J. of the Chosun Natural Science
Vol. 4, No. 4 (2011) pp. 289 – 293

Sobolev Estimates for Certain Singular Curves

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Abstract

In this paper we obtain some Sobolev estimates for the integral operator over singular curves (t, t^m) on $\mathbb{R}^2$ for $m \geq 2$.

Key words : Sobolev Estimates, Singular Curves, Boundedness

1. Introduction

For $m \geq 2$ we consider the integral operator

$$Tf(x) = \int_{-\infty}^{\infty} f(x_1 - t, x_2 - t^m)\sigma(t)dt,$$

where σ is a smooth function with a compact support near the origin with $\sigma(0) \neq 0$.

For $\sigma \geq 0$ and $1 < p < \infty$ let $L^p_\alpha(\mathbb{R}^2)$ denote the L^p Sobolev space with the norm

$$\|f\|_{L^p_\alpha(\mathbb{R}^2)} = \left\| \left[(1 + |\bullet|^2)^{\frac{\alpha}{2}} \hat{f} \right]^{\vee} \right\|_{L^p(\mathbb{R}^2)}$$

It is well known that T maps $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ if $1/p - 1/q = 1/1+m$ and $(1/p, 1/q)$ only if and belongs to the closed triangle belongs with vertices (0,0), (1,1) and $(2/1+m, 1/1+m)$ $(m/1+m, m-1/1+m)$, (see [1,3,4]). The localized operator of T maps $L^p(\mathbb{R}^2)$ to $L^p_{1/m}(\mathbb{R}^2)$ if $m/m-1 < p < m$ in view of M. Christ in [2].

The purpose of this paper is to determine the exact range of $(1/p, 1/q, \alpha)$ for which T maps $L^p(\mathbb{R}^2)$ to $L^q_\alpha(\mathbb{R}^2)$ when $0 < \alpha < 1/m$ and $p < q$. We shall prove the following :

Theorem 1. Let $0 < \alpha < 1/m$. The operator T maps $L^p(\mathbb{R}^2)$ to $L^q_\alpha(\mathbb{R}^2)$ if and only if $(1/p, 1/q, \alpha)$ lies on or in the interior of the closed trapezoid with vertices

$$A = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}\right),$$

$$B_\alpha = \left(\frac{\alpha(1-m)+2}{1+m}, \frac{\alpha+1}{1+m}, \alpha\right),\ B'_\alpha,\ A'$$

except the edge AA', where A' and B'_α are the symmetries of A and B_α with respect to the non principal diagonal, respectively.

For an even function $\chi \in C_0^\infty(\mathbb{R})$ such that supp $\chi \subset \{t \in \mathbb{R} : 2^{1/m} \leq t \leq 2^{4/m}\}$, $0 \leq \chi \leq 1$ and $\sum_{l \in \mathbb{Z}} \chi(2^{l/m}t) = 1$ for $t \neq 0$.
We may decompose the operator

$$Tf(x) = \sum_l f \times d\sigma_l$$

where

$$< d\sigma_l, f > = \int f(t, t^m)\chi(2^{-l/m}t)dt.$$

Following the approach in M. Christ in [1], we introduce C^∞ partition of unity $\{\eta_l\}$ in $\mathbb{R}^2$ minus the coordinate axes, with η_l homogeneous of degree zero (with respect to the Euclidean dilation $\mathbb{R}^2$) such that

$$\eta_l(\xi_1, \xi_2) = \eta(2^{-l/m}\xi_1, 2^{-l}\xi_2)$$

and the support of is a subset of

$$\left\{(\xi_1, \xi_2) : 2^{-\frac{l}{m}-1}|\xi_1| \leq 2^{-l}|\xi_2| \leq 2^{-\frac{l}{m}+2}|\xi_1|\right\}.$$

Let Q_l be the operator with multiplier η_l and C_0 be a constant such that $\tilde{\eta}_l = \sum_{|i-l| \leq C_0} \eta_l$ is identically one on the support of η_l. We define $Q_l = \sum_{|i-l| \leq C_0} Q_i$ and

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(Received : September 27, 2011, Revised : December 15, 2011, Accepted : December 22, 2011)

denote by $\tilde{\eta}_l$ its multiplier.

