

Probabilistic Modeling of Fiber Length Segments within a Bounded Area of Two-Dimensional Fiber Webs

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Abstract

Statistical and probabilistic behaviors of fibers forming fiber webs of all kinds are of great significance in the determination of the uniformity and physical properties of the webs commonly found in many industrial products such as filters, membranes and non-woven fabrics. However, in studying the spatial geometry of the webs the observations must be theoretically as well as experimentally confined within a specified unit area. This paper provides a general theory and framework for computer simulation for quantifying the fiber segments bounded by the unit area in consideration of the “edge effects” resulting from the truncated length segments within the boundary. The probability density function and the first and second moments of the length segments found within the counting region were derived by properly defining the seeding region and counting region.

Keywords: Fiber segment length distribution, fiber intersections, fiber web, edge effect, seeding region, counting region.

1. Introduction

An ideal two-dimensional fiber web may be defined as a sheet of fibers formed by depositing fibers at random onto a specified area by a random process. The geometrical structures and mechanical properties of random fiber webs have often been discussed in terms of ideal two dimensional fiber arrangements under simplified assumptions in order to study the basic inter-relationships among the fiber lengths and the resulting structural and physical properties of the webs. Some of these details are well explained by Kallmes and Corte (1960) and Yi *et al.* (2004). Such geometrical properties, such as the number of fiber intersections and pore size distributions formed by the fibers are known to be important for predicting the mechanical, porous and optical properties of a two-dimensional random fiber web. It has been recognized that the fiber length plays a crucial role in determining the number of intersections and pore sizes in every random fiber web (Rawal *et al.*, 2007)). In forming a two-dimensional random fiber web, the manufacturers of various industrial products have known that the number of fibers and their aggregate length determine most of the properties of the resulting web. However, the exact relationship has not been well quantified due to the random nature of the fibers and their distribution in a plane or volume, often resorting on experimental trials only. Many studies have shown that fiber length is the most dominant variable influencing the end-use properties and the manufacturing processes (Krifa, 2006).

In examining the prior studies, however, the effects originating the edges (borders) of the study area with respect to the fiber lengths have not been properly addressed, making it difficult to validate a theory as to whether the discrepancies, if any, come from the theory or the simulation or the “edge effects”. If a fiber is to be found in a confined area of study called a counting region, the segment to

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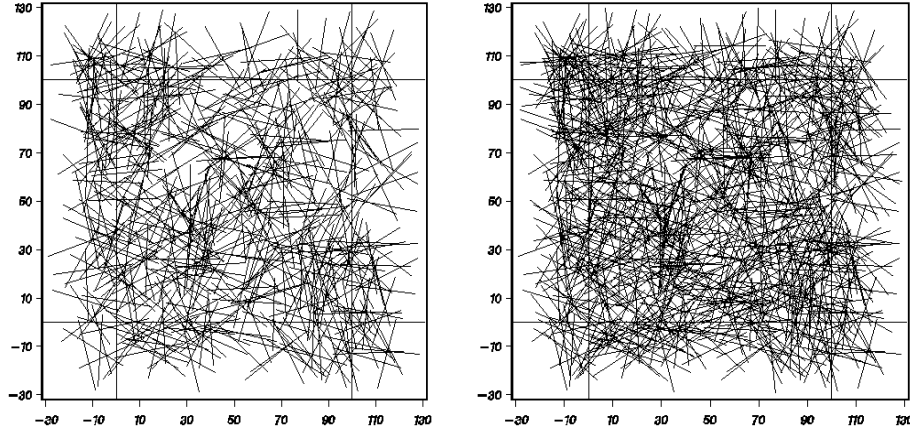


Figure 1: Simulated random fiber webs ($n = 600, 1000$ for $l = 30$)

be found in the region partially or in its entirety is a basic underlying statistical random variable for studying the intersection and the pore size geometries of the web especially when the observation area is not sufficiently larger than the fiber lengths. Fiber length segment distribution is essential factor of determining the exact mean and variance formula of the fiber intersection in a bounded area (Suh *et al.*, 2010).

In studying the physical and functional properties of nonwoven fabrics, filters, and papers, it is essential to understand the orientation of the fibers and the resulting mean and variance of the number of fiber intersections within a unit area, and the pore sizes formed as a consequence. These geometrical properties are directly related to the aggregate fiber lengths, fiber length distributions, and their orientations. However, uniformities in the mechanical and geometrical properties of these important industrial products cannot be understood or predicted without studying the random nature of the fibers and their geometrical distributions in two and three dimensional spaces. Because of the historic significance of these subjects, Suh *et al.* (2010) published a paper on the number of fiber intersections points to be found within a confined area. In the paper, the authors derived the mean and variance using an innovative approach based on the concept of a seeding region and a counting region for proving theories by computer simulations. The results show,

$$E(Y) = \frac{n(n-1)l^2|C|}{\pi|S|^2},$$

$$\text{Var}(Y) = \frac{n(n-1)l^2|C|}{\pi|S|^2} \left(1 - \frac{2l^2|C|}{\pi|S|^2}\right) + \frac{4n(n-1)(n-2)l^2\text{Var}(L)}{\pi^2|S|^2},$$

where n fibers are randomly thrown onto the seeding area “S”.

For an illustration of random fiber intersections, Figure 1 shows two simulated two-dimensional random fiber webs which are formed by 600 and 1000 staple fibers, respectively, of length 30 thrown at random within a seeding area of 130×130 defined as the region where the midpoints of all fibers are to be located or thrown at random. While the fibers are of different types, the intersection and pore size behaviors of the webs appear strikingly similar to each other. We are interested in obtaining the distribution of fiber length segments found in a region (counting region) within which the number of fiber intersections will be counted based on the known number of fiber midpoints thrown onto the seeding area.

