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On Quasi-Conformally Recurrent Manifolds with Harmonic Quasi-Conformal Curvature Tensor

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Abstract. The main objective of the paper is to provide a full classification of quasi-conformally recurrent Riemannian manifolds with harmonic quasi-conformal curvature tensor. Among others it is shown that a quasi-conformally recurrent manifold with harmonic quasi-conformal curvature tensor is any one of the following:
(i) quasi-conformally symmetric, (ii) conformally flat, (iii) manifold of constant curvature, (iv) vanishing scalar curvature, (v) Ricci recurrent.

1. Introduction

Let M be an n-dimensional connected Riemannian manifold with Riemannian metric g and Levi-Civita connection ∇. Let R (resp. S , κ) be the Riemannian curvature tensor (resp. the Ricci tensor, the scalar curvature) of the manifold M.

Let T be a tensor field of type $(0,k), k \geq 1$, on M. The tensor field T is said to be recurrent [15], if the following condition holds on M

$$(1.1)\qquad (\nabla T)(X_1,...,X_k;X)T(Y_1,...,Y_k) = (\nabla T)(Y_1,...,Y_k;X)T(X_1,...,X_k),$$

where X, Y, X_1, Y_1, . . ., X_k, $Y_k \in TM$, the tangent bundle of M. From (1.1) it follows that at a point $p \in M$ if the tensor T is non-zero, then there exists a unique 1-form A defined on a neighborhood U of p, such that

$$\nabla T = A \otimes T, \quad A = d(log\,\|T\|),$$

holds on U, where $\|T\|$ denotes the norm of T, $\|T\|^2 = g(T,T)$. Hence a non-flat Riemannian manifold is recurrent [15] if its curvature tensor satisfies $\nabla R = A \otimes R$, A being a unique non-zero 1-form on M.

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Let $\{E_i : i = 1, ..., n\}$ be an orthonormal frame of the tangent bundle TM. Then the codifferential δT of T is defined by

$$\delta T(X_1, ..., X_{k-1}) = \sum_{i=1}^{k} (\nabla_{E_i} T)(E_i, X_1, ..., X_{k-1})$$

for any vector fields $X_1, X_2, ..., X_{k-1} \in TM$. If $\delta R = 0$, then M is said to have harmonic curvature tensor [6].

During the study of conformal transformation group on a Riemannian manifold, in 1968 Yano and Sawaki [16] defined a generalized curvature tensor, called quasi-conformal curvature tensor which includes both the conformal and concircular curvature tensor as special cases. The quasi-conformal curvature tensor W of type $(1, 3)$ on a Riemannian manifold M of dimension $n > 3$ [this condition is assumed throughout the paper as for $n = 3$, the conformal curvature tensor vanishes] is defined by

$$W^h_{ijk} = -(n-2)bC^h_{ijk} + [a + (n-2)b]\tilde{C}^h_{ijk}, \tag{1.2}$$

where a, b are arbitrary constants not simultaneously zero, C^h_{ijk} and $\tilde{C}^h_{ijk}$ are conformal and concircular curvature tensor of type $(1, 3)$ respectively.

Especially, if $a = 1$ and $b = -\frac{1}{n-2}$, then the quasi-conformal curvature tensor turns into conformal curvature tensor, whereas for $a = 1$ and $b = 0$ such a tensor turns into concircular curvature tensor.

A non-flat Riemannian manifold M is said to be quasi-conformally recurrent if it satisfies $\nabla W = A \otimes W$, where A is a unique non-zero 1-form. In particular, if $\nabla W = 0$, it is said to be quasi-conformally symmetric. Quasi-conformally recurrent manifold is studied by De and Shaikh [5]. The manifold M is said to have harmonic quasi-conformal curvature tensor if it satisfies $\delta W = 0$, that is

$$\sum_i W_{ijkli} = 0, \quad \text{where } W_{ijkl} = g_{hl} W^h_{ijk} \text{ and } W_{ijklp}$$

are the components of ∇W. Here all the indices run from 1 to n.

