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On geometric ergodicity and $\beta-$ mixing property of asymmetric power transformed threshold GARCH(1,1) process†

Oesook Lee[1]

[1]Department of Statistics, Ewha Womans University

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Abstract

We consider an asymmetric power transformed threshold GARCH(1.1) process and find sufficient conditions for the existence of a strictly stationary solution, geometric ergodicity and $\beta-$mixing property. Moments conditions are given. Box-Cox transformed threshold GARCH(1.1) process is also considered as a special case.

Keywords: Asymmetric power transformed threshold GARCH, $\beta-$mixing, geometric ergodicity, Markov chain, stationarity.

1. Introduction

Since the introduction by Engle (1982) of autoregressive conditional heteroscedastic (ARCH) models and their generalization by Bollerslev (1986), numerous GARCH-type models have been developed and successfully applied in various fields. Classical GARCH(1,1) process $\{\varepsilon_t\}$ is defined as

$$\varepsilon_t = \sqrt{h_t}e_t, \quad h_t - \beta h_{t-1} = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2, \tag{1.1}$$

where $\beta > 0, \alpha_0 > 0, \alpha_1 \geq 0$, and $\{e_t\}$ is a sequence of independent and identically distributed random variables with zero mean. e_t is independent of $\varepsilon_{t-1}, \varepsilon_{t-2}, \cdots$.

Classical GARCH process fails to explain the asymmetric phenomena, since in the model, the conditional variance is a function of only the magnitudes of the lagged residual but not their signs. Threshold GARCH model is a model that accounts for the asymmetric effects.

Li and Li (1996) introduce a class of threshold ARCH process where the asymmetry in conditional variances is represented via threshold:

$$h_t = \alpha_0 + \alpha_{11}(\varepsilon_{t-1}^{+2}) + \alpha_{12}(\varepsilon_{t-1}^{-2}),$$

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[1] Professor, Department of Statistics, Ewha Womans University, Seoul 120-750, Korea.
E-mail:oslee@ewha.ac.kr

where $\varepsilon_t^+ = max(0, \varepsilon_t)$, $\varepsilon_t^- = max(0, -\varepsilon_t)$, $\varepsilon_t^{+2} = (\varepsilon_t^+)^2$, and $\varepsilon_t^{-2} = (\varepsilon_t^-)^2$.

As a nonlinear asymmetric model, power transformed threshold models are suggested and studied by many authors (Ding *et al.*, 1993; Ling and McAleer, 2002; Hwang and Basawa, 2004; Liu, 2006; Lee, 2006, 2007a, 2007b).

He and Terasvirta (1999) suggest the general GARCH(1,1) model:

$$\varepsilon_t = h_t e_t, h_t^\gamma = g(e_{t-1}) + c(e_{t-1})h_{t-1}^\gamma. \tag{1.2}$$

Ling and McAleer (2002) show the existence of moments and a unique $\alpha\gamma$-order stationary solution of (1.2), i.e., there exists a unique strictly stationary solution h_t of (1.2) with $Eh_t^{\alpha\gamma} < \infty$ under some moments conditions on e_t.

Hwang and Basawa (2004) propose a Box-Cox transformed threshold GARCH(1,1) model for the time series $\{\varepsilon_t\}$ which is defined by

$$\varepsilon_t = \sqrt{h_t}e_t, \quad h_t^\gamma - \beta h_{t-1}^\gamma = \alpha_0 + \alpha_{11}(\varepsilon_{t-1}^{+2})^\gamma + \alpha_{12}(\varepsilon_{t-1}^{-2})^\gamma, \tag{1.3}$$

where $\gamma > 0,\ \beta \geq 0,\ \alpha_0 > 0,\ \alpha_{11} \geq 0,\ \alpha_{12} \geq 0$. They show that if $\beta + \alpha_{11}E[(e_t^{+2})^\gamma] + \alpha_{12}E[(e_t^{-2})^\gamma] < 1$, then the process h_t^γ has a unique strictly stationary solution $h_t^{\gamma *} = \alpha_0 + \alpha_0 \sum_{k=1}^\infty \Pi_{i=1}^k(\beta + \alpha_{11}(e_{t-i}^{+2})^\gamma + \alpha_{12}(e_{t-i}^{-2})^\gamma)$ whose infinite sum is finite almost surely. Lee (2007c) show the geometric ergodicity of the process of (1.3) under some additional assumption on e_t. Liu (2006) and Meitz (2006) prove that the process has a unique strictly stationary ergodic solution if and only if $E[\ln(\beta + \alpha_{11}(e_t^{+2})^\gamma + \alpha_{12}(e_t^{-2})^\gamma)] < 0$. Moments conditions and tail behavior are also considered.