This paper attempted the following: Section 2 develops the basic geometrical and probabilistic model for defining the fiber segments found within the counting area where the number of fiber intersections is counted for any studies leading to the intersection geometry or pore size distributions. Section 3 derives the first two moments of the fiber length segment. Section 4 and Section 5 show the exact distributions of fiber length segment derived within a circle and within a rectangle, respectively, applying geometrical probability. In the final section, the second moments of two fiber length segment distributions obtained in a circle and a rectangle are compared based on the theoretical expressions.

2. Geometrical and Probabilistic Model of Two-Dimensional Random Fiber Web

The probabilistic model relating to the number of fiber intersections has originated from the Buffon's needle problem (Klain and Rota, 1997; Schuster, 1974; Solomon, 1978). First, a fixed number of fiber midpoints n is considered followed by distributions of the fiber midpoints. Let F be a bounded convex set called a seeding region within which n fiber midpoints are independently and uniformly located. Let $|\cdot|$ denote area, that is, the Lebesgue measure on the plane. For a given Borel set S ($S \subset F$), which is also called a seeding region. Let $N(S)$ be the number of midpoints contained in S . The geometric probabilities will be considered for the midpoints (X, Y) of the fibers thrown at random on the seeding region. In order to generate a probabilistic model for the geometry of an ideal two-dimensional random fiber web, the following assumptions are made by Stoyan and Stoyan (1994) and Stoyan *et al.* (1995).

Assumption 1. *The n midpoints of fibers m_1, \dots, m_n are stochastically independent, i.e. the probability that m_i lies in the Borel set $S_i \subset F$, $i = 1, 2, \dots, n$ satisfies*

$$\Pr[m_1 \in S_1, \dots, m_n \in S_n] = \Pr[m_1 \in S_1] \cdots \Pr[m_n \in S_n].$$

Assumption 2. *Each midpoints of fibers m_1, \dots, m_n is uniformly distributed in F , i.e. for any $i = 1, \dots, n$ and for any Borel set $S \subset F$,*

$$\Pr[m_i \in S] = \frac{|S|}{|F|}.$$

The probability that m_i lies in S is proportional to the area of S .

From Assumption 2, the probability that more than two fibers cross at a single point is 0 and that the midpoints of all fibers are to be found within an area $dxdy$ equally likely within the plane. A random fiber web is assumed to be constructed by fibers whose midpoints are generated on the seeding region S . Each fiber is assumed to be a straight-line segment with its midpoint located at (x, y) and forming an angle randomly with respect to the x -axis. By designating C to be the counting region where the fiber intersections will be counted, it is also assumed to be a bounded convex set. The counting region C should be at a distance $l/2$ inside the boundary of the seeding region S in order to take into account the edge effect (see Figure 2).

3. Moments of Fiber Length Distribution

In developing a probabilistic model, the length of the fiber contained in the counting region C can be less than l and becomes a random variable even though we begin with a fixed fiber length for all fibers. By letting L be a random variable defined as the length segment of a fiber that falls within the counting region C in part or in its entirety, we are interested in obtaining the first two moments of L .

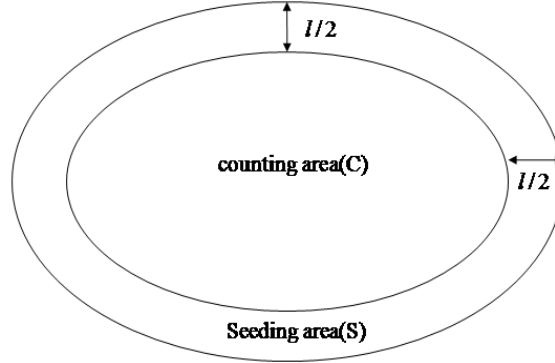


Figure 2: Seeding region and counting region

Theorem 1. Consider a random straight fiber web according to the probabilistic model generated in binomial field. Then,

$$E(L) = l \frac{|C|}{|S|} \quad \text{and} \quad E(L^2) = \int_0^\pi \int_{-l}^l (l - |u|) \frac{|C \cap C_{u,\theta}|}{|S|} \frac{1}{\pi} du d\theta,$$

where $C_{u,\theta} = C - (u \cos \theta, u \sin \theta) \subset S$ for all $u \in [-l, l]$ and for all $\theta \in [0, \pi)$.

Proof: Suppose that (X, Y) is the position of the fiber midpoint in S and θ is an angle that the fiber forms with the x -axis. We define

$$1_C(x, y) = \begin{cases} 1, & (x, y) \in C, \\ 0, & \text{otherwise.} \end{cases}$$

The fiber length segment L contained in C is expressed as

$$L = \int_{-\frac{l}{2}}^{\frac{l}{2}} 1_C(X + t \cos \theta, Y + t \sin \theta) dt.$$

The first moment given θ is

$$\begin{aligned} E(L | \theta) &= \int_{-\frac{l}{2}}^{\frac{l}{2}} E[1_C(X + t \cos \theta, Y + t \sin \theta | \theta)] dt \\ &= \int_{-\frac{l}{2}}^{\frac{l}{2}} \Pr[(X + t \cos \theta, Y + t \sin \theta) \in C | \theta] dt \\ &= \int_{-\frac{l}{2}}^{\frac{l}{2}} \Pr[(X, Y) \in C - (t \cos \theta, t \sin \theta) | \theta] dt. \end{aligned}$$

Since counting region C is at a distance at least $l/2$ from the boundary of the seeding region, for all $t \in [-l/2, l/2]$ and for all $\theta \in [0, \pi)$,

$$C_{t,\theta} = C - (t \cos \theta, t \sin \theta) \subset S.$$

Thus, it follows that

$$\Pr[(X, Y) \in C_{t,\theta} | \theta] = \frac{|C_{t,\theta}|}{|S|} = \frac{|C|}{|S|}.$$