Let $h_l \in C^\infty(\mathbb{R})$ be identically one in a neighborhood of the origin, and let P_l be the Fourier multiplier operator with symbol

$$h_l(\xi_1, \xi_2) = h(2^{-l/m}\xi_1, 2^{-l}\xi_2).$$

For the estimates we split the operator into

$$\begin{aligned}(I-\Delta)^{\alpha/2}Tf &= (I-\Delta)^{\alpha/2}\sum_l d\sigma_l \times f \\ &= (I-\Delta)^{\alpha/2}\sum_l P_l \; d\sigma_l \times f \\ &\quad + (I-\Delta)^{\alpha/2}\sum_l (I-P_l)\tilde{Q}_l \; d\sigma_l \times f \\ &\quad + (I-\Delta)^{\alpha/2}\sum_l (I-P_l)(I-\tilde{Q}_l) d\sigma_l \times f\end{aligned} \tag{1.1}$$

To treat the first and the third integrals in (1.1), we prove the following Lemmas 1 and 2.

Lemma 1. The kernel K^1 of the convolution operators

$$(I-\Delta)^{\alpha/2}\sum_l P_l \; d\sigma_l \times f$$

satisfies

$$|K^1(x)| \le C\frac{1}{(1+|x_1^m + x_2|)^{1+\alpha}}.$$

Proof. Denote K_l by the kernel of the operators $(I-\Delta)^{\alpha/2} P_l \, d\sigma_l \times f$. A computation shows that the kernel K_l is

$$\begin{aligned}&\int e^{i\{(x_1-t)\xi_1 + (x_2 - t^m)\xi_2\}}(1+|\xi|^2)^{\alpha/2} \\ &\quad \times h(2^{-l/m}\xi_1, 2^{-l}\xi_2)\chi(2^{-l/m}t)\,dt\,d\xi_1 d\xi_2 \\ &= 2^l \int e^{i\{(2^{l/m}x_1 - t)\xi_1 + (2^l x_2 - t^m)\xi_2\}} h(\xi)\chi(t) \\ &\quad \times (1+(2^{l/m}\xi_1)^2 + (2^l\xi_2)^2)^{\alpha/2}\,dt\,d\xi_1 d\xi_2\end{aligned} \tag{1.2}$$

Since 2^l is more contributive than $2^{l/m}$ in the kernel estimate (1.2), we have

$$|K_l(x)| \le C 2^{l(1+\alpha)} |G_l(2^{l/m}x_1, 2^l x_2)|$$

where $G_l \in S(\mathbb{R}^2)$ and $G_l = [d\hat{\sigma}_l h_l]^\vee$.

We integrate by parts to obtain

$$|G_l(2^{l/m}x_1, 2^l x_2)| \le C\frac{1}{(1+2^l|x_1^m + x_2|)^N}.$$

Thus, we have

$$\begin{aligned}|K^1(x)| &\le \sum_l |K_l(x)| \\ &\le C\left(\sum_{2^l|x_1^m + x_2| \le 1} 2^{l(1+\alpha)} + \sum_{2^l|x_1^m+x_2|>1} \frac{2^{l(1+\alpha)}}{(2^l|x_1^m + x_2|)^N}\right) \\ &\le C\frac{1}{(1+|x_1^m+x_2|)^{1+\alpha}}\end{aligned}$$

Lemma 2. The kernel K^3 of the convolution operators

$$(I-\Delta)^{\alpha/2}\sum_l (I-P_l)(I-\tilde{Q}_l) d\sigma_l \times f$$

satisfies

$$|K^3(x)| \le C\frac{1}{(1+|x_1^m+x_2|)^{1+\alpha}}.$$

Proof. As in Lemma 1 the kernel of

$$(I-\Delta)^{\alpha/2}(I-P_l)(I-\tilde{Q}_l)d\sigma_l \times f$$

is bounded by

$$C2^{l(1+\alpha)}|G_l(2^{l/m}x_1, 2^l x_2)|,$$

where $G_l = [d\hat{\sigma}_l(1-h_l)(1-\tilde{\eta}_l)]^\vee$.

