Vol. 10, No. 2, pp. 10-15, December 2011

INTERNATIONAL
JOURNAL OF SAFETY
www.kosos.or.kr

Reliability Analysis of Interconnected Dynamical Systems with Reconfigurable Control

Satoshi Matsumoto* and Takehisa Kohda

Department of Aeronautics and Astronautics, Kyoto University, Kyoto 606-8501, Japan

(Received October 28, 2011; Accepted December 9, 2011)

Abstract : The reconfigurable control systems based on feedback controls are utilized to compensate for damages of actuators in control systems. Such systems have multiple feedback controls and switch them in accordance with the degrees of the failures of the actuators. In order to be able to assess those systems, this paper develops a method to obtain reliabilities of reconfigurable dynamical systems which are interconnected in parallel / series configuration. By calculating reliabilities of interconnected dynamical systems, it is possible to assess many dynamical systems by comparing their reliabilities. Firstly, reliabilities of subsystems are obtained according to the definitions of the failures in terms of robust reliability for each subsystem. Then, the reliability of overall system is calculated from reliabilities of subsystems, using the methodology employed for probabilistic safety assessment. Failure rates of subsystems with feedbacks for reconfiguration change in certain time periods because of the switching of feedback controls. In order to deal with this, combinations of subsystems which compose overall system for each time period are derived by the methodology mentioned above and then integrated to calculate the reliability of overall system for a specific time. An illustrative example shows the validity and details of the proposed method.

Key words : dynamical system, interconnected, reconfigurable control, robust reliability, mission reliability

1. Introduction

The reconfigurable control systems based on feedback controls are utilized to compensate for damages of actuators in control systems [1]. Such systems have multiple feedback controls and switch them in accordance with the degrees of the failures of the actuators. In order to be able to assess those systems, methodologies in terms of robust reliability have been introduced [2]. On those methodologies, failures of dynamical systems are defined firstly. In this situation, the dynamical systems are regarded as failure if outputs of the dynamical systems exceed the designated limit or the convex safe region [2]. Then, failure probabilities of the dynamical systems are derived from their covariance matrices. Finally, parameters of the dynamical systems are determined so as to minimize the failure probabilities.

Those failure probabilities can be applied as criteria for the designs of different dynamical systems. For instance, by comparing failure probabilities of the different dynamical systems, we can determine which dynamical systems are more reliable than others.

In this paper, the methodology for calculating unreliabilities (failure probabilities) of dynamical systems which are interconnected in parallel / series configuration, and have reconfigurable control consisted of multiple feedbacks is considered.

By interconnecting the dynamical systems, dependencies of covariance matrices between them arise. In addition, failure rates of subsystems with feedbacks for reconfiguration change in certain time periods because of the switching of feedback controls.

In order to cope with these situations, we propose the method to calculate unreliabilities of the dynamical systems which have stochastic inputs and are interconnected in parallel / series configuration by applying the methodology employed for probabilistic safety assessment. As for multiple feedback controls, those systems can be regarded as phased mission systems [3-4] due to the changes of the failure rates in certain time periods. Thus, phased mission systems approach is introduced to calculate unreliablity of overall system. In other words, combinations of subsystems which compose overall system for each time period are derived firstly and then

*Corresponding author: msatoshi@t03.mbox.media.kyoto-u.ac.jp

integrated to calculate unreliability of overall system for a specific time. Finally, an example of the dynamical system which has multiple feedback controls for reconfigurable control is presented. This example shows the validity and the details of the proposed method, by illustrating how to calculate unreliabilities of overall systems.

The rest of the paper is organized as follows. Section 2 describes the relationship between subsystems and covariance matrices. Section 3 presents how to derive failure probabilities of subsystems using their covariance matrices. In section 4, the method to calculate unreliabilities of overall systems from failure probabilities of subsystems is presented. Section 5 is an example. In section 6, conclusions are shown.

2. Subsystems and covariance matrices

We consider subsystem k :

$$\Sigma_k : \begin{cases} \dot{x}_k(t) = A_k x_k(t) + B_k u_k(t) + E_k w_k(t) \\ z_k(t) = C_k x_k(t) \end{cases} \tag{1}$$

where the state covariance matrix is,

$$E\{x_k x_k^T\} = P_k \tag{2}$$

the input covariance matrix is,

$$E\{u_k u_k^T\} = Q_k \tag{3}$$

and the intensity of the white noise is S_k.