Therefore,

$$E(L|\theta) = l \frac{|C|}{|S|}.$$

Since this conditional expectation does not depend on θ , we have

$$E(L) = l \frac{|C|}{|S|}.$$

Similarly, L^2 is represented as

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} 1_C(X + t \cos \theta, Y + t \sin \theta) \times 1_C(X + t' \cos \theta, Y + t' \sin \theta) dt dt'.$$

The second moment with a given θ is

$$\begin{aligned} E(L^2 | \theta) &= \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} E[1_C(X + t \cos \theta, Y + t \sin \theta) \times 1_C(X + t' \cos \theta, Y + t' \sin \theta) | \theta] dt dt' \\ &= \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} \Pr[(X, Y) \in C_{t,\theta} \cap C_{t',\theta}] dt dt' \\ &= \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{|C_{t,\theta} \cap C_{t',\theta}|}{|S|} dt dt'. \end{aligned} \quad (3.1)$$

Consider transformation $u = t - t'$ and $v = t + t'$. Then

$$t = \frac{u + v}{2} \quad \text{and} \quad t' = \frac{u - v}{2}. \quad (3.2)$$

Let us consider a rectangular set C as shown in Figure 3. Note that

$$|C| = |C_{t,\theta}| = |C_{t',\theta}|.$$

$|C_{t,\theta} \cap C_{t',\theta}|$ is the grayed area of Figure 3 identical to $|C \cap C_{u,\theta}|$. Substituting the expression (3.2) for t and t' into Equation (3.1) and using the Jacobian of transformation $|J| = 1/2$, we have

$$\begin{aligned} E(L^2 | \theta) &= \frac{1}{2} \int_0^l \int_{u-l}^{l-u} \frac{|C \cap C_{u,\theta}|}{|S|} dv du + \frac{1}{2} \int_{-l}^0 \int_{-l-u}^{l+u} \frac{|C \cap C_{u,\theta}|}{|S|} dv du \\ &= \int_0^l (l - u) \frac{|C \cap C_{u,\theta}|}{|S|} du + \int_{-l}^0 (l + u) \frac{|C \cap C_{u,\theta}|}{|S|} du \\ &= \int_{-l}^l (l - |u|) \frac{|C \cap C_{u,\theta}|}{|S|} du. \end{aligned}$$

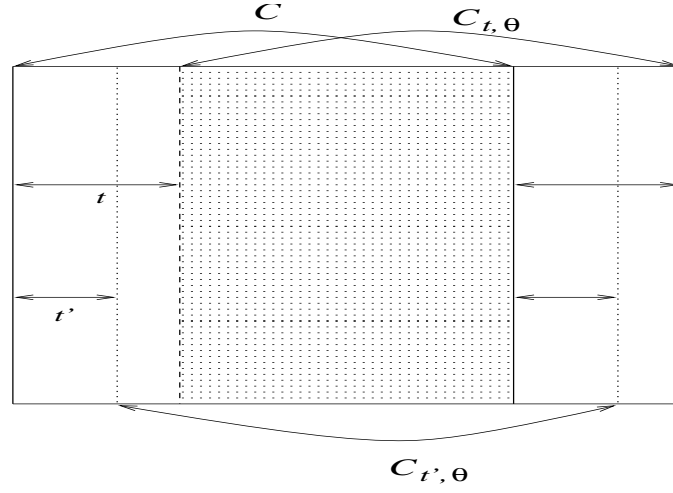


Figure 3: Translation

Therefore, it follows

$$E(L^2) = \int_0^\pi \int_{-l}^l (l - |u|) \frac{|C \cap C_{u,\theta}|}{|S|} \frac{1}{\pi} du d\theta. \quad (3.3)$$

□

Note that the second moment of Theorem 1 depends on S only through an overall factor of $1/|S|$. The exact expression of the second moment of L does not appear to simplify in any way that is similar to the integral for the first moment. However, because Theorem 1 is not a closed form rendering to an easy calculation, the Corollary 1 is provided for the second moment of L for the cases where computation of exact value of $E(L^2)$ is rather difficult.

Corollary 1. *Let us denote P the perimeter of the counting region C . The second moment of Theorem 1 is approximated as*

$$E(L^2) \approx l^2 \frac{|C|}{|S|} \left(1 - \frac{lP}{3\pi|C|} \right).$$

Proof: From Equation (3.3)

$$\begin{aligned} |C \cap C_{u,\theta}| &= |C| - |C \setminus (C \cap C_{u,\theta})| \\ &= |C_{u,\theta}| - |C_{u,\theta} \setminus (C \cap C_{u,\theta})| \\ &= |C| - |C_{u,\theta} \setminus (C \cap C_{u,\theta})| \\ &= |C| - \frac{1}{2} \{ |C \setminus (C \cap C_{u,\theta})| + |C_{u,\theta} \setminus (C \cap C_{u,\theta})| \} \\ &= |C| - \frac{1}{2} |[C \setminus (C \cap C_{u,\theta})] \cup [C_{u,\theta} \setminus (C \cap C_{u,\theta})]| \\ &= |C| - \frac{1}{2} |(C \cup C_{u,\theta}) \setminus (C \cap C_{u,\theta})| \end{aligned} \quad (3.4)$$

and for small u the area of the symmetric difference $(C \cup C_{u,\theta}) \setminus (C \cap C_{u,\theta})$ is approximately the area swept out by the boundary of C as it is translated to $C_{u,\theta}$. In Equation (3.4), $|(C \cup C_{u,\theta}) \setminus (C \cap C_{u,\theta})|$ is approximated by