Since $G_l \in S(\mathbb{R}^2)$, we apply the same argument as above to obtain the desired bound.

2. Proof of Theorem1

We begin with the sufficiency. We introduce the Littlewood-Paley decomposition. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ be supported in $\{\xi : 1/8 \le |\xi| \le 8\}$ such that $\phi(\xi) = 1$, if $1/2 \le |\xi| \le 2$. Then Littlewood-Paley operator L_k is given by $\hat{L_k f} = [\phi(2^{-k}|\cdot|\hat{f})]\}$ and denote by $\phi(2^{-k}|\cdot|) = \phi_k(|\cdot|)$.

For fixed k the operator $L_k T$ does not map $L^1 \to L^\infty$ and $L^m \to L^m_{1/m}$. However, it holds $\|L_k T\|_{L^\infty} \le 2^k \|f\|_{L^1}$ and the following estimate:

Lemma 3. For fixed k the operator $L_k T$ is of restricted weak type (m, m).

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be supported in (-1,1) and $\chi(s) = 1$ if $|s| \le 1/2$. Fix k. We decompose $L_k T$ into

$$L_k Tf(x) = \sum_{l>0} L_k T_l f(x),$$

where

$$T_k f(x) = \int_{-\infty}^{\infty} f(x_1 - t, x_2 - t^m)\chi(K(t)2^l)\sigma(t)dt$$

and K is the Gaussian curvature of the surface (t,t^m). From $K(t) = t^{m-2} \approx 2^{-l}$, we have $t^{-\frac{l}{m-2}}$. Write $L_k T_l = T_{k,l}$ and $f = \chi_E$, where E is a measurable set of finite measure in $\mathbb{R}^2$. We want to show that for $\beta > 0$

$$\left|\left\{x : \sum_{l>0} |T_{k,l}\chi_E(x)| > \beta\right\}\right| \le \left(\frac{2^{-k/m}\|\chi_E\|_m}{\beta}\right)^m .$$

The left-hand side of the above is bounded by

$$\left|\left\{x : \sum_{2^l > \delta} |T_{k,l}\chi_E(x)| > \beta/2\right\}\right| + \left|\left\{x : \sum_{2^l \le \delta} |T_{k,l}\chi_E(x)| > \beta/2\right\}\right|$$

In view of $\|T_{k,l}\chi_E\|_{L^\infty} \le 2^{-\frac{l}{m-2}}$,

we may assume $2^{-\frac{l}{m-2}} < \, < \beta/2$. Then

$$\sum_{2^l > \delta} |T_{k,l}\chi_E(x)| \le \sum_{2^l > \delta} 2^{-\frac{l}{m-2}} < \delta^{-\frac{l}{m-2}} < \frac{\beta}{2}.$$

If we take $\delta^{-\frac{l}{m-2}} = \frac{\beta}{10}$, then

$\left\{x : \sum_{2^l > \delta} |T_{k,l}\chi_E(x)| > \frac{\beta}{2}\right\}$ is an empty set. We proceed to the case $2^l \le \delta$. By van der Corput lemma, the multiplier corresponding to $T_{k,l}$ is bounded by

$$\phi_k(|\xi|)\left(\frac{\chi(2^l K(t))}{\left(|\xi||t^{m-2}|^{\frac{1}{2}}\right)} \le C2^{\frac{l-k}{2}}\right)$$

and so by Plancherel's theorem

$$\|T_{k,l}\|_{L^2 \to L^2} \le 2^{\frac{l-k}{2}} .$$

Therefore by Chebyshev's inequality and L^2 boundedness, we obtain

$$\left|\left\{x : \sum_{2^l \le \delta} |T_{k,l}\chi_E(x)| > \frac{\beta}{2}\right\}\right| \le \frac{1}{\beta^2}\left(2^{-\frac{k}{2}}|E|^{\frac{1}{2}} \sum_{2^l \le \delta} 2^{\frac{l}{2}}\right)$$

where $|E|$ denotes the Lebesgue measure of E. Since $\delta^{-\frac{1}{m-2}} = \frac{\beta}{10}$, this completes the proof.