Q_k and S_k are already given.

Then, the state covariance matrix P_k for stationary response is given by the solution of the Lyapunov equation,

$$A_k P_k + P_k A_k^T + E_k S_k E_k^T = 0 \tag{4}$$

This equation has the same form when the input covariance matrix Q_k does not exist.

The proof of this equation is shown in the appendix section.

The output covariance matrix $E\{z_k z_k^T\}$ is derived as:

$$E\{z_k z_k^T\} = E\{C_k x_k x_k^T C_k^T\} = C_k P_k C_k^T \tag{5}$$

The covariance matrix of the differentiation of z_k is derived as

$$\begin{aligned} &E\{\dot{z}_k \dot{z}_k^T\} \\ &= E\{C_k \dot{x}_k \dot{x}_k^T C_k^T\} \\ &= E\{C_k (A_k x_k + B_k u_k + E_k w_k)(A_k x_k + B_k u_k + E_k w_k)^T C_k^T\} \\ &= C_k A_k E\{x_k x_k^T\} A_k^T C_k^T + C_k B_k E\{u_k u_k^T\} B_k^T C_k^T \\ &= C_k A_k P_k A_k^T C_k^T + C_k B_k Q_k B_k^T C_k^T \end{aligned} \tag{6}$$

where $C_k E_k = 0$ is assumed.

If subsystem k has multiple feedback controls for reconfigurable control,

$$u_k(t) = -F_{k,j} x_k(t) \quad (j = 1, 2, \ldots) \tag{7}$$

is substituted for Eq. (1). The feedback controls change in accordance with the degrees of failures of actuators [1]. If the degrees of failures of actuators worsen, feedback control switches from status $j = 1$ to status $j = 2$. If the degrees of failures aggravate more, the number of the status increases accordingly.

In such cases, the equation

$$\Sigma_k : \begin{cases} \dot{x}_k(t) = (A_k - B_k F_{k,j}) x_k(t) + E_k w_k(t) \\ z_k(t) = C_k x_k(t) \end{cases} \tag{8}$$

can be yielded from Eq. (1) for subsystem k with reconfigurable control. Then, the Lyapunov equation for Eq. (8) becomes

$$(A_k - B_k F_{k,j}) P_k + P_k (A_k - B_k F_{k,j})^T + E_k S_k E_k^T = 0 \tag{9}$$

Also, the covariance matrix of the differentiation of z_k is derived as

$$E\{\dot{z}_k \dot{z}_k^T\} = C_k (A_k - B_k F_{k,j}) P_k (A_k - B_k F_{k,j})^T C_k^T \tag{10}$$

where $C_k E_k = 0$ is assumed.

3. Failure probabilities of subsystems

In this section, the failure probability of subsystem k is presented [2]. Subsystem k is regarded as failure if outputs z_k of subsystem k exceed the designated limit or the convex safe region in this situation.

The failure probability of subsystem k is approximately derived as

$$PF_k(D_{sk}, T) \approx 1 - exp\{-\nu_{zk}^{+}(S_{Dk}) T\} \tag{11}$$

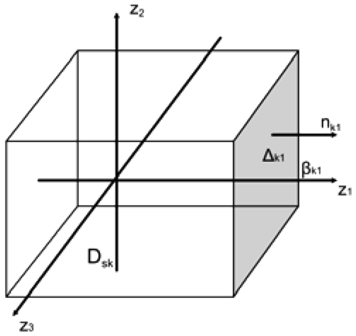

Fig. 1. Safe region for a three-dimensional space.

The failure rate or the mean out crossing rate of subsystem k is obtained as

$$v^+_{z_k}(S_{Dk}) = \sum_{i=1}^{n} v_{z_{ki}}(|\beta_{ki}|) \int_{\Delta_{ki}} p(w_{ki}|z_{ki} = \beta_{ki})dw_{ki} \quad (12)$$

where the notation $i(i=1,\ldots,n)$ denotes the element of the output z_k(n-dimension), D_{sk} denotes the hypercubic safe region for subsystem k, S_{Dk} denotes the boundary of D_{sk}, $(|z_{ki}|<)\ \beta_{ki}$ denotes the value of the safe region for the definition of the failure of the subsystem k. Fig. 1 shows an example for a three-dimensional space.