$$\int |u \sin(\psi - \theta)| ds,$$

where ds is the length of an element of the boundary and ψ is the angle between the tangent at that point and the x -axis. Thus,

$$\begin{aligned} E(L^2 | \theta) &\approx \int_{-l}^l (l - |u|) \frac{|C| - |u|/2 \int |\sin(\psi - \theta)| ds}{|S|} du \\ &= \int_{-l}^l l \frac{|C|}{|S|} du - \int_{-l}^l |u| \frac{|C|}{|S|} du - \int_{-l}^l \frac{l|u|}{2} \frac{\int |\sin(\psi - \theta)| ds}{|S|} du + \int_{-l}^l \frac{u^2}{2} \frac{\int |\sin(\psi - \theta)| ds}{|S|} du \\ &= l^2 \frac{|C|}{|S|} - l^3 \frac{\int |\sin(\psi - \theta)| ds}{6|S|}. \end{aligned} \quad (3.5)$$

Since $E |\sin(\psi - \theta)|$ is $2/\pi$ for any ψ and Equation (3.5) does not depend on θ ,

$$E(L^2) \approx l^2 \frac{|C|}{|S|} - l^3 \frac{\int ds}{3\pi|S|} = l^2 \frac{|C|}{|S|} \left(1 - \frac{lP}{3\pi|C|}\right),$$

where $P = \int ds$ is the perimeter of the counting region C . □

The second moment of fiber length segment is a crucial factor for calculating the variance of fiber length segment in a bounded area and finally determining the variance of the number of fiber intersections.

4. Fiber Length Segment Distribution in a Circle

Consider a fiber of length l , which is randomly deposited on a plane with its midpoint (X, Y) uniformly distributed over a seeding region S with an angle uniformly distributed on $[0, \pi]$. S is at distance at least $l/2$ apart from the boundary of the counting region C which is a circle of radius r centered at the origin.

Theorem 2. *Let L be the length segment of the fiber that falls within the counting region, a circle with radius r . Suppose that r is greater than or equal to $l/2$. Then, the cumulative distribution function(cdf) of L is*

$$F_L(t) = \begin{cases} 1, & t \geq l, \\ \frac{|S| - 2r^2 \cos^{-1}(t/2r) - r(2l - 3t) \sqrt{1 - t^2/4r^2}}{|S|}, & 0 \leq t < l, \\ 0, & t < 0. \end{cases}$$

The probability density function (pdf) of L is

$$\begin{aligned}\Pr[L = l] &= \frac{2r^2 \cos^{-1}(l/2r) - lr \sqrt{1 - l^2/4r^2}}{|S|}, \\ f_L(t) &= \frac{8r^2 - 3t^2 + lt}{2r|S| \sqrt{1 - t^2/4r^2}}, \quad 0 < t < l, \\ \Pr[L = 0] &= \frac{|S| - r^2\pi - 2rl}{|S|} = \frac{|S| - (|C| + lP/\pi)}{|S|}.\end{aligned}$$

The first two moments are

$$E(L) = \frac{|C|}{|S|} l = \frac{\pi r^2}{|S|} l$$

and

$$E(L^2) = \frac{1}{|S|} \left\{ 2r^2(l^2 + r^2) \cos^{-1}\left(\frac{l}{2r}\right) - \pi r^4 + 16\frac{r^3 l}{3} - rl\left(\frac{l^2}{6} + 13\frac{r^2}{3}\right) \sqrt{1 - \frac{l^2}{4r^2}} \right\}.$$

Proof: Since fiber length segment distribution in a circle is independent of the angle's distribution,

$$\Pr[L \geq t | \theta] = \Pr[L \geq t].$$

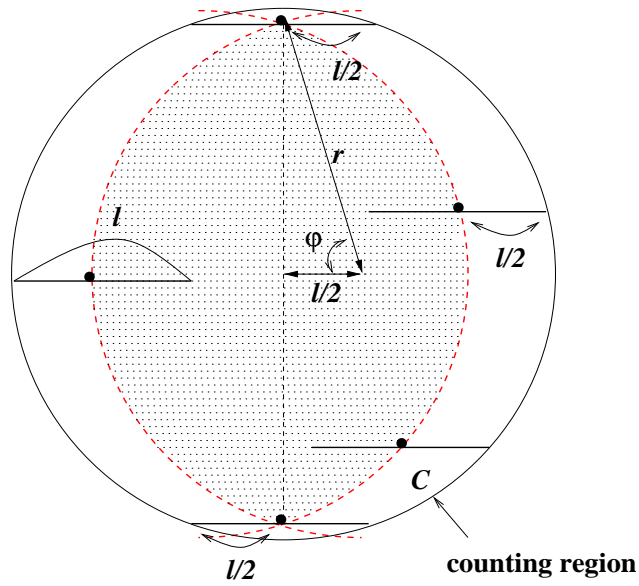
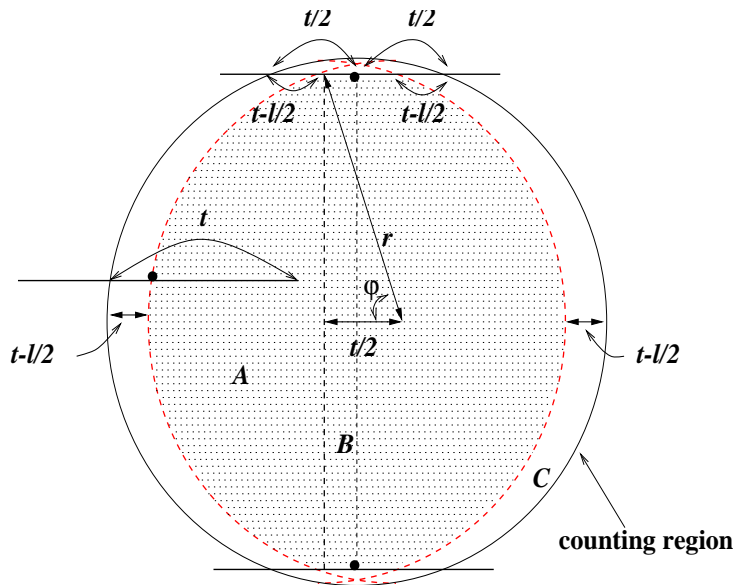
We may fix θ at a particular angle, say, $\theta = 0$.