In [3] Chaki and Gupta, in [10] Miyazawa and also in [4] Derdziński and Roter studied conformally symmetric manifolds and they obtained full classification of such a manifold. Again in [1] Adati and Miyazawa studied conformally recurrent manifolds. Recently in [14] Suh et. al. gave the full classification of conformally recurrent manifolds with harmonic conformal curvature tensor. Generalizing the results of [14], in the present paper a full classification of quasi-conformally recurrent manifolds with harmonic quasi-conformal curvature tensor is given. The paper is organized as follows:

Section 2 deals with preliminaries where we have discussed some notations and properties of quasi-conformally flat manifolds along with necessary and sufficient conditions for a Riemannian manifold to be of harmonic quasi-conformal curvature

tensor (see, Proposition 2.2, 2.3). The last section is devoted to the study of quasi-conformally recurrent manifolds with harmonic quasi-conformal curvature tensor and it is shown that such a manifold is any one of the following:
(i) quasi-conformally symmetric, (ii) quasi-conformally flat, (iii) Ricci recurrent.
Again, it is proved that a quasi-conformally recurrent manifold with harmonic quasi-conformal curvature tensor and non-vanishing scalar curvature is either locally symmetric or quasi-conformally flat. Finally, it is shown that a quasi-conformally recurrent manifold with harmonic quasi-conformal curvature tensor is either quasi-conformally flat or recurrent. As the particular case of our results we can obtain the results of [14], where conformally recurrent manifold with harmonic conformal curvature tensor is classified.

2. Preliminaries

Let R_{ijkl}, S_{ij} denote respectively the components of the curvature tensor R of type $(0,4)$ and the Ricci tensor S of type $(0,2)$ where $S_{ij} = \sum_k R_{kijk}$. The scalar curvature of the manifold is given by $\kappa = \sum_i S_{ii}$. The conformal curvature tensor C of type $(0,4)$ with components C_{ijkl} and the concircular curvature tensor $\tilde{C}$ of type $(0,4)$ with components $\tilde{C}_{ijkl}$ are respectively given by

$$(2.1)\qquad C_{ijkl} = R_{ijkl} - \frac{1}{n-2}[S_{jk}g_{il} - S_{ik}g_{jl} + S_{il}g_{jk} - S_{jl}g_{ik}] + \frac{\kappa}{(n-1)(n-2)}[g_{jk}g_{il} - g_{ik}g_{jl}]$$

and

$$(2.2)\qquad \tilde{C}_{ijkl} = R_{ijkl} - \frac{\kappa}{n(n-1)}[g_{jk}g_{il} - g_{ik}g_{jl}].$$

In view of (2.1) and (2.2), (1.2) takes the form

$$(2.3)\qquad \begin{aligned} W_{ijkl} &= aR_{ijkl} + b\{(g_{jk}S_{il} - g_{ik}S_{jl}) + (g_{il}S_{jk} - g_{jl}S_{ik})\} \\ &\quad -\frac{\kappa}{n}\left(\frac{a}{n-1} + 2b\right)(g_{il}g_{jk} - g_{jl}g_{ik}), \end{aligned}$$

where W_{ijkl} denotes the components of the quasi-conformal curvature tensor W of type $(0,4)$. If M is Einstein ($S_{ij} = \frac{\kappa}{n}g_{ij}$) with $a = 1$, then

$$W_{ijkl} = \tilde{C}_{ijkl}.$$

Let θ_i, θ_{ij} and Θ_{ij} be the canonical form, the connection form and the curvature form on M respectively, with respect to the local orthonormal frame field

$\{E_i : i = 1, ..., n\}$. Then we have the structure equations

$$d\theta_i + \sum_j \theta_{ij} \wedge \theta_j = 0, \qquad \theta_{ij} + \theta_{ji} = 0,$$
$$d\theta_{ij} + \sum_k \theta_{ik} + \theta_{kj} = \Theta_{ij},$$
$$\Theta_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \theta_k \wedge \theta_l.$$