By duality the operator $L_k T$ is of restricted weak type $m-1/m$, $m-1/m$. We also note that $\|L_k T\|_{L^2} \le 2^{-\frac{k}{m}}\|f\|_{L^2}$.

We interpolate between the points ($C=(1/2,1/2,1/m)$) and $O = (1,0,0)$ to obtain $L^p \to L^q_\alpha$ boundedness of T. We thus obtain that on the open line segment CO the operator T maps L^p to L^q_α such that

$$\|Tf\|_{L^q_\alpha} \le \sum_{k>0} 2^{k\alpha}\|L_k T\|_{L^q} \le C\sum_{k>0} 2^{k\left\{\alpha - \left(\frac{1}{m}+1\right)\theta + 1\right\}}\|f\|_{L^p} \tag{2.1}$$

where for $\theta = 1 - 1/p + 1/q$ for $0 < \theta < 1$.

The last term in (2.1) is convergent when $\alpha - \left(\frac{1}{m}+1\right)\theta + 1 < 0$, which is equivalent to the condition

$$\Im : \frac{1}{p} - \frac{1}{q} < \frac{1-ma}{1+m}$$

Similarly, if we interpolate between the points $A=(1/m,1/m,1/m)$ and O, we also obtain the condition $\Im$. Thus $(1/p,1/q,\alpha)$ must lie on or above the line joining AO and CO with $\Im$, except the point A. By duality $(1/p,1/q,\alpha)$ must also lie on or above the line joining $A'O$ with $\Im$ without the point A'. The intersection points of the line segments AO, $A'O$ with $\Im$ are $B_\alpha = (\alpha(1-m)+2/1+m, \alpha+1/1+m, \alpha)$ and $B_\alpha' = (m-\alpha/1+m, m-\alpha+\alpha m-1/1+m, \alpha)$.

Consequently, if we let Ξ_α be the closed trapezoid with vertices A, B_α, B_α' and A', then T maps L^p to L^q_α in the interior of Ξ_α union the open line segments AB_α and $A'B_\alpha'$. Thus there is only remained the case of the edge $B_\alpha B_\alpha'$, which is on the line $1/p-1/q=1-m\alpha/1+m$.

Therefore, we shall show that

$$\|Tf\|_{L^q_\alpha} \le C\|f\|_{L^p}$$

on the line $1/p-1/q=1-m\alpha/1+m$.

We consider the first integral in (1.1). In view of Lemma 1, we write

$$(I-\Delta)^{\alpha/2}\sum_l P_l d\sigma_l \times f = K^1 \times f.$$

For the estimates

$$\left\|(I-\Delta)^{\alpha/2}\sum_l P_l d\sigma_l \times f\right\|_{L^q} \le C\|f\|_{L^p,} \tag{2.1}$$

we shall show that $K^{-1} \in L^{\frac{1+m}{m(1+\alpha)},\infty}$.

Let λ be a positive number and set $\gamma = \lambda^{\frac{1}{1+\alpha}}$. Using Lemma 1, we have

$$\left|\{x:|K^1(x)|>\lambda\}\right| \le \left|\{x:C|x_1^m+x_2|^{-(1+\alpha)}>\lambda\}\right|$$
$$\le \left|\{x:C|(\gamma^{1/m}x_1)^m+(\gamma x_2)|^{-(1+\alpha)}>1\}\right|$$
$$\le \gamma^{-\left(\frac{1}{m}+1\right)}\left|\{x:C|x_1^m+x_2|^{-(1+\alpha)}>1\}\right| \le C\lambda^{-\frac{1+\frac{1}{m}}{1+\alpha}}.$$