The following equations hold.

$$w_{ki} = z\text{-}z_{ki}n_{ki} \quad (13)$$

$$v_{z_{ki}}(|\beta_{ki}|) = \frac{\sigma_{\dot{z}_{ki}}}{\pi\sigma_{z_{ki}}} exp\left(-\frac{\beta_{ki}^2}{2\sigma_{z_{ki}}^2}\right) \quad (14)$$

$p(w_{ki}|z_{ki}=b_{ki})$ may be readily calculated since the distribution is gaussian [5]. n_{ki} is the unit outward normal vector at the boundary. The stationary variances $\sigma_{z_{ki}}$, $\sigma_{\dot{z}_{ki}}$ can be calculated from Eqs. (5) and (6).

4. Unreliabilities of overall systems

In this section, unreliabilities of overall systems are derived from failure probabilities of subsystems. Since the failure rates of subsystems which contain multiple feedback controls change every certain time periods (phases), structures of overall systems also change every phases. Thus the interconnected dynamical systems with multiple feedbacks for reconfigurable control can be regarded as a kind of phased mission systems [3-4]. Using phased mission systems approach, the way to derive unreliabilities of overall systems is presented.

4.1 State variables

In order to represent statuses of subsystems and to calculate unreliabilities of overall systems at specific time t, the following binary variable or state variable is introduced.

$$X_k(t) = \begin{cases} 1 \text{ if subsystem } k \text{ is in failure at time } t \\ 0 \text{ otherwise} \end{cases} \quad (15)$$

Then the following simplification rules hold.

$$X_k(t_1)X_k(t_2) = X_k(t_1) \quad for \quad t_1 \le t_2 \quad (16)$$

$$X_k(t_1) \vee X_k(t_2) = X_k(t_2)\ for \ \ t_1 \le t_2 \quad \text{‘}(17)$$

The term $\overline{X_k(t_1)}X_k(t_2)$ represents the occurrence of failure of subsystem k between time t_1 and time t_2. ($\overline{X_k(t_1)}$ is the negated term of $X_k(t_1)$.)

Thus, relationships between failure probabilities and state variables are established as follows.

$$\Pr\{\overline{X_k(t_1)}X_k(t_2) = 1\} = PF_k(D_{Sk}, t_2) - PF_k(D_{Sk}, t_1) \quad (18)$$

$$\Pr\{X_k(t_1) = 1\} = PF_k(D_{Sk}, t_1) \quad (19)$$

4.2 System states for phase *l*

The failure rates of subsystems which contain multiple feedback controls change every phases (i.e. when one feedback control switches to another feedback control).

So, structures of overall systems change among phases. To deal with this, structures of overall systems for each phase must be described in terms of state variables. We introduce system states $Y_l(t)$ in order to describe structures of overall systems for phase l (l=1,2,3,…). The number of the phase (l) changes when the feedback control is switched due to aggravation of the actuators.

$$Y_l(t) = \begin{cases} 1 \text{ if overall system is in failure at time } t \text{ in phase } l \\ 0 \text{ otherwise} \end{cases} \quad (20)$$

Interconnected systems considered are consisted of the combinations of series systems and parallel systems. Then, system states of parallel systems and series systems are derived as follows.

4.2.1 System states of parallel systems

We consider an interconnected system consisted of M subsystems in parallel configuration. Then, the system state of the overall system at time t during phase l becomes

$$Y_l(t) = \overset{M}{\underset{k=1}{\wedge}} X_k(t) \tag{21}$$

4.2.2 System states of series systems

We consider an interconnected system consisted of M subsystems in series configuration. Then, the system state of the overall system at time t during phase l becomes

$$Y_l(t) = \overset{M}{\underset{k=1}{\vee}} X_k(t) \tag{22}$$

4.3 Unreliabilities of overall systems at time *t*

Unreliabilities of overall systems at specific time t during phase n can be derived from combinations of system states at each phase. The failure of overall system which occurs at time t in phase n requires that {failure does not occur before the phase n} AND {failure firstly occurs at phase n}. Thus, failure occurrence conditions of overall systems at time t during phase n, denoted by binary variable $\phi_n(t)$, can be represented in terms of $Y_l(t)$, as :

$$\phi_n(t) = \left(\overset{n-1}{\underset{l=1}{\wedge}} \overline{Y_l(t_l^{\,E})} \right) Y_n(t) \tag{23}$$

where t_l^E denotes the end time of phase l.