Kim and Hwang (2005) examine a class of models possessing threshold asymmetric conditional variance to which distinct power transformation parameters are applied according to the sign of e_t. The model is given by

$$h_t^{\gamma_1} = \alpha_0 + \alpha_{11}(\varepsilon_{t-1}^2)^{\gamma_1}, \quad \text{if} \quad \varepsilon_{t-1} \geq 0 \tag{1.4}$$

$$h_t^{\gamma_2} = \alpha_0 + \alpha_{12}(\varepsilon_{t-1}^2)^{\gamma_2}, \quad \text{if} \quad \varepsilon_{t-1} < 0 \tag{1.5}$$

where $\alpha_0 > 0,\ \alpha_{11}, \alpha_{12} \geq 0,\ \gamma_1, \gamma_2 > 0$. Parameter estimations and comparative data analysis are studied and it is observed that for certain data, (1.4)-(1.5) is better than some other traditional models.

In this paper, we consider the asymmetric power transformed threshold GARCH(1,1) process defined by;

$$\varepsilon_t = \sqrt{h_t}e_t, \tag{1.6}$$

$$h_t^{\gamma_1} - \beta h_{t-1}^{\gamma_1} = \alpha_0 + \alpha_{11}(\varepsilon_{t-1}^2)^{\gamma_1}, \quad \text{if} \quad \varepsilon_{t-1} \geq 0 \tag{1.7}$$

$$h_t^{\gamma_2} - \beta h_{t-1}^{\gamma_2} = \alpha_0 + \alpha_{12}(\varepsilon_{t-1}^2)^{\gamma_2}, \quad \text{if} \quad \varepsilon_{t-1} < 0 \tag{1.8}$$

where $\alpha_0 > 0,\ \alpha_{11}, \alpha_{12}, \beta \geq 0,\ \gamma_1, \gamma_2 > 0$.

We aim to find sufficient conditions under which the given process is strictly stationary, geometrically ergodic and beta-mixing with exponential decay. Existence of moments is also examined.

We let $\{X_t : t \geq 0\}$ be a temporarily homogeneous Markov chain taking values in $(E, \mathcal{E})$, where E is a set and $\mathcal{E}$ is a countably generated $\sigma-$algebra of subsets of E, with transition probabilities given by $p^{(t)}(x,A) = P(X_t \in A|X_0 = x),\ x \in E,\ A \in \mathcal{E}$. In this paper $E = R^+$ and $\mathcal{E}$ is the $\sigma-$algebra of Borel sets.

The Markov chain $\{X_t\}$ is $\phi-$irreducible if, for some $\sigma-$finite measure ϕ on $(E,\mathcal{E})$, $\sum_t p^{(t)}(x,A) > 0$ for all $x \in E$, whenever $\phi(A) > 0$. A set $B \in \mathcal{E}$ is said to be small (with respect to ϕ) if $\phi(B) > 0$ and for every $A \in \mathcal{E}$ with $\phi(A) > 0$, there exists $j \geq 1$ such that $\inf_{x\in B} \sum_{t=1}^{j} p^{(t)}(x,A) > 0$.

$\{X_t\}$ is ergodic if there exists a probability measure π on $(E,\mathcal{E})$ such that $\lim_{t\to\infty} \|p^{(t)}(x,\cdot) - \pi(\cdot)\| = 0$ for all $x \in E$, where $\|\cdot\|$ denotes the total variation norm. If $\{X_t\}$ is ergodic and there exists a $\rho,\ 0 < \rho < 1$ such that $\lim_{t\to\infty} \rho^{-t}\|p^{(t)}(x,\cdot) - \pi(\cdot)\| = 0$ for all $x \in E$, then $\{X_t\}$ is said to be geometrically ergodic.

If $\{X_t\}$ is a Markov process with initial distribution as its invariant measure $\pi(dx)$, then $\{X_t\}$ is stationary $\beta-$mixing with exponential decay if there exist $0 < \rho < 1$ and $c > 0$ such that $\int \|p^{(t)}(x,\cdot) - \pi(\cdot)\|\pi(dx) \leq c\rho^t,\ \ \forall t \in N$.