Let $D^k M$ be the vector bundle consisting of smooth k-forms and $DM = \sum_{k=0}^{n} D^k M$, where $D^0 M$ is the algebra of smooth functions on M. For any tensor field K in $D^k M$ the components K_{ijklh} of the covariant derivative ∇K of K of type $(0, 4)$ are defined by

$$\sum_h K_{ijlmh}\theta_h = dK_{ijlm} - \sum_h (K_{hjlm}\theta_{hi} + K_{ihlm}\theta_{hj} + K_{ijhm}\theta_{hl} + K_{ijlh}\theta_{hm}).$$

The symmetric tensor K of type $(0, 2)$ with components K_{ij} is called the *Weyl tensor* [14], if it satisfies

$$K_{ijl} - K_{ilj} = \frac{1}{2(n-1)}[\kappa_l g_{ij} - \kappa_j g_{il}],$$

where $\kappa = TrK$, K_{ijl} and κ_j denote respectively the components of the covariant derivative ∇K and $\nabla\kappa$. Again, if K_{ij} is the Ricci tensor of type $(0, 2)$ then the manifold is called nearly conformally symmetric introduced by Roter [11]. The manifold M is said to be quasi-conformally flat if its quasi-conformal curvature tensor vanishes identically. In [2] Amur and Maralabhavi studied quasi-conformally flat Riemannian manifold. In fact, they obtained the following:

Theorem A. *A quasi-conformally flat Riemannian manifold M is any one of the following:*
(i) *conformally flat,*
(ii) *manifold of constant curvature,*
(iii) *manifold of vanishing scalar curvature.*

From (2.3), it follows that

$$(2.4) \qquad \sum_r W_{rjkmr} = (a+b)(S_{jkm} - S_{jmk}) + \left\{ \frac{(n-1)(n-4)b - 2a}{2n(n-1)} \right\} (\kappa_m g_{jk} - \kappa_k g_{jm}).$$

If M has harmonic quasi-conformal curvature tensor, then (2.4) yeilds

$$(2.5) \qquad (S_{jkm} - S_{jmk}) = \left\{ \frac{2a - (n-1)(n-4)b}{2n(n-1)(a+b)} \right\} (\kappa_m g_{jk} - \kappa_k g_{jm}), \quad \text{provided that } a + b \neq 0.$$

From (2.5), it follows that the Ricci tensor is the Weyl tensor if

$$\frac{2a-(n-1)(n-4)b}{2n(n-1)(a+b)} = \frac{1}{2(n-1)},$$

i.e., if $\quad a+(n-2)b=0.$

Hence we can state the following:

Proposition 2.1. *If a Riemannian manifold M has harmonic quasi-conformal curvature tensor, then the Ricci tensor is the Weyl tensor provided that $a+b\neq 0$ and $a+(n-2)b=0$.*

Again contracting (2.5) over m and k, we get $\kappa_j = 0$ provided that $a+(n-2)b \neq 0$, which implies that the manifold is of constant scalar curvature. Again a Riemannian manifold is said to be of Codazzi type ([7], [12]) Ricci tensor if its Ricci tensor satisfies $S_{jkm} = S_{jmk}$ for all j,k,m. It is then obvious that the scalar curvature is constant. Hence if the Ricci tensor is of Codazzi type, then from (2.4) it follows that the quasi-conformal curvature tensor is harmonic. Hence we can state the following:

Proposition 2.2. *A Riemannian manifold M with $a+b\neq 0$ and $a+(n-2)b\neq 0$ has harmonic quasi-conformal curvature tensor if and only if its Ricci tensor is of Codazzi type.*

Again a Riemannian manifold has harmonic curvature tensor if and only if its Ricci tensor is of Codazzi type. Hence by virtue of Proposition 2.2, we can state the following:

Proposition 2.3. *A Riemannian manifold M with $a+b\neq 0$ and $a+(n-2)b\neq 0$ has harmonic quasi-conformal curvature tensor if and only if its curvature tensor is harmonic.*

We note that in [13] Shaikh and Binh studied Riemannian manifolds with Codazzi type Ricci tensor and proved the existence of such a class by several non-trivial examples.