Case I (where $t = l$): Consider the case where the fiber length segment included in C will be the full length l . If the midpoint of the fiber with length l is within the shaded area shown in Figure 4, then the length of the fiber included in the counting area C is equal to l .

To obtain the shaded area, take the left half of the circle of the counting area, and move it a distance $l/2$ to the right (translation of the left semicircle at distance $l/2$ apart horizontally to the right). The other side of the shaded area can be similarly defined by repeating the process and translating the right semicircle at distance $l/2$ apart to the left. The shaded area in Figure 4 is the enclosed region established by this overlap. Consider an arbitrary fiber of length l placed in seeding region S . If the midpoint of this fiber falls within the shaded area in Figure 4, the length of the part of the fiber contained in C is always l . Since $\cos \varphi = (l/2)/r$,

$$\Pr[L = l] = \frac{\text{shaded area}}{|S|} = \frac{2r^2 \cos^{-1}(l/2r) - lr \sqrt{1 - l^2/4r^2}}{|S|}. \quad (4.1)$$

Case II (where $l/2 < t < l$): Consider the case where the fiber length contained in the counting region C will be greater than or equal to t ($l/2 < t < l$). In order to obtain the shaded area of Figure 5, first, place two fibers horizontally on top and on bottom in such a way that the part of the fibers contained in C has the fixed length t . Second, take the left half of the circle defined as the counting area, and move it at distance $t - l/2$ to the right (translation of the left semicircle at distance $l/2$ apart horizontally to the right). Third, the other side of the shaded area can be similarly obtained by repeating the process and translating the right semicircle at distance $t - l/2$ to the left. Finally, the shaded area in Figure 5 is the region surrounded by the shifted two semi-circles and the two fibers


 Figure 4: Case I ($L \geq t$, where $t = l$)

 Figure 5: Case II ($L \geq t$, where $l/2 < t < l$)

placed on top and bottom. Consider an arbitrary fiber of length l placed in the seeding region S . If the midpoint of this fiber falls within the shaded area in Figure 5, the length of the part of the fiber in C is always greater than and equal to t . To calculate the shaded area, consider the left part from the central vertical line. The half shaded area can be divided into two pieces of A and B as shown Figure 5. The

area of A is

$$\begin{aligned} |A| &= |\text{fan-shape}| - |\text{triangle}| = \frac{1}{2}r^2 2\varphi - \frac{t}{2}\sqrt{r^2 - \frac{t^2}{4}} \\ &= r^2 \cos^{-1}\left(\frac{t}{2r}\right) - \frac{t}{2}\sqrt{r^2 - \frac{t^2}{4}}, \end{aligned}$$

since $\cos \varphi = t/2r$. The area of B is

$$|B| = (l - t)\sqrt{r^2 - \frac{t^2}{4}}.$$

Then the shaded area is

$$2|A| + 2|B| = 2r^2 \cos^{-1}\left(\frac{t}{2r}\right) + r(2l - 3t)\sqrt{1 - \frac{t^2}{4r^2}}.$$

It follows that, for $l/2 \leq t < l$,

$$\Pr[L \geq t] = \frac{\text{shaded area}}{|S|} = \frac{2r^2 \cos^{-1}(t/2r) + r(2l - 3t)\sqrt{1 - t^2/4r^2}}{|S|}. \quad (4.2)$$

Case III (where $0 < t \leq l/2$): Consider the case that the fiber length included in the counting C will have a length greater than or equal to t ($0 < t \leq l/2$). The event that the fiber length is less than $l/2$ is realized when a midpoint of fiber is outside C . If the midpoint is outside the shaded area in the Figure 6, the random variable L will be less than t . The shaded area in Figure 6 can be obtained in a similar manner to Case II above, and $\Pr[L \geq t]$ ends up with the same answer as Case II above.

If we substitute l for t in Equation (4.2), it becomes Equation (4.1). Therefore, by Case I, II and III,

$$\Pr[L \geq t] = \frac{2r^2 \cos^{-1}(t/2r) + r(2l - 3t)\sqrt{1 - t^2/4r^2}}{|S|}, \quad \text{for } 0 < t \leq l. \quad (4.3)$$

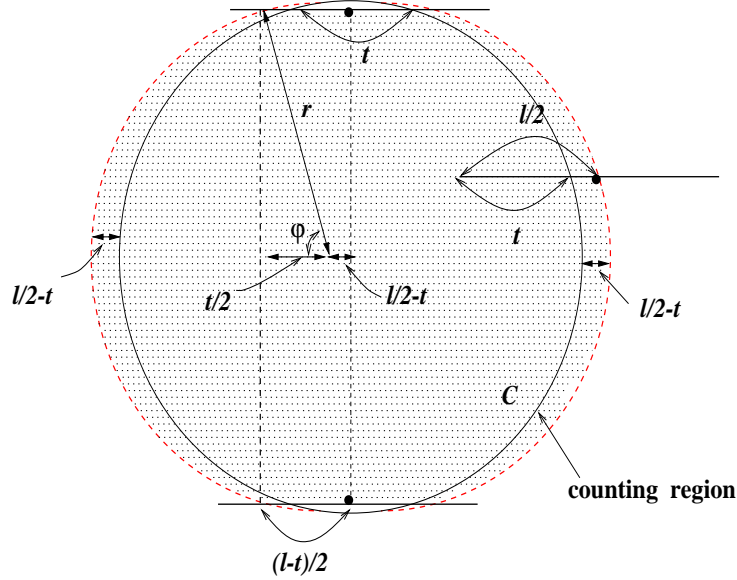
And

$$\Pr[L = 0] = 1 - \Pr[L > 0] = 1 - \frac{r^2\pi + 2rl}{|S|} = \frac{|S| - (|C| + lP/\pi)}{|S|}.$$