To obtain our main result, we owe the following theorem to Tweedie (1983a, 1983b).

Theorem 1.1 Suppose that $\{X_t\}$ is a $\phi-$irreducible aperiodic Markov chain with one-step transition probability function $p(x,dy)$. If there exist, for some small set A, a nonnegative measurable function $g,\ \rho,\ 0 < \rho < 1$ and $\varepsilon > 0$ satisfying

$$\int p(x,dy)g(y) \leq \rho g(x) - \varepsilon, x \in A^c, \tag{1.9}$$

and

$$\sup_{x\in A} \int p(x,dy)g(y) < \infty, \tag{1.10}$$

then $\{X_t\}$ is geometric ergodic. If $\{X_t\}$ is initialized from an invariant initial distribution, say π, it is strictly stationary and $\beta-$mixing with exponential decay. Moreover, $E_\pi g(X_1) < \infty$.

Readers are referred to Meyn and Tweedie (1993) for additional definitions and properties in Markov chain context.

2. Main results

$\{h_t\}$ given in (1.6)-(1.8) can be rewritten as

$$\begin{aligned} h_t = &(\alpha_0 + (\alpha_{11}e_{t-1}^{2\gamma_1} + \beta)h_{t-1}^{\gamma_1})^{1/\gamma_1}\ I_{1t-1} \\ &+(\alpha_0 + (\alpha_{12}e_{t-1}^{2\gamma_2} + \beta)h_{t-1}^{\gamma_2})^{1/\gamma_2}\ I_{2t-1}, \end{aligned} \tag{2.1}$$

where $I_{1t} = I(e_t \geq 0),\ I_{2t} = 1 - I_{1t}$, and $I(A)$ is the indicator function of A.

$\{h_t\}$ given by (2.1) is a Markov chain with $t-$step transition probability function $p^{(t)}(x,A) = P(h_t \in A|h_0 = x)$ and $p^{(1)}(x,A) = p(x,A)$.

Throughout this paper, we assume that e_t has an absolutely continuous distribution whose probability density function is positive everywhere on R and $E|e_t|^2 < \infty$. For simplicity of notations, let $p = P(e_t \geq 0),\ q = P(e_t < 0),\ e_t^{+2\gamma} = (e_t^+)^{2\gamma},\ e_t^{-2\gamma} = (e_t^-)^{2\gamma}$.

Lemma 2.1 $\{h_t\}$ generated by (2.1) is $\mu-$irreducible with some $\sigma-$finite measure μ on R^+ if one of the following conditions holds:

(c1) $\gamma_1 > 1, \quad \gamma_2 > 1$, and
$\beta^{1/\gamma_1}p + \beta^{1/\gamma_2}q + \alpha_{11}^{1/\gamma_1}E(e_t^{+2}) + \alpha_{12}^{1/\gamma_2}E(e_t^{-2}) < 1;$

(c2) $\gamma_1 > 1, \ 0 < \gamma_2 \leq 1$, and
$\beta q + \beta^{\gamma_2/\gamma_1}p + \alpha_{11}^{\gamma_2/\gamma_1}E(e_t^{+2\gamma_2}) + \alpha_{12}E(e_t^{-2\gamma_2}) < 1;$

(c3) $0 < \gamma_1 \leq 1, \ \gamma_2 > 1$, and
$\beta p + \beta^{\gamma_1/\gamma_2}q + \alpha_{11}E(e_t^{+2\gamma_1}) + \alpha_{12}^{\gamma_1/\gamma_2}E(e_t^{-2\gamma_1}) < 1;$

(c4) $0 < \gamma_1 \leq 1, \ 0 < \gamma_2 \leq 1$, and
$\beta^{\gamma_1}q + \beta^{\gamma_2}p + \alpha_{11}^{\gamma_2}E(e_t^{+2\gamma_1\gamma_2}) + \alpha_{12}^{\gamma_1}E(e_t^{-2\gamma_1\gamma_2}) < 1.$

Lemma 2.2 Consider a Markov chain $\{h_t\}$ given by (2.1). If one of (c1)-(c4) holds, $\{h_t\}$ is aperiodic and $[c, d]$ with $0 \leq c < d < \infty$ and $\mu([c, d]) > 0$ is a small set. Here μ is a $\sigma-$finite measure defined in the proof of lemma 2.1.