The above results will be used in the sequel.

3. Quasi-conformally recurrent manifolds with harmonic quasi-conformal curvature tensor

Let M be an $n(>3)$-dimensional Riemannian manifold with Riemannian metric g and Riemannian connection ∇. To obtain our main theorem we require some lemmas which will be derived at first.

Lemma 3.1. *If a Riemannian manifold M has harmonic quasi-conformal curvature tensor with $a+b\neq 0$, then the following relation holds*

$$\sum_r (R_{rikm}S_{rj} + R_{rimj}S_{rk} + R_{rijk}S_{rm}) = 0. \tag{3.1}$$

Proof. To prove this Lemma we need the following identity due to Lovelock ([9], page 289):

$$\sum_r (R_{rjkmri} + R_{rkimrj} + R_{rijmrk}) = \sum_r (S_{ir}R_{jkmr} + S_{jr}R_{kimr} + S_{kr}R_{ijmr}), \tag{3.2}$$

where $R_{ijkm} = g_{hm}R^h_{ijk}$, R_{ijkml} and R_{ijkmln} are respectively the components of ∇R and $\nabla\nabla R$. By performing the covariant derivative of (2.4), we obtain

$$\sum_r W_{rjkmri} = (a+b)(S_{jkmi} - S_{jmki}) + \left\{ \frac{(n-1)(n-4)b - 2a}{2n(n-1)} \right\} (\kappa_{mi}g_{jk} - \kappa_{ki}g_{jm}).$$

Summing over a cyclic permutations of the indices i, j, k and then using (3.2), we obtain

$$\begin{aligned} &\sum_r (W_{rjkmri} + W_{rkimrj} + W_{rijmrk}) \\ = \quad &(a+b)\sum_r (S_{ir}R_{jkmr} + S_{jr}R_{kimr} + S_{kr}R_{ijmr}). \end{aligned} \tag{3.3}$$

From (3.3) it follows that if M has harmonic quasi-conformal curvature tensor and $a + b \neq 0$, then we obtain (3.1). □

Lemma 3.2. *If a Riemannian manifold M is quasi-conformally recurrent with harmonic quasi-conformal curvature tensor such that $a + b \neq 0$, then the following relations hold:*

$$\sum_r (W_{rikm}S_{rj} + W_{rimj}S_{rk} + W_{rijk}S_{rm}) = 0, \tag{3.4}$$

$$\sum_r (W_{rikm}S_{rjh} + W_{rimj}S_{rkh} + W_{rijk}S_{rmh}) = 0. \tag{3.5}$$

Proof. In view of (2.3) and (3.1), we can easily obtain (3.4). Differentiating (3.4) covariantly we obtain

$$\sum_r (W_{rikmh}S_{rj} + W_{rimjh}S_{rk} + W_{rijkh}S_{rm} + W_{rikm}S_{rjh} + W_{rimj}S_{rkh} + W_{rijk}S_{rmh}) = 0.$$

Since the manifold under consideration is quasi-conformally recurrent, by virtue of (3.4) the last relation turns into (3.5). □

We now state and prove our main theorem.

Theorem 3.1. *If a Riemannian manifold M is quasi-conformally recurrent with harmonic quasi-conformal curvature tensor such that $a+b \neq 0$ and $a+(n-2)b \neq 0$, then the manifold is any one of the following:*

(i) *quasi-conformally symmetric,*
(ii) *quasi-conformally flat,*
(iii) *Ricci recurrent.*