Thus, the cumulative distribution function of L is

$$\Pr[L \leq t] = \begin{cases} 1, & t \geq l, \\ \frac{|S| - 2r^2 \cos^{-1}(t/2r) - r(2l - 3t)\sqrt{1 - t^2/4r^2}}{|S|}, & 0 \leq t < l, \\ 0, & t < 0. \end{cases}$$

From this cumulative distribution function or from the Equation (4.3), the probability density function and its moments can be obtained by simple calculations. \square


 Figure 6: Case III ($L \geq t$, where $0 < t \leq l/2$)

5. Fiber Length Segment Distribution in a Rectangle

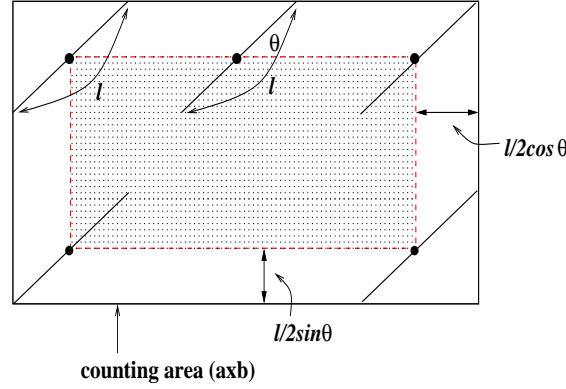
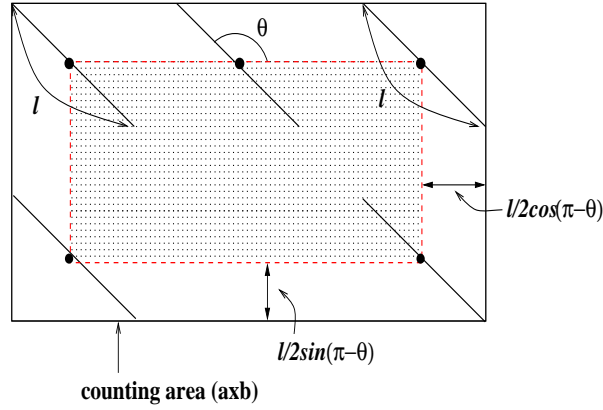
Consider a rectangle as a special case for convenience. Suppose that the seeding region S is any bounded convex set at distance at least $l/2$ apart from the counting region C formed by a rectangle with side a (parallel to the x -axis) and side b (parallel to the y -axis) with left lower corner with the origin: $C = [0, a] \times [0, b]$. Suppose that the midpoint of a fiber of length l is randomly dropped over S . Then, the length of the fiber segment contained in C is of particular interest for obtaining the intersection probability.

Theorem 3. Let L be the length of the fiber segment that falls within C , rectangle of side lengths a and b . Suppose the fiber length l is less than $\min(a, b)$. Then, the cumulative distribution function(cdf) of L is

$$F_L(t) = \begin{cases} 1, & t \geq l, \\ \frac{|S| - \{|C| + (6t^2 - 2t(P + l) + Pl)/\pi\}}{|S|}, & 0 \leq t < l, \\ 0, & t < 0. \end{cases}$$

The probability density function(pdf) of L is

$$\begin{aligned} \Pr[L = l] &= \frac{|C| + (2l^2 - Pl)/\pi}{|S|}, \\ f_L(t) &= \frac{2(l + Pb - 6t)/\pi}{|S|}, \quad 0 < t < l, \\ \Pr[L = 0] &= \frac{|S| - (|C| + lP/\pi)}{|S|}. \end{aligned}$$

Figure 7: Case I: $0 \leq \theta \leq \pi/2$ ($L \geq t$, where $t = l$)Figure 8: Case I: $\pi/2 \leq \theta \leq \pi$ ($L \geq t$, where $t = l$)

The first two moments are,

$$E(L) = \frac{|C|}{|S|} l = \frac{ab}{|S|} l$$

and

$$E(L^2) = \frac{3\pi ab l^2 + (l^4 - 2(a+b)l^3)}{3\pi|S|} = l^2 \frac{|C|}{|S|} \left(1 - \frac{lP}{3\pi|C|}\right) + \frac{l^4}{3|S|},$$

where P is the perimeter of the rectangle counting region C , which is $2(a+b)$.

Proof: Since the angle of a fiber is not symmetric in a square, fix an angle θ that the fiber forms with respect to the x -axis. After we compute $\Pr[L \geq l \mid \theta]$, we take the expectation with respect to θ , and obtain $\Pr[L \geq l]$.

Case I (where $t = l$): Consider the case that the fiber length segment included in the counting region C will be the full fiber length l given θ . Whenever the midpoint of a fiber with angle θ lies within the shaded area shown in Figure 7 and Figure 8, the length segment of the fibers included in the covering

area are l . Given an angle θ with respect to x -axis, the shaded area is the area of the inner rectangle that is at a distance $l/2 \cos \theta$ for $\theta < \pi/2$ or $l/2 \cos(\pi - \theta)$ for $\theta > \pi/2$ horizontally and at a distance $l/2 \sin \theta$ for $\theta < \pi/2$ or $l/2 \sin(\pi - \theta)$ for $\theta > \pi/2$ vertically from the boundary of the counting rectangle area.