We make the following assumptions:

(d1) $E(e_t^{2m}) < \infty, \gamma_1 > 1, \gamma_2 > 1$, and
$E(\beta^{1/\gamma_1}I_{1t} + \beta^{1/\gamma_2}I_{2t} + \alpha_{11}^{1/\gamma_1}e_t^{+2} + \alpha_{12}^{1/\gamma_2}e_t^{-2})^m < 1;$

(d2) $E(e_t^{2\gamma_2 m}) < \infty, \gamma_1 > 1, \ 0 < \gamma_2 \leq 1$, and
$E(\beta I_{2t} + \beta^{\gamma_2/\gamma_1}I_{1t} + \alpha_{11}^{\gamma_2/\gamma_1}e_t^{+2\gamma_2} + \alpha_{12}e_t^{-2\gamma_2})^m < 1;$

(d3) $E(e_t^{2\gamma_1 m}) < \infty, 0 < \gamma_1 \leq 1, \ \gamma_2 > 1$, and
$E(\beta I_{1t} + \beta^{\gamma_1/\gamma_2}I_{2t} + \alpha_{11}e_t^{+2\gamma_1} + \alpha_{12}^{\gamma_1/\gamma_2}e_t^{-2\gamma_1})^m < 1;$

(d4) $E(e_t^{2\gamma_1\gamma_2 m}) < \infty, 0 < \gamma_1 \leq 1, \ 0 < \gamma_2 \leq 1$, and
$E(\beta^{\gamma_1}I_{2t} + \beta^{\gamma_2}I_{1t} + \alpha_{11}^{\gamma_2}e_t^{+2\gamma_1\gamma_2} + \alpha_{12}^{\gamma_1}e_t^{-2\gamma_1\gamma_2})^m < 1.$

Theorem 2.1 If one of the conditions (d1)-(d4) holds for some integer $m \geq 1$, then $\{h_t\}$ given by (1.6)-(1.8) is geometrically ergodic and $\{h_t\}$ initialized from invariant probability π is strictly stationary and $\beta-$mixing with exponential decay. If one of (d1)-(d4) holds for some integer $m \geq 1$, then $E(h_t^m) < \infty, E(h_t^{\gamma_1 m}) < \infty, E(h_t^{\gamma_2 m}) < \infty$ or $E(h_t^{\gamma_1\gamma_2 m}) < \infty$, respectively.

Corollary 2.1 If $\gamma_1 = \gamma_2 = \gamma > 0$, (2.1) reduces to the Box-Cox transformed threshold GARCH(1,1) process (1.3) and if one of the following (2.2) and (2.3) holds for some positive integer $m \geq 1$

$$\gamma \geq 1, \quad E(\beta^{1/\gamma} + \alpha_{11}^{1/\gamma}e_t^{+2} + \alpha_{12}^{1/\gamma}e_t^{-2})^m < 1, \tag{2.2}$$

$$0 < \gamma < 1, \quad E(\beta^{\gamma} + \alpha_{11}^{\gamma}e_t^{+2\gamma} + \alpha_{12}^{\gamma}e_t^{-2\gamma})^m < 1, \tag{2.3}$$

then the conclusion of theorem 2.1 holds.

Remark 2.1 Consider the Box-Cox transformed threshold GARCH(1.1) process given by (1.3). It is proved that if

$$\beta + \alpha_{11}E(e_t^{+2\gamma}) + \alpha_{12}E(e_t^{-2\gamma}) < 1, \tag{2.4}$$

then $\{h_t^{\gamma}\}$ is geometric ergodic and $\beta-$mixing process. Proof can be found in Lee (2007c).

Remark 2.2 Note that (2.4) or one of (2.2) and (2.3) with $m = 1$ is not superior to each other.

3. Proofs

Proof of Lemma 2.1 : Recall that $(a+b)^\gamma \le a^\gamma + b^\gamma$ if $a>0, b>0$ and $0 \le \gamma \le 1$.

(c1) Suppose that $\gamma_1 > 1$ and $\gamma_2 > 1$. We may assume without loss of generality that $\alpha_{11} > 0$.

