $A \oplus \mathrm{Hom}_R(M, X) \cong B \oplus C$, i.e., $A \prec_a B$.

$\Leftarrow$ The same as the above. $\square$

Theorem 1.19. *Let R be a refinement ring. The following conditions are equivalent*:

(1) *R satisfies almost comparability*;

(2) *For each finitely generated projective R-module P, $\text{End}_R(P)$ satisfies almost comparability;*

(3) *Every ring S which is Morita equivalent to R satisfies almost comparability;*

(4) *For all positive integers n, $M_n(R)$ satisfies almost comparability;*

(5) *There exists a positive integer n such that $M_n(R)$ satisfies almost comparability.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 1.17, and Proposition 1.18, R satisfies almost comparability if and only if every finitely generated projective R-module P satisfies almost comparability, if and only if $\text{End}_R(P)$ satisfies almost comparability.

(1) $\Leftrightarrow$ (3) Let S be a ring which is Morita equivalent to R, and let F and G be inverse equivalences of the categories of R-modules and S-modules. By Theorem 1.17, R satisfies almost comparability if and only if every finitely generated projective R-module P satisfies almost comparability, and if and only if $F(P)$ satisfies almost comparability, and if and only if S satisfies almost comparability.

(1) $\Leftrightarrow$ (4), (1) $\Leftrightarrow$ (5) Since R is Morita equivalent to $M_n(R)$, the assertions are true by (1) $\Leftrightarrow$ (3). $\square$

We remark that Theorems 1.15, 1.17 and 1.19 are motivated by [12, Propostion 1.8, Thereom 1.9 and Thereom 1.11], respectively. The assertions of Theorem 1.17, Proposition 1.18 and Theorem 1.19 are true if the ring R is an exchange ring, since any exchange ring is a refinement ring by [3, Proposition 1.2].

2. Cancellation property of refinement rings

Recall that a module M is called directly finite if M is not isomorphic to any proper direct summand of itself, otherwise, M is called directly infinite. A ring R is said to be directly finite provided R_R is a directly finite module. R is called stably finite if all finitely generated projective R-modules are directly finite. It is well-known that for some positive integer s refinement rings satisfying s-comparability have cancellation of separativity [14, Theorem 1.9], and that directly finite refinement rings satisfying s-comparability have cancellation of strong separativity [14, Corollary 1.11]. In [12, Proposition 2.1], Kutami proved that for any regular ring satisfying almost comparability $A \oplus C \prec_\oplus B \oplus C$ implies that $A \prec_\oplus B$ for any nonzero finitely generated directly finite projective R-modules A, B, C. We consider the similar property of refinement rings. In order to do this, we need the following lemma which is well-known for regular rings.

Lemma 2.1. *Let R be a directly finite refinement ring satisfying s-comparability for some positive integer s. Then R is stably finite.*

Proof. Since R is a refinement ring satisfying s-comparability, by [14, Theorem 1.9], R is separative. Since R is directly finite, then by [3, Proposition 2.3] R is stably finite. $\square$

Theorem 2.2. *Let R be a directly finite refinement ring satisfying almost comparability. Then $A \oplus C \oplus X \cong B \oplus C$ implies that $A \prec_{\oplus} B$ for any nonzero finitely generated projective R-modules A, B, C and any simple projective R-module X.*

Proof. At first, note that R is stably finite by Lemma 2.1. Since R satisfies almost comparability, by Proposition 1.7, $X \lesssim_{\oplus} A$. Let $X \oplus A^* \cong A$ for some finitely generated projective R-module A^*. So $A^* \oplus C \oplus 2X \cong B \oplus C$. By Lemma 1.10, there is a nonzero principal right ideal X^* of R such that $X^* \lesssim_{\oplus} X$, A^* and B. Because R satisfies almost comparability, by Theorem 1.17 and Proposition 1.9, either $B \prec_a A^* \oplus X^*$ or $A^* \oplus X^* \prec_a B$. If $B \prec_a A^* \oplus X^*$, then $A^* \oplus C \oplus X \oplus X^* \lesssim_{\oplus} A^* \oplus C \oplus 2X \cong B \oplus C \prec A^* \oplus C \oplus X^* \oplus X$, which contradicts the direct finiteness of $A^* \oplus C \oplus X^* \oplus X$, since R is stably finite. So we have $A^* \oplus X^* \prec_a B$. And then $A^* \oplus X^* \prec_{\oplus} B \oplus X^*$. Since R is separative by [14, Theorem 1.9] and $X^* \lesssim_{\oplus} A^*$, B, we have $A^* \prec_{\oplus} B$ by [3, Lemma 2.1]. Let $A^* \oplus Y \cong B$ for some nonzero finitely generated projective R-module Y. By Proposition 1.7, we have $X \lesssim_{\oplus} Y$. Thus $A \cong X \oplus A^* \lesssim_{\oplus} A^* \oplus Y \cong B$. If $A \cong B$, it follows from $A \oplus C \oplus X \cong B \oplus C$ that $B \oplus C \oplus X \cong B \oplus C$, which contradicts the direct finiteness of $B \oplus C$, since R is stably finite. Thus $A \prec_{\oplus} B$. $\square$