Proof. Let W_{ijkmhp} be the components of the covariant derivative $\nabla^2 W$ of ∇W. Then we have

$$W_{ijkmhp} = (A_h A_p + A_{hp}) W_{ijkm}. \tag{3.6}$$

Now we define by f the scalar product of W, namely, we put $f = g(W, W)$. Let $M' = \{x \in M : f(x) = 0\}$. Then $M' \subset M$. On the open subset $M - M'$, we have

$$\nabla f = 2g(\nabla W, W) = 2Af,$$

and hence $A = \nabla f / 2f$, from which it follows that

$$2A = \nabla \log |f|.$$

This implies that

$$A_{ij} = A_{ji} \text{ on } M - M'. \tag{3.7}$$

So, on $M - M'$, by (3.6) and (3.7) we have $W_{ijkmhp} = W_{ijkmph}$. Therefore by the Ricci identity we get

$$\sum_r (R_{phir} W_{rjkm} + R_{phjr} W_{irkm} + R_{phkr} W_{ijrm} + R_{phmr} W_{ijkr}) = 0. \tag{3.8}$$

Differentiating above covariantly and taking account of $W_{ijkmh} = A_h W_{ijkm}$, we get

$$\begin{aligned} &\sum_r [(R_{phirq} W_{rjkm} + R_{phjrq} W_{irkm} + R_{phkrq} W_{ijrm} + R_{phmrq} W_{ijkr}) \\ &+ A_q (R_{phir} W_{rjkm} + R_{phjr} W_{irkm} + R_{phkr} W_{ijrm} + R_{phmr} W_{ijkr})] = 0. \end{aligned}$$

Hence by virtue of (3.8) we get

$$\sum_r (R_{phirq} W_{rjkm} + R_{phjrq} W_{irkm} + R_{phkrq} W_{ijrm} + R_{phmrq} W_{ijkr}) = 0. \tag{3.9}$$

On the other hand, by (2.3) and $A_h W_{ijkl} = W_{ijklh}$, we have

$$\begin{aligned} &A_h [a R_{ijkm} + b \{(g_{jk} S_{im} - g_{ik} S_{jm}) + (g_{im} S_{jk} - g_{jm} S_{ik})\} \\ &- \frac{\kappa}{h} \left(\frac{a}{n-1} + 2b \right) (g_{im} g_{jk} - g_{jm} g_{ik})] \\ &= a R_{ijkmh} + b \{(g_{jk} S_{imh} - g_{ik} S_{jmh}) + (g_{im} S_{jkh} - g_{jm} S_{ikh})\} \\ &- \frac{\kappa_h}{n} \left(\frac{a}{n-1} + 2b \right) (g_{im} g_{jk} - g_{jm} g_{ik}), \end{aligned}$$

and hence we get

$$
\begin{aligned}
(3.10)\quad aR_{ijkmh} &= A_h a R_{ijkm} \\
&+b\left\{A_h(g_{jk}S_{im}-g_{ik}S_{jm})-(g_{jk}S_{imh}-g_{ik}S_{jmh})\right\} \\
&+b\left\{A_h(g_{im}S_{jk}-g_{jm}S_{ik})-(g_{im}S_{jkh}-g_{jm}S_{ikh})\right\} \\
&-\frac{\kappa}{n}\left(\frac{a}{n-1}+2b\right)A_h(g_{im}g_{jk}-g_{jm}g_{ik}), \text{ since } \kappa_h=0.
\end{aligned}
$$