Then, unreliabilities of overall systems at specific time t during phase n, $PF_{sys,n}(t)$ can be obtained from the failure occurrence conditions of overall system as :

$$PF_{sys,n}(t) = \Pr\{\phi_n(t) = 1\} = \mathrm{E}\left\{ \left(\overset{n-1}{\underset{l=1}{\wedge}} \overline{Y_l(t_l^{\,E})} \right) Y_n(t) \right\} \tag{24}$$

where E{} denotes the expectation operation.

5. Example

We consider an interconnected dynamical system consisted of three subsystems as shown in Fig. 2 for example. Subsystem 1 has multiple feedback controls $F_{1,j}$ for reconfigurable control. Those multiple feedback controls have three stages (j=1, 2, 3). Subsystem 1 operates with the feedback control j=1 at first. If the failures of the actuators of subsystem 1 worsen to a certain level, the feedback control of subsystem 1 switches from j=1 to j=2. If the failures of the actuators of subsystem 1 aggravate more, the feedback control of subsystem 1 switches from j=2 to j=3.

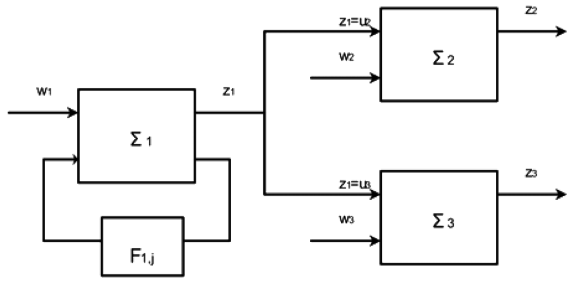

Fig. 2. Block diagram of interconnected system.

In this example, the time period when the feedback control j=1 operates is denoted as phase 1, the time period when the feedback control j=2 operates is denoted as phase 2 and the time period when the feedback control j=3 operates is denoted as phase 3.

Each subsystem is assigned as

$$\Sigma_k : \begin{cases} \dot{x}_k(t) = A_k x_k(t) + B_k u_k(t) + \begin{bmatrix} 1 \\ -2 \end{bmatrix} w_k(t) \\ z_k(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} x_k(t) \end{cases} \tag{25}$$

and

$$A_1 = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix},\quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\quad S_1 = 3,$$
$$A_2 = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix},\quad B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad S_2 = 5,$$
$$A_3 = \begin{bmatrix} -1 & -3 \\ 0 & -4 \end{bmatrix},\quad B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad S_3 = 1.$$

The definitions of the failures, or the safe regions of each subsystem are determined as β_{11}=20, β_{21}=30, β_{31}=20.

The durations of the phases are 10[s] for each. As for subsystem 1,

$$u_1(t) = -F_{1,j} x_1(t) \quad (j = 1,2,\ldots) \tag{26}$$

is substituted, and

$$F_{1,1} = \begin{bmatrix} 0 \\ -1.5 \end{bmatrix}^T,\quad F_{1,2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}^T,\quad F_{1,3} = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}^T$$

Then, system states $Y_l(t)$ for each phase can be derived as :

$$Y_1(t) = X_{1,1}(t) \vee X_2(t) X_3(t) \tag{27}$$

$$Y_2(t) = X_{1,2}(t)X_{1,2}(t_2^s) \vee X_2(t)X_3(t) \tag{28}$$

$$Y_3(t) = X_{1,3}(t)X_{1,3}(t_3^s) \vee X_2(t)X_3(t) \tag{29}$$

where $X_{1,j}(t)$ denotes the failure of subsystem 1 with the feedback control j, and t_l^S, t_l^E denote the start / end time of phase l.

Using state variable $Y_l(t)$, failure occurrence conditions of overall system at time t during phase 3 can be expressed as:

$$\begin{aligned}\phi_3(t) &= \overline{Y_1(t_1^{\ E})}\,\overline{Y_2(t_2^{\ E})}Y_3(t) \\ &= \overline{X_{1,1}(t_1^{\ E})} \wedge \overline{X_{1,2}(t_2^{\ E})X_{1,2}(t_2^{\ S})} \wedge X_{1,3}(t)X_{1,3}(t_3^{\ S}) \\ &\quad \vee X_2(t)X_3(t)\overline{X_2(t_2^{\ E})X_3(t_2^{\ E})}\end{aligned} \tag{30}$$