Since now the kernel belongs to $L^{\frac{1+m}{m(1+\alpha)},\infty}$, from Young's inequality it follows that convolution with K^1 maps L^p to L^w, where

$$\frac{1}{w} = \frac{1}{p} + \frac{m(1+\alpha)}{1+m} - 1 = \frac{1}{q}.$$

The same goes for the third integral in (1.1), because the kernel K^3 of the convolution operators

$$(I-\Delta)^{\alpha/2}\sum_l (I-P_l)(I-\tilde{Q}_l)d\sigma_l \times f$$

also belongs to $L^{\frac{1+m}{m(1+\alpha)},\infty}$. Thus we have

$$\left\|(I-\Delta)^{\alpha/2}\sum_l (I-P_l)(I-\tilde{Q}_l)d\sigma_l \times f\right\|_{L^q} \le C\|f\|_{L^p} \quad (2.2)$$

Therefore, it remains to estimate the second term in (1.1). By replacing C_0 by a large constant in the definition of $\tilde{Q}_l$, we may define Q_l' with the same properties such that $Q_l' \circ \tilde{Q}_l = \tilde{Q}_l$ for all l. Then by Littlewood-Paley inequalities

$$\begin{aligned}&\left\|(I-\Delta)^{\alpha/2}\sum_l (I-P_l)\tilde{Q}_l d\sigma_l \times f\right\|_{L^q}\\ &= \left\|\sum_l Q_l'(I-\Delta)^{\alpha/2}(I-P_l)\tilde{Q}_l d\sigma_l \times f\right\|_{L^q}\\ &\le \left\|\left(\sum_l |(I-\Delta)^{\alpha/2}(I-P_l)d\sigma_l \times f_l|^2\right)^{1/2}\right\|_{L^q}\end{aligned} \quad (2.3)$$

where $\tilde{Q}_l\, f = f_l$. Now the kernel of

$$(I-\Delta)^{\alpha/2}(I-P_l)\tilde{Q}_l d\sigma_l \times f$$

is not positive. Thus we cannot directly apply the method of M. Christ in [1]. We decompose the operator $I–P_l$ by telescoping series

$$I-P_l = \sum_{k=1}^{\infty}(P_{k+l}-P_{k+l-1})$$

Set $(P_{k+l} - P_{k+l-1}) = R_{k+l}$. Likewise Q_l we define operators such that $R_n' \circ R_n = R_n$ for all $n \ge l+1$. Then Littlewood-Paley inequalities acting on $L^p(l^2(\mathbb{Z}^2))$, the last term in (2.3) is bounded by

$$\begin{aligned}&\left\|\left(\sum_l\left|\sum_{m=l+1}^{\infty}(I-\Delta)^{\alpha/2}R_nR_n'd\sigma_l \times f_l\right|^2\right)^{1/2}\right\|_{L^q}\\ &\le \left\|\left(\sum_l\sum_n |2^{l\alpha}R_n d\sigma_l \times f_{l,n}|^2\right)^{1/2}\right\|_{L^q}\end{aligned} \quad (2.4)$$

where $f_{l,n} = R_nf_l$.

For and $p \le 2$ by $0 < \theta < 1$, observing $1/2 = \theta/p + 1\text{-}\theta/\infty = \theta/p$, we shall use complex interpolation to estimate (2.4).

Let us denote M_{HL} by nonisotropic Hardy-Littlewood maximal function and M_C by the maximal function of f, respectively.

We note that and $|R_n f| \le M_{HL} f$ $|2^{l\alpha}R_n d\sigma_l \times f_{l,n}| \le M_{HL}(M_C f_{l,n})$.