If the midpoint of an arbitrary fiber falls within the shaded area in Figure 7 and Figure 8, the length of the part of the fiber contained in C is always l . Since the probability that L is equal to l given θ is the ratio of the shaded area to the seeding area S ,

$$\Pr[L = l | \theta] = \begin{cases} \frac{(a - l \cos \theta)(b - l \sin \theta)}{|S|}, & 0 \leq \theta < \frac{\pi}{2}, \\ \frac{(a - l \cos(\pi - \theta))(b - l \sin(\pi - \theta))}{|S|}, & \frac{\pi}{2} \leq \theta < \pi. \end{cases}$$

Thus, it follows that

$$\begin{aligned} \Pr[L = l] &= \int \Pr[L = l | \theta] f_{\theta}(\theta) d\theta \\ &= \frac{\frac{1}{2} \int_0^{\pi/2} \Pr[L = l | \theta]_{\pi}^2 d\theta + \frac{1}{2} \int_{\pi/2}^{\pi} \Pr[L = l | \theta]_{\pi}^2 d\theta}{|S|} \\ &= \frac{ab + \frac{2}{\pi} \{l^2 - (a + b)l\}}{|S|} \\ &= \frac{|C| + (2l^2 - Pl)/\pi}{|S|}. \end{aligned} \quad (5.1)$$

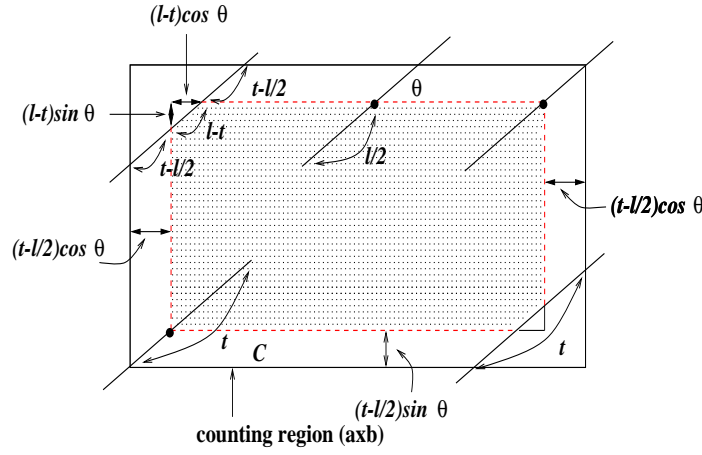
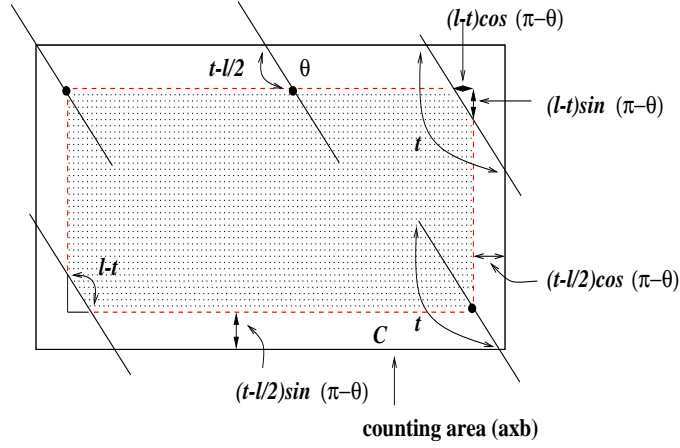
Case II (where $l/2 < t < l$): Consider the case that the fiber length segment contained in the counting region C will be greater than or equal to t ($l/2 < t < l$), given θ . When the midpoint of a fiber is within the shaded region in Figure 9 and Figure 10, the segment of the fiber included in the counting area C is greater than or equal to the fixed length t .

In order to obtain the shaded area of Figure 9, we first place two fibers with an angle θ ($0 \leq \theta < \pi/2$), one at the upper left corner and other at the lower right corner of the rectangle C in a such a way that the part of the fibers contained in C has a fixed length t . Second, we take the two vertical sides of the rectangle defined as the counting area, and move them a distance $(t - l/2) \cos \theta$ in an inward direction. Third, take the two horizontal sides of the counting region, and move them a distance $(t - l/2) \sin \theta$ in an inward direction. Finally, the shaded area in Figure 9 is the area surrounded by the shifted four sides and two fibers of the upper left and lower right corners. The shaded area of Figure 10 where $\pi/2 \leq \theta < \pi$ can be obtained in a similar manner. Thus, whenever the midpoint of a fiber falls within the shaded regions of Figure 9 or Figure 10, L is always greater than or equal to t . Otherwise, L will be less than t , where $l/2 \leq t \leq l$, producing a shaded area in Figure 9 of size

$$\{a - (2t - l) \cos \theta\} \{b - (2t - l) \sin \theta\} - (l - t)^2 \sin \theta \cos \theta.$$

The shaded area in Figure 10 can also be expressed as

$$\{a - (2t - l) \cos(\pi - \theta)\} \{b - (2t - l) \sin(\pi - \theta)\} - (l - t)^2 \sin(\pi - \theta) \cos(\pi - \theta).$$

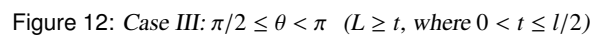
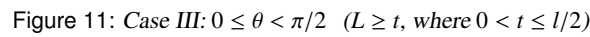
Figure 9: Case II: $0 \leq \theta < \pi/2$ ($L \geq t$, where $l/2 < t < l$)Figure 10: Case II: $\pi/2 \leq \theta < \pi$ ($L \geq t$, where $l/2 < t < l$)