For any $x \in R^+$,

$$\begin{aligned} p(x,A) &= P(h_t \in A | h_{t-1} = x) \\ &= P(e_{t-1} \ge 0) P((\alpha_0 + (\alpha_{11} e_{t-1}^{+2\gamma_1} + \beta) x^{\gamma_1})^{1/\gamma_1} \in A) \\ &\quad + P(e_{t-1} < 0) P((\alpha_0 + (\alpha_{12} e_{t-1}^{-2\gamma_2} + \beta) x^{\gamma_2})^{1/\gamma_2} \in A). \end{aligned} \tag{3.1}$$

Define $\mu(A) = \lambda(A^{\gamma_1} \cap [\alpha^* + (\alpha^*(1-r)^{-1}+1)^{\gamma_1},\ \infty))$ where λ is a Lebesgue measure on R^+, $\alpha^* = \max\{\alpha_0, E(\alpha_{0t})\}$, $r = E(\beta_t + \eta_t) < 1$, $\alpha_{0t} = \alpha_0^{1/\gamma_1} I_{1t} + \alpha_0^{1/\gamma_2} I_{2t}$, $\beta_t = \beta^{1/\gamma_1} I_{1t} + \beta^{1/\gamma_2} I_{2t}$, $\eta_t = \alpha_{11}^{1/\gamma_1} e_t^{+2} + \alpha_{12}^{1/\gamma_2} e_t^{-2}$.

Let A be a Borel set with $\mu(A) > 0$ and let $a = \max\{\inf A^{\gamma_1}, \alpha^* + (\alpha^*(1-r)^{-1}+1)^{\gamma_1}\}$, where $A^{\gamma_1} = \{x^{\gamma_1} | x \in A\}$ and $\inf A^{\gamma_1} = \inf\{x^{\gamma_1} | x \in A\}$.

For any x with $0 < x^{\gamma_1} < a - \alpha_0$, the fact $x^{-\gamma_1}(a-\alpha_0) - \beta > 0$ yields that $\lambda(B) > 0$ where $B = x^{-\gamma_1}(A^{\gamma_1} - \alpha_0) - \beta$. Hence we have that

$$\begin{aligned} p(x,A) &\ge P(e_{t-1} \ge 0) P((\alpha_0 + (\alpha_{11} e_{t-1}^{+2\gamma_1} + \beta) x^{\gamma_1})^{1/\gamma_1} \in A) \\ &= p\ P(\alpha_{11} e_{t-1}^{+2\gamma_1} \in B) \\ &= p \int_B q(y) dy \\ &> 0, \end{aligned} \tag{3.2}$$

where $q(\cdot)$ is a probability density function of $\alpha_{11} e_t^{+2\gamma_1}$ which is positive on R^+.

Note that the following inequality holds:

$$h_t \le \alpha_{0t-1} + \sum_{k=1}^{t-1} \Pi_{i=1}^k (\beta_{t-i} + \eta_{t-i}) \alpha_{0,t-i-1} + \Pi_{i=1}^t (\beta_{t-i} + \eta_{t-i}) h_0. \tag{3.3}$$

From (3.3), we have that for any $x \in R^+$,

$$\begin{aligned} E[h_t | h_0 = x] &\le E(\alpha_{0t})(1 + r + r^2 + \cdots + r^{t-1}) + r^t x \\ &\le \frac{E(\alpha_{0t})}{1-r} + 1, \end{aligned} \tag{3.4}$$

for sufficiently large t.

Since $a \ge \alpha^* + (\alpha^*(1-r)^{-1}+1)^{\gamma_1}$, for any x satisfying $x^{\gamma_1} \ge a - \alpha_0$, we have that

$$\begin{aligned} P(h_{t_0}^{\gamma_1} \le a - \alpha^* | h_0 = x) &= P(h_{t_0} \le (a-\alpha^*)^{1/\gamma_1} | h_0 = x) \\ &\ge P(h_{t_0} \le \alpha^*(1-r)^{-1} + 1 | h_0 = x) \\ &\ge P(h_{t_0} \le \frac{E(\alpha_{0t})}{1-r} + 1 | h_0 = x) \\ &> 0 \end{aligned} \tag{3.5}$$

for some $t_0 = t_0(x) \ge 1$. The last inequality in (3.5) is obtained from (3.4).

Let $\{h_t(x) : t \geq 0\}$ denote $\{h_t\}$ in (2.1) if $h_0 = x,\ x \in R^+$.