Recall that a ring has cancellation of the strict unperforation property provided that $nA \prec_{\oplus} nB$ implies $A \prec_{\oplus} B$ for any finitely generated projective R-modules A and B. Ara et al. [5, Example 4.11] showed that unit-regular rings with 2-comparability do not necessarily have cancellation of the strictly unperforation property. But some directly finite refinement rings satisfying almost comparability have the cancellation property (Theorem 2.4).

Lemma 2.3. *Let R be a refinement ring satisfying almost comparability with a simple right ideal $S \lesssim_{\oplus} R$. If $A \prec_a B$, then $A \lesssim_{\oplus} B$ for any nonzero finitely generated projective R-modules A and B.*

Proof. By Proposition 1.7, $S \lesssim_{\oplus} A$ and $S \lesssim_{\oplus} B$. Let $A \cong S \oplus A^*$, $B \cong S \oplus B^*$ for some finitely generated projective R-modules A^* and B^*. If $A^* = 0$, it is clear that $A \lesssim_{\oplus} B$. So we may assume that $A^* \neq 0$. Note that $A \prec_a B$, then we have $S \oplus A^* \cong A \prec_{\oplus} S \oplus B$. Since R is separative by [14, Theorem 1.9] and $S \lesssim_{\oplus} A$ and $S \lesssim_{\oplus} B$, by [3, Lemma 2.1], we have that $A^* \prec_{\oplus} B$. Let $B \cong A^* \oplus X$ for some nonzero finitely generated projective R-module X. So $A \cong S \oplus A^* \lesssim_{\oplus} A^* \oplus X \cong B$, i.e., $A \lesssim_{\oplus} B$. $\square$

We remark that there are plenty of examples of rings which have a simple right ideal as a direct summand but which are not regular. For example, if V is an infinite dimensional vector space over a field, E is the ring of all linear

transformations on V, and J is the set of linear transformations in E with finite rank (i.e., having finite dimensional image), then any subring R of E which contains J satisfies the conditions. The author thanks K. R. Goodearl for his giving the example.

Theorem 2.4. *Let R be a directly finite refinement ring satisfying almost comparability with a simple right ideal $S \lesssim_\oplus R$. For any two nonzero finitely generated projective R-modules A and B and positive integer n, $nA \prec_\oplus nB$ implies $A \prec_a B$.*

Proof. Since R satisfies almost comparability, either $A \prec_a B$ or $B \prec_a A$, by Theorem 1.17 and Proposition 1.9. Assume that the latter is true. By Lemma 2.3, $B \lesssim_\oplus A$. Let $B \oplus C \cong A$ for some finitely generated projective R-module C. So $nB \oplus nC \cong nA$. Since $nA \prec_\oplus nB$, we have $nA \oplus D \cong nB$ for some nonzero finitely generated projective R-module D. Therefore, $nA \oplus D \oplus nC \cong nA$, which contradicts the direct finiteness of nA, since R is stably finite. $\square$

Recall that a ring R is said to satisfy weak comparability if for each nonzero $x \in R$, there is $n = n(x)$ such that $n(yR) \lesssim_\oplus R$ implies that $yR \lesssim_\oplus xR$ for all $y \in R$. There exists a non-simple directly infinite refinement ring with both 1-comparability (hence almost comparability) and weak comparability [11, p. 3145]. For the cancellation property of special case of exchange rings with weak comparability, we have the following results.

Theorem 2.5. *Let R be an exchange ring with weak comparability. Then $A \oplus C \lesssim_\oplus B \oplus C$ implies $A \lesssim_\oplus B$ for any nonzero finitely generated projective R-modules A, B and C with B directly infinite.*