By the same argument in Case I,

$$\begin{aligned}
 \Pr[L \geq t] &= \frac{\frac{1}{2} \int_0^{\pi/2} \Pr[L \geq l | \theta] \frac{2}{\pi} d\theta + \frac{1}{2} \int_{\pi/2}^{\pi} \Pr[L \geq l | \theta] \frac{2}{\pi} d\theta}{|S|} \\
 &= \frac{ab + \frac{2}{\pi} \{3t^2 - 2t(a+b+l) + (a+b)l\}}{|S|} \\
 &= \frac{|C| + \{6t^2 - 2t(P+l) + Pl\} / \pi}{|S|}, \tag{5.2}
 \end{aligned}$$

where $l/2 < t < l$.

Case III (where $0 < t \leq l/2$): Consider the case that the fiber length segment to be found in the counting region C will have a value greater than or equal to t . The event that the length segment of fiber within C is less than $l/2$ will occur when the midpoint of a fiber is placed outside C . The shaded


$$\{a + (l - 2t) \cos \theta\} \{b + (l - 2t) \sin \theta\} - (l - t)^2 \sin \theta \cos \theta,$$
$$\{a + (l - 2t) \cos(\pi - \theta)\} \{b + (l - 2t) \sin(\pi - \theta)\} - (l - t)^2 \sin(\pi - \theta) \cos(\pi - \theta),$$
$$\Pr[L \geq t] = \frac{|C| + \{6t^2 - 2t(P + l) + Pl\} / \pi}{|S|}, \quad t > 0,$$

and

$$\Pr[L = 0] = 1 - \Pr[L > 0] = \frac{|S| - (|C| + lP/\pi)}{|S|}.$$

Thus, the cumulative distribution function(cdf) of L is

$$\Pr[L \leq t] = \begin{cases} 1, & t \geq l, \\ \frac{|S| - \{|C| + (6t^2 - 2t(P + l) + Pl)/\pi\}}{|S|}, & 0 \leq t < l, \\ 0, & t < 0. \end{cases}$$

From this, the probability density function(pdf) and its moments can be obtained easily. \square

6. Remarks and Conclusion

Now we would like to check that the exact expression for $E(L^2)$ where C is a circle is consistent with the approximation in Corollary 1. The second moment of Theorem 2 may be approximated by Taylor series expansion. If we consider $E(L^2)$ of Theorem 2 as a function of l and expand it up to order l^7 at $l = 0$, Taylor series expansion of the exact moment is

$$\begin{aligned} E(L^2) &= \frac{1}{|S|} \left\{ 2r^2(l^2 + r^2) \cos^{-1}\left(\frac{l}{2r}\right) - \pi r^4 + \frac{16r^3 l}{3} - rl\left(\frac{l^2}{6} + 13\frac{r^2}{3}\right) \sqrt{1 - \frac{l^2}{4r^2}} \right\} \\ &= \frac{r^2 l^2 \pi - \frac{2rl^3}{3}}{|S|} + \frac{l^5}{120r|S|} + O(l^7) \\ &= l^2 \frac{|C|}{|S|} \left(1 - \frac{lP}{3\pi|C|}\right) + \frac{l^5}{120r|S|} + O(l^7), \end{aligned} \quad (6.1)$$

where $O(l^7)$ is the remaining term including the higher orders than order 7 of l . The difference between Equation (6.1) and Corollary 1 is

$$\frac{1}{120} \frac{l^5}{r|S|} + O(l^7). \quad (6.2)$$

Note that the lowest order of l in the error term in case of circle is l^5 , showing that the coefficient of l^4 is zero.

In order to check that the exact expression for $E(L^2)$ is consistent with the approximation in Corollary 1, write the second moment of L of Theorem 3, where C is a rectangle, as

$$\begin{aligned} E(L^2) &= \frac{3\pi ab l^2 + (l^4 - 2(a+b)l^3)}{3\pi|S|} \\ &= l^2 \frac{|C|}{|S|} \left(1 - \frac{lP}{3\pi|C|}\right) + \frac{l^4}{3\pi|S|}. \end{aligned} \quad (6.3)$$

In Equation (6.3), the error term using Corollary 1 is

$$\frac{l^4}{3\pi|S|}. \quad (6.4)$$

This shows that the remainder term in case of circle is of smaller (higher) order than l^4 in case of rectangle. The case where C is a rectangle can be generalized to any polygon, with error term that is proportional to l^4 for all l less than the smallest side length. By Equation (6.2) and Equation (6.4), circle and rectangle both provide the error term of order l^4 . Corollary 1 gives a very concise approximation to $E(L^2)$ that is explicitly of order l^3 .

In this study, we developed a foundation for studying the geometrical and probabilistic properties of two-dimensional random fiber webs applicable to the proper derivation of the distributions of fiber length segments in a confined area of our interest and the resulting number of fiber intersection points that are crucial for the simulation of geometrical, physical, and mechanical properties of nonwoven random fiber webs. The exact numerical derivation of the edge effects is found to be the most important aspects of the theory development as well as any computer simulation as they provide a basis for validating the theories relating to all properties. The probability models and distribution of the fiber length segment found within a counting region facilitated the computation of the first moment and an estimate of the second moment that lay foundations for the study of the number of fiber intersection points. The exact distributions of fiber length segment were derived for circular and rectangular counting regions as special cases. Finally, the general expression of the second moment derived theoretically in this paper is shown to be almost flawless in that the order of magnitudes for the errors were negligible.

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