By Minkowski's inequality, L^q boundedness of M_{HL} and $L^p \to L^q$ boundedness of M_C we obtain we obtain

$$\begin{aligned}&\left\|\left(\sum_l\sum_n |2^{l\alpha}\ R_n d\sigma_l \times f_{l,n}|^p\right)^{1/p}\right\|_{L^q}\\ &\le \left(\sum_l\sum_n \|2^{l\alpha}\ R_n d\sigma_l \times f_{l,n}\|_q^p\right)^{1/p}\\ &\le \left(\sum_l\sum_n \|M_{HL}(M_C f_{l,n})\|_q^p\right)^{1/p}\\ &\le \left(\sum_l\sum_n \|M_C f_{l,n}\|_q^p\right)^{1/p}\\ &\le M_0\left(\sum_l\sum_n \|f_{l,n}\|_p^p\right)^{1/p} = M_0\left\|\left(\sum_l\sum_n \|f_{l,n}\|^p\right)^{1/p}\right\|_{L^p}\end{aligned} \quad (2.5)$$

We use the positivity and boundedness of to have

$$\begin{aligned}&\left\|\sup_{l,n\in\mathbb{Z}}|2^{l\alpha}\ R_n d\sigma_l \times f_{l,n}|\right\|_{L^q}\\ &\le \left\|M_{HL}(\sup_{l,n\in\mathbb{Z}}|2^{l\alpha}\ d\sigma_l \times f_{l,n}|)\right\|_{L^q}\\ &\le \left\|\sup_{l,n\in\mathbb{Z}}|2^{l\alpha}\ d\sigma_l \times f_{l,n}|\right\|_{L^q}\\ &\le \left\|\sum_l 2^{l\alpha}d\sigma_l \times (\sup_{l,n\in\mathbb{Z}}|f_{l,n}|)\right\|_{L^q}\\ &\le \|T\|_{L^p \to L^q_\alpha}\left\|\sup_{l,n\in\mathbb{Z}}|f_{l,n}|\right\|_{L^p}\end{aligned} \quad (2.6)$$

We now interpolate (2.5) and (2.6) to obtain

$$\begin{aligned}&\left\|(I-\Delta)^{\alpha/2}\sum_l (I-P_l)\tilde{Q}_l d\sigma_l \times f\right\|_{L^q}\\ &\le M_0^{\frac{p}{2}}\ \|T\|_{L^p \to L^q_\alpha}^{1-\frac{p}{2}}\left\|\left(\sum_l\sum_n |f_{l,n}|^2\right)^{1/2}\right\|_{L^p}\end{aligned} \quad (2.7)$$

Consequently, combining (2.1) through (2.4) and (2.7), we obtain the desired bound.

We turn to the proof of the necessity. Let $\delta > 0$ be small and f_δ be the characteristic function of the rectangle with dimensions $\delta^{1/m} \times \delta$ centered at the origin. Since away from the parabola the kernel of $(I-\Delta)^{\alpha/2}T$ looks like $C|x_1^m + x_2|^{-(1+\alpha)}$, $|(I-\Delta)^{\alpha/2}Tf_\delta|$ looks like

$$|(I-\Delta)^{\alpha/2}Tf_\delta(x)| \sim |x_1^m + x_2|^{-(1+\alpha)}\delta^{1+\frac{1}{m}}$$

on the set

$$B_\delta = \{x : x_1 \sim \delta^{1/m}, |x_1^m + x_2| > 10\delta\}.$$

Therefore,

$$\left(\int_{B_\delta} |(I-\Delta)^{\alpha/2}Tf_\delta(x)|^q dx\right)^{1/q} \sim \delta^{1+\frac{1}{m}}\delta^{-(1+\alpha)+\left(1+\frac{1}{m}\right)\frac{1}{q}}.$$

Since $\| f_\delta \|_{L^p} = \delta^{\left(1+\frac{1}{m}\right)\frac{1}{p}}$, letting $\delta \to 0$, and comparing the exponent, no inequality of the form

$$\|(I-\Delta)^{\alpha/2}Tf_\delta\|_{L^q} \le C\| f_\delta \|_{L^p}$$

is possible when

$$1 + \frac{1}{m} - (1+\alpha) + \left(1+\frac{1}{m}\right)\frac{1}{q} < \left(1+\frac{1}{m}\right)\frac{1}{p},$$

which is equal to $\frac{1}{p} - \frac{1}{q} > \frac{1-m\alpha}{1+m}$.

This finishes the proof of Theorem 1.

Acknowledgment

This study was supported (in part) by research funds from Chosun University, 2010.

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