Proof. By [15, Theorem 1], we may assume that C is cyclic. Since B is directly infinite, there is a nonzero cyclic projective R-module W such that $B \oplus W \lesssim_\oplus B$. Since R satisfies weak comparability, there is a positive integer m such that $mX \lesssim_\oplus R$ implies that $X \lesssim_\oplus W$ for any cyclic projective R-module X. Since $A \oplus C \lesssim_\oplus B \oplus C$, we have a decomposition $B \oplus C = A' \oplus C' \oplus D$ for some projective R-modules A', C' and D such that $A' \cong A$, $C' \cong C$. Let f be the isomorphism from C to C'. Since R is an exchange ring, the finitely generated projective R-module A' has the exchange property. Thus there exist decompositions $B = B_1 \oplus B_1^*$ and $C = C_1 \oplus C_1^*$ such that $B \oplus C = A' \oplus B_1^* \oplus C_1^*$. And so $A' \cong B_1 \oplus C_1$. Since $A' \oplus fC_1$ is the direct summand of $B \oplus C$, it has the exchange property. Thus there are decompositions $B_1^* = B_2 \oplus B_2^*$ and $C_1^* = C_2 \oplus C_2^*$ such that $B \oplus C = A' \oplus fC_1 \oplus B_2^* \oplus C_2^*$. So $fC_1 \cong B_2 \oplus C_2$. Since $C_1 \cap C_2 = 0$, we have that $A' \oplus fC_1 \oplus fC_2$ is a direct summand of $B \oplus C$. Continuing the above procedure, we have decompositions $B_n^* = B_{n+1} \oplus B_{n+1}^*$ and $C_n^* = C_{n+1} \oplus C_{n+1}^*$ such that $B \oplus C = A' \oplus fC_1 \oplus \cdots \oplus fC_n \oplus B_{n+1}^* \oplus C_{n+1}^*$ and $fC_n \cong B_{n+1} \oplus C_{n+1}$. If $C_k = 0$ for some positive integer k, we have that $A \cong A' \cong B_1 \oplus C_1 \cong B_1 \oplus fC_1 \cong B_1 \oplus B_2 \oplus C_2 \cong \cdots \cong B_1 \oplus B_2 \oplus \cdots \oplus B_{k-1} \oplus C_{k-1} \cong B_1 \oplus B_2 \oplus \cdots \oplus B_k \leq_\oplus B$. If $C_n \neq 0$ for all positive integers n, since

$C_m \lesssim_\oplus \cdots \lesssim_\oplus C_1 \leq_\oplus C$, we have that $mC_m \lesssim_\oplus C \lesssim_\oplus R$, and so $C_m \lesssim_\oplus W$. Thus we have that

$$A \cong A' \cong B_1 \oplus C_1 \cong B_1 \oplus fC_1 \cong B_1 \oplus B_2 \oplus C_2$$
$$\cong \cdots \cong B_1 \oplus B_2 \oplus \cdots \oplus B_m \oplus C_m$$
$$\lesssim_\oplus B_1 \oplus B_2 \oplus \cdots \oplus B_m \oplus W$$
$$\leqslant_\oplus B \oplus W \lesssim_\oplus B,$$

and we complete the proof. $\square$

Theorem 2.6. *Let R be an exchange ring with weak comparability. Then $A \oplus C \prec_\oplus B \oplus C$ implies $A \prec_\oplus B$ for any nonzero finitely generated projective R-modules A, B and C, if C has the property that $\aleph_0 X \not\lesssim_\oplus C$ for any nonzero finitely generated projective R-module X.*