After all, unreliability of overall system at time t during phase 3, $PF_{sys,3}(t)$ is calculated as:

$$\begin{aligned}PF_{sys,3}(t) &= \mathrm{E}\{\phi_3(t)\} \\ &= \{1 - PF_{1,1}(t_1^{\ E})\}\{1 - PF_{1,2}(t_2^{\ E}) + PF_{1,2}(t_2^{\ S})\}\{PF_{1,3}(t) - PF_{1,3}(t_3^{\ S})\} \\ &\quad + \{PF_2(t) - PF_2(t_2^{\ E})\}\{PF_3(t) - PF_3(t_2^{\ E})\} \\ &\quad - \{1 - PF_{1,1}(t_1^{\ E})\}\{1 - PF_{1,2}(t_2^{\ E}) + PF_{1,2}(t_2^{\ S})\}\{PF_{1,3}(t) - PF_{1,3}(t_3^{\ S})\} \\ &\quad \times \{PF_2(t) - PF_2(t_2^{\ E})\}\{PF_3(t) - PF_3(t_2^{\ E})\}\end{aligned} \tag{31}$$

The relationship between time and unreliability is obtained as shown in Fig. 3 through procedures mentioned above.

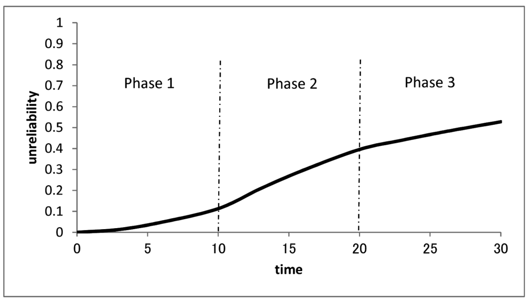

Fig. 3. Relationship between time and unreliability.

6. Conclusions

In this paper, we proposed the method to calculate unreliabilities of the dynamical systems which are interconnected in parallel / series configuration with reconfigurable control. By introducing stochastic inputs for each subsystem and the method employed for probabilistic safety assessment, it enables us to assess not only a single dynamical system which does not have any inputs, but interconnected dynamical systems which have stochastic inputs. Interconnected dynamical systems with reconfigurable control have multiple feedback controls which are switched in accordance with the degrees of the failures of the actuators. So, failure rates of subsystems which have multiple feedback controls change every certain time periods (phases). To assess unreliabilities of such systems, phased mission systems approach has been introduced. System states at each phase have been derived firstly and then unreliabilities for each phase have been obtained from system states and failure probabilities of subsystems. By introducing phased mission systems approach, it became possible to calculate unreliabilities of interconnected systems with reconfigurable control at specific time t. This approach will be applicable to systems whose overall structures change among phases to derive unreliabilities, and the analysis of such systems will be a next step for the research.

Appendix

Discrete system of Eq. (1) yields the following state covariance matrix,

$$\begin{aligned}&E[\{x_k(t_{i+1}) - \overline{x_k(t_{i+1})}\}\{x_k(t_{i+1}) - \overline{x_k(t_{i+1})}\}^T] \\ &= E[\{A_{kd}(x_k(t_i) - \overline{x_k(t_i)}) + B_{kd}(u_k(t_i) - \overline{u_k(t_i)}) \\ &+ E_{kd}(w_k(t_i) - \overline{w_k(t_i)})\} \cdot \{(x_k(t_i) - \overline{x_k(t_i)})^T A_{kd}^{\ T} + \\ &(u_k(t_i) - \overline{u_k(t_i)})^T B_{kd}^{\ T} + (w_k(t_i) - \overline{w_k(t_i)})^T E_{kd}^{\ T}\}] \\ &= A_{kd}E[\{x_k(t_i) - \overline{x_k(t_i)}\}\{x_k(t_i) - \overline{x_k(t_i)}\}^T]A_{kd}^{\ T} \\ &+ B_{kd}E[\{u_k(t_i) - \overline{u_k(t_i)}\}\{u_k(t_i) - \overline{u_k(t_i)}\}^T]B_{kd}^{\ T} \\ &+ E_{kd}E[\{w_k(t_i) - \overline{w_k(t_i)}\}\{w_k(t_i) - \overline{w_k(t_i)}\}^T]E_{kd}^{\ T}\end{aligned} \tag{32}$$

where $\overline{x_k(t_{t+1})}$ denotes the means of $x_k(t_{t+1})$ and

$$A_{kd} = I + A_k\Delta, \tag{33}$$

$$B_{kd} = B_k\Delta, \tag{34}$$

$$E_{kd} = E_k\Delta, \tag{35}$$

$\Delta = t_{i+1} - t_i$ are assumed [5]. (36)