Combining (3.2) and (3.5), we have that for any $x^{\gamma_1} \geq a - \alpha_0$,

$$\begin{aligned} p^{(t_0+1)}(x, A) &= P(h_{t_0+1} \in A | h_0 = x) \\ &\geq P(h_{t_0}^{\gamma_1}(x) \leq a - \alpha^*) P(h_{t_0+1}(x) \in A | h_{t_0}^{\gamma_1}(x) \leq a - \alpha^*) \\ &> 0. \end{aligned} \tag{3.6}$$

Thus, from (3.2) and (3.6), irreducibility of $\{h_t\}$ under the assumption (c1) is proved.

Suppose that (c2) holds. In this case, we define that $\mu(A) = \lambda(A^{\gamma_2} \cap [\alpha^* + \alpha^*(1-r)^{-1} + 1,\ \infty))$ where $\alpha^* = \max\{\alpha_0, E(\alpha_{0t})\},\ r = E(\beta_t + \eta_t) < 1,\ \alpha_{0t} = \alpha_0^{\gamma_2/\gamma_1} I_{1t} + \alpha_0 I_{2t},\ \beta_t = \beta^{\gamma_2/\gamma_1} I_{1t} + \beta I_{2t},\ \eta_t = \alpha_{11}^{\gamma_2/\gamma_1} e_t^{+2\gamma_2} + \alpha_{12} e_t^{-2\gamma_2}$. Take $a = \max\{\inf A^{\gamma_2}, \alpha^* + \alpha^*(1-r)^{-1} + 1\}$.

For the case (c3), let $\mu(A) = \lambda(A^{\gamma_1} \cap [\alpha^* + \alpha^*(1-r)^{-1} + 1,\ \infty))$ where $\alpha^* = \max\{\alpha_0, E(\alpha_{0t})\},\ r = E(\beta_t + \eta_t) < 1,\ \alpha_{0t} = \alpha_0^{\gamma_1/\gamma_2} I_{1t} + \alpha_0 I_{2t},\ \beta_t = \beta^{\gamma_1/\gamma_2} I_{1t} + \beta I_{2t},\ \eta_t = \alpha_{11}^{\gamma_1/\gamma_2} e_t^{+2\gamma_1} + \alpha_{12} e_t^{-2\gamma_2}$. Take $a = \max\{\inf A^{\gamma_1}, \alpha^* + \alpha^*(1-r)^{-1} + 1\}$.

Under the assumption (c4), we define $\mu(A) = \lambda(A^{\gamma_1\gamma_2} \cap [\alpha^* + \alpha^*(1-r)^{-1} + 1,\ \infty))$ where $\alpha^* = \max\{\alpha_0, E(\alpha_{0t})\},\ r = E(\beta_t + \eta_t) < 1,\ \alpha_{0t} = \alpha_0^{\gamma_2} I_{1t} + \alpha_0^{\gamma_1} I_{2t},\ \beta_t = \beta^{\gamma_2} I_{1t} + \beta^{\gamma_1} I_{2t},\ \eta_t = \alpha_{11}^{\gamma_2} e_t^{+2\gamma_1\gamma_2} + \alpha_{12}^{\gamma_1} e_t^{-2\gamma_1\gamma_2}$. Let $a = \max\{\inf A^{\gamma_1\gamma_2}, \alpha^* + (\alpha^*(1-r)^{-1} + 1)\}$.

Since the remaining parts of the proof of (c2)-(c4) are basically the same as those of the case (c1), details are omitted.

Proof of Lemma 2.2: We first consider the case (c1). Suppose that A is a Borel set with $\mu(A) > 0$ and let $\mu([c, d]) > 0$.

Let $d^{\gamma_1} < a - \alpha_0$. If $x \in [c, d]$, then $x^{\gamma_1} < d^{\gamma_1} < a - \alpha_0$ and

$$\inf_{x \in [c,d]} p(x, A) \geq \int_{B(d)} g(y) > 0, \tag{3.7}$$

where $B(d) = d^{-\gamma_1}(A^{\gamma_1} - \alpha_0) - \beta$. Note that $B(y) \subset B(x)$ if $x < y$.