Proof. By [15, Theorem 1], we may assume that C is cyclic. Since $A \oplus C \prec_\oplus B \oplus C$, we have a decomposition $B \oplus C = A' \oplus C' \oplus D$ for some nonzero projective R-modules A', C' and D such that $A' \cong A$, $C' \cong C$. Let f be the isomorphism from C to C'. Since R is an exchange ring, the finitely generated projective R-module D has the exchange property. Thus there exist decompositions $B = B_1 \oplus B_1^*$ and $C = C_1 \oplus C_1^*$ such that $B \oplus C = D \oplus B_1^* \oplus C_1^*$. And so $D \cong B_1 \oplus C_1$. Since $D \oplus fC_1$ is isomorphic to the direct summand of $B \oplus C$, it has the exchange property. Thus there are decompositions $B_1^* = B_2 \oplus B_2^*$ and $C_1^* = C_2 \oplus C_2^*$ such that $B \oplus C = D \oplus fC_1 \oplus B_2^* \oplus C_2^*$. So $C_1 \cong fC_1 \cong B_2 \oplus C_2$. Since $C_1 \cap C_2 = 0$, we have that $D \oplus fC_1 \oplus fC_2$ is isomorphic to a direct summand of $B \oplus C$. Continuing the above procedure, we have decompositions $B_n^* = B_{n+1} \oplus B_{n+1}^*$ and $C_n^* = C_{n+1} \oplus C_{n+1}^*$ such that $B \oplus C = D \oplus fC_1 \oplus \cdots \oplus fC_n \oplus B_{n+1}^* \oplus C_{n+1}^*$ and $C_n \cong fC_n \cong B_{n+1} \oplus C_{n+1}$ for each $n = 1, 2, \ldots$. We claim that $B_k \neq 0$ for some positive integer k. In fact, if $B_k = 0$ for all k, $0 \neq \aleph_0 D \cong C_1 \oplus C_2 \oplus \cdots \leq_\oplus C$, which contradicts the hypothesis. Since $B \oplus C = A' \oplus C' \oplus D$, we have that $A' \oplus fC_k^* \oplus (D \oplus fC_1 \oplus \cdots \oplus fC_{k-1}) \leq_\oplus A' \oplus fC_1 \oplus \cdots \oplus fC_{k-1} \oplus fC_k \oplus fC_k^* \oplus D = B_k^* \oplus C_k^* \oplus (D \oplus fC_1 \oplus \cdots \oplus fC_{k-1})$. Since $A' \oplus fC_k^*$ is a finitely generated projective R-module, according to the above procedure, there exist decompositions $B_k^* = B_{k1} \oplus B_{k1}^*, \ldots, B_{kn}^* = B_{k,n+1} \oplus B_{k,n+1}^*$ and $C_k^* = C_{k1} \oplus C_{k1}^*, \ldots, C_{kn}^* = C_{k,n+1} \oplus C_{k,n+1}^*$ such that $A' \cong B_{k1} \oplus C_{k1}$ and $C_{kn} \cong B_{k,n+1} \oplus C_{k,n+1}$. Then we have that $nC_{kn} \lesssim_\oplus C \lesssim_\oplus R$ for each positive integer n. If $C_{km} = 0$ for some m, then

$$A \cong A' \cong B_{k1} \oplus C_{k1} \cong B_{k1} \oplus B_{k2} \oplus C_{k2}$$
$$\cong \cdots \cong B_{k1} \oplus \cdots \oplus B_{k,m-1} \oplus C_{k,m-1}$$
$$\cong B_{k1} \oplus \cdots \oplus B_{km}$$
$$\prec_\oplus B_k \oplus B_{k1} \oplus \cdots \oplus B_{km}$$
$$\leqslant_\oplus B.$$

If $C_{km} \neq 0$ for all positive integers m. Since $B_k \neq 0$, there is a nonzero cyclic submodule Z of B_k as a direct summand. Thus using the weak comparability

for R, we have that $C_{kl} \lesssim_{\oplus} Z \leq_{\oplus} B_k$ for some positive integer l. We claim that $B_{ks} \neq 0$ for some positive integer $s > l$. In fact, if $B_{ks} = 0$ for all positive integers $s > l$, then $C_{kl} \cong C_{k,l+1} \cong \cdots$ and $C_{kl} \oplus C_{k,l+1} \oplus \cdots \leqslant_{\oplus} C$, so $\aleph_0 C_{kl} \leq_{\oplus} C$, which contradicts the hypothesis. Thus

$$\begin{aligned}
A \cong A' \cong B_{k1} \oplus C_{k1} &\cong B_{k1} \oplus B_{k2} \oplus C_{k2} \\
&\cong \cdots \cong B_{k1} \oplus \cdots \oplus B_{k,l} \oplus C_{k,l} \\
&\prec_{\oplus} B_k \oplus B_{k1} \oplus \cdots \oplus B_{kl} \oplus B_{ks} \\
&\leq_{\oplus} B.
\end{aligned}$$
$\qquad\square$

Remark 2.7. (1) In Kutami's 2005 paper, he proved that for a directly infinite regular ring R with weak comparability, $A \oplus C \lesssim_{\oplus} B \oplus C$ implies $A \prec_{\oplus} B$ for any nonzero finitely generated projective R-modules A, B and C with $B \neq 0$, see [11, Proposition 1.5]. So the idea of Theorem 2.5 is due to Kutami.

(2) Recall that a module M is said to have finite uniform dimension if it contains no infinite direct sum of nonzero submodules. It is clear that any module with finite uniform dimension (hence any uniform module and Noetherian module) has the hypothesis of Theorem 2.6.

(3) In 2003, Kutami proved that for a regular ring R with weak comparability, $A \oplus C \prec_{\oplus} B \oplus C$ implies $A \prec_{\oplus} B$ for any finitely generated projective R-modules A, B and C with C directly finite, see [10, Theorem 2.2]. Theorem 2.6 is motivated by [10, Theorem 2.2].

We end this note by raising the following questions:

Question 2.8. (1) Does there exist a refinement ring with almost comparability which is not regular and without strictly unperforated property?

(2) Does there exist a refinement ring with weak comparability that is not a regular ring without almost comparability?

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