Substituting Eqs. (33)-(36) into Eq. (32) yields

$$
\begin{aligned}
&E[\{x_k(t_{i+1})-\overline{x_k(t_{i+1})}\}\{x_k(t_{i+1})-\overline{x_k(t_{i+1})}\}^T] \\
&= A_k E[\{x_k(t_i)-\overline{x_k(t_i)}\}\{x_k(t_i)-\overline{x_k(t_i)}\}^T]\Delta \\
&+ E[\{x_k(t_i)-\overline{x_k(t_i)}\}\{x_k(t_i)-\overline{x_k(t_i)}\}^T] \\
&+ E[\{x_k(t_i)-\overline{x_k(t_i)}\}\{x_k(t_i)-\overline{x_k(t_i)}\}^T]A_k{}^T\Delta \\
&+ A_k E[\{x_k(t_i)-\overline{x_k(t_i)}\}\{x_k(t_i)-\overline{x_k(t_i)}\}^T]A_k{}^T\Delta^2 \\
&+ B_k E[\{u_k(t_i)-\overline{u_k(t_i)}\}\{u_k(t_i)-\overline{u_k(t_i)}\}^T]B_k{}^T\Delta^2 \\
&+ E_k \cdot E[\{w_k(t_i)-\overline{w_k(t_i)}\}\{w_k(t_i)-\overline{w_k(t_i)}\}^T]E_k{}^T\Delta^2
\end{aligned}
\tag{37}
$$

Let us define $P_k(t_i)$ and $S_k(t_i)$ and substitute them into Eq. (37).

$$P_k(t_i) = E[\{x_k(t_i)-\overline{x_k(t_i)}\}\{x_k(t_i)-\overline{x_k(t_i)}\}^T] \tag{38}$$

$$S_k(t_i) = E[\{w_k(t_i)-\overline{w_k(t_i)}\}\{w_k(t_i)-\overline{w_k(t_i)}\}^T] \tag{39}$$

Then the following equation holds.

$$
\begin{aligned}
&P_k(t_{i+1}) = A_k P_k(t_i)\Delta + P_k(t_i) + P_k(t_i)A_k{}'\Delta^2 \\
&+ A_k P_k(t_i)A_k{}' + B_k Q_k(t_i)B_k{}'\Delta^2 + E_k S_k(t_i)E_k{}'\Delta^2
\end{aligned}
\tag{40}
$$

The covariance matrix of the white noise for discrete system must be established as follows in order to represent equally as it is seen in continuous system [5].

$$S_k(t_i)\Delta = S_k(t) \tag{41}$$

Finally, Eq. (4) holds from Eqs. (40) and (41) when is applied and the steady state is considered.

References

[1] Kemin Zhou *et al.*, "Fault Diagnosis and Reconfigurable Control for Flight Control Systems with Actuator Failures," *43rd IEEE Conf. on Decision and Control*, pp. 5266-5271, 2004.

[2] Taflanidis AA and Beck JL." Analytical approximation for stationary reliability of certain and uncertain linear dynamic systems with higher dimensional output," *Earthquake Eng. Struct.Dynam.*,Vol. 35, pp. 1247-1267, 2006.

[3] T. Kohda, S. Matsumoto and M. Nakagawa, "Risk Analysis of Phased Mission Systems with Multiple Failure Modes," In *Proceedings of Annual Reliability and Maintainability Symposium*, IEEE, 2008.

[4] J. D. Esary and H. Ziehms, "Reliability analysis of phased missions," *Reliability and Fault Tree Analysis*, SIAM, 1975, pp. 213-236.

[5] Kanichiro Kato, "Introduction to Optimal Control" University of Tokyo Press, 1987.

[6] E. Duke, "Combining and Connecting Linear, Multi-Input, Multi-Output Subsystem Models," NASA, 1986.

[7] E. J. Henley and H. Kumamoto, "Probabilistic risk assessment : reliability engineering, design, and analysis," IEEE Press ,1991.

[8] Y. Ben-Haim, "Robust Reliability," Springer-Verlag, 1996

[9] H. Kwakernaak and R. Sivan, "Linear Optimal Control Systems," Wiley Interscience, 1972.

[10] A.E. Bryson, Jr and Y. C. Ho, "Applied Optimal Control," John Wiley and Sons, 1975.