Now assume that $a - \alpha_0 < d^{\gamma_1}$. By virtue of (3.5), there exists $t_0 = t_0(d)$ such that $P(h_{t_0}^{\gamma_1} \leq a - \alpha^* | h_0 = d) > 0$ and hence using(3.3)-(3.5), we obtain that for any $x < d$, $P(h_{t_0}^{\gamma_1} \leq a - \alpha^* | h_0 = x) > 0$.

Therefore we have that for any $x < d$,

$$\begin{aligned} p^{(t_0+1)}(x, A) &\geq P(h_{t_0}^{\gamma_1}(d) \leq a - \alpha^*) P(h_{t_0+1}(x) \in A | h_{t_0}^{\gamma_1}(d) \leq a - \alpha^*) \\ &> 0. \end{aligned} \tag{3.8}$$

Consequently from (3.7) and (3.8), for any A with $\mu(A) > 0$, we may choose t_0 such that

$$\inf_{x \in [c,d]} \sum_{t=1}^{t_0+1} p^{(t)}(x, A) > 0,$$

which implies that $[c, d]$ with $\mu([c, d]) > 0$ is a small set.

Moreover, if $t > t_0(d)$, then (3.5) holds for all $x \in [c, d]$, which implies that

$$P(h_t^{\gamma_1}(x) \in [c, d]) > 0 \ \text{ and } \ P(h_{t+1}^{\gamma_1}(x) \in [c, d]) > 0, \tag{3.9}$$

for every $x \in [c, d]$. Aperiodicity of $\{h_t\}$ follows from, together with (3.9), the fact that $[c, d]$ is a small set.

Proof of Theorem 2.1: Recall that $[E(X)]^m \leq E(X^m), m \geq 1$.

(d1) Suppose that (d1) holds for some integer $m \geq 1$. Define a test function $g : R^+ \to R^+$ by $g(x) = x^m + 1$. Then

$$\begin{aligned}
& E[g(h_t)|h_{t-1} = x] \\
& \leq 1 + E[(\alpha_{0t-1} + \xi_{t-1}x)^m] \\
& = 1 + E(\xi_{t-1}^m x^m + \sum_{i=0}^{m-1} \binom{m}{i} (\xi_{t-1}^i \alpha_{0t-1}^{m-i}) x^i \\
& \leq 1 + E(\xi_{t-1}^m) x^m + \sum_{i=0}^{m-1} \binom{m}{i} E[\xi_{t-1}^i \alpha_{0t-1}^{m-i}] (1+x)^{m-1} \\
& = (1 + x^m)(\frac{(E(\xi_{t-1}^m) - 1)x^m + K(1+x)^{m-1}}{1 + x^m}) \\
& \leq \rho(1 + x^m), \quad x \geq M
\end{aligned} \tag{3.10}$$

for some $\rho < 1$ and sufficiently large $M < \infty$, where $\alpha_{0t} = \alpha_0^{1/\gamma_1} I_{1t} + \alpha_0^{1/\gamma_2} I_{2t}$, $\xi_t = \beta_t^{1/\gamma_1} I_{1t} + \beta^{1/\gamma_2} I_{2t} + \alpha_{11}^{1/\gamma_1} e_t^{+2} + \alpha_{12}^{1/\gamma_2} e_t^{-2}$, and $K = \sum_{i=0}^{m-1} \binom{m}{i} E[\xi_{t-1}^i \alpha_{0t-1}^{m-i}] < \infty$.

Now let $\varepsilon > 0$ be fixed. Since $g(x)$ increases as x increases, (3.10) yields that there exist $\rho', 0 < \rho < \rho' < 1$, $B < \infty$ and $M < M' < \infty$ so that $\mu([0, M']) > \infty$,

$$E[g_i(h_t)|h_{t-1} = x] \leq \rho' g(x) - \varepsilon, \quad x > M' \tag{3.11}$$

and

$$E[g_i(h_t)|h_{t-1} = x] \leq B < \infty, \quad x \leq M'. \tag{3.12}$$

Applying Lemma 2.1, Lemma 2.2 and Theorem 2.1 together with (3.11) and (3.12), we can deduce the desired results.

For the case (d2), (d3) and (d4), we take $g(x) = x^{\gamma_2 m} + 1$, $g(x) = x^{\gamma_1 m} + 1$ and $g(x) = x^{\gamma_1 \gamma_2 m} + 1$, respectively. Then we obtain the results for each case by using the same method adopted for the proof of the case (d1).

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