From (3.9) and (3.10), it follows that

$$
\begin{aligned}
&\sum_r [A_q(R_{phir}W_{rjkm}+R_{phjr}W_{irkm}+R_{phkr}W_{ijrm}+R_{phmr}W_{ijkr})] \\
&+\frac{b}{a}\sum_r [\{A_q(g_{hi}S_{pr}-g_{pi}S_{hr})-(S_{prq}g_{hi}-S_{hrq}g_{pi})\}W_{rjkm} \\
&+\{A_q(g_{pr}S_{hi}-g_{hr}S_{pi})-(S_{hiq}g_{pr}-S_{piq}g_{hr})\}W_{rjkm} \\
&+\{A_q(g_{hj}S_{pr}-g_{pj}S_{hr})-(S_{prq}g_{hj}-S_{hrq}g_{pj})\}W_{irkm} \\
&+\{A_q(g_{pr}S_{hj}-g_{hr}S_{pj})-(S_{hjq}g_{pr}-S_{pjq}g_{hr})\}W_{irkm} \\
&+\{A_q(g_{hk}S_{pr}-g_{pk}S_{hr})-(S_{prq}g_{hk}-S_{hrq}g_{pk})\}W_{ijrm} \\
&+\{A_q(g_{pr}S_{hk}-g_{hr}S_{pk})-(S_{hkq}g_{pr}-S_{pkq}g_{hr})\}W_{ijrm} \\
&+\{A_q(g_{hm}S_{pr}-g_{pm}S_{hr})-(S_{prq}g_{hm}-S_{hrq}g_{pm})\}W_{ijkr} \\
&+\{A_q(g_{pr}S_{hm}-g_{hr}S_{pm})-(S_{hmq}g_{pr}-S_{pmq}g_{hr})\}W_{ijkr}] \\
&-\frac{\kappa}{n}\left(\frac{1}{n-1}+\frac{2b}{a}\right)A_q[(g_{hi}W_{pjkm}+g_{hj}W_{ipkm}+g_{hk}W_{ijpm}+g_{hm}W_{ijkp}) \\
&-(g_{pi}W_{hjkm}+g_{pj}W_{ihkm}+g_{pk}W_{ijhm}+g_{pm}W_{ijkh})=0, \qquad a\neq 0.
\end{aligned}
$$

Now using (3.8), we obtain

$$
\begin{aligned}
&(S_{hiq}W_{pjkm}+S_{hjq}W_{ipkm}+S_{hkq}W_{ijpm}+S_{hmq}W_{ijkp}) \\
&-(S_{piq}W_{hjkm}+S_{pjq}W_{ihkm}+S_{pkq}W_{ijhm}+S_{pmq}W_{ijkh}) \\
&-\sum_r (g_{pi}W_{rjkm}+g_{pj}W_{irkm}+g_{pk}W_{ijrm}+g_{pm}W_{ijkr})S_{hrq} \\
&+\sum_r (g_{hi}W_{rjkm}+g_{hj}W_{irkm}+g_{hk}W_{ijrm}+g_{hm}W_{ijkr})S_{prq} \\
&+A_q(S_{pi}W_{hjkm}+S_{pj}W_{ihkm}+S_{pk}W_{ijhm}+S_{pm}W_{ijkh}) \\
&-A_q(S_{hi}W_{pjkm}+S_{hj}W_{ipkm}+S_{hk}W_{ijpm}+S_{hm}W_{ijkp}) \\
&+A_q\sum_r (g_{pi}W_{rjkm}+g_{pj}W_{irkm}+g_{pk}W_{ijrm}+g_{pm}W_{ijkr})S_{hr} \\
&-A_q\sum_r (g_{hi}W_{rjkm}+g_{hj}W_{irkm}+g_{hk}W_{ijrm}+g_{hm}W_{ijkr})S_{pr} \\
&+\frac{\kappa}{n}\left(\frac{a}{(n-1)b}+2\right)A_q[(g_{hi}W_{pjkm}+g_{hj}W_{ipkm}+g_{hk}W_{ijpm}+g_{hm}W_{ijkp})
\end{aligned}
$$

$$-(g_{pi}W_{hjkm}+g_{pj}W_{ihkm}+g_{pk}W_{ijhm}+g_{pm}W_{ijkh})]=0, \qquad a,b\neq 0.$$

Accordingly, we have

$$\begin{aligned}(3.11)\quad & [(S_{hiq}-A_qS_{hi})W_{pjkm}+(S_{hjq}-A_qS_{hj})W_{ipkm}\\ &+(S_{hkq}-A_qS_{hk})W_{ijpm}+(S_{hmq}-A_qS_{hm})W_{ijkp}]\\ &-[(S_{piq}-A_qS_{pi})W_{hjkm}+(S_{pjq}-A_qS_{pj})W_{ihkm}\\ &+(S_{pkq}-A_qS_{pk})W_{ijhm}+(S_{pmq}-A_qS_{pm})W_{ijkh}]\\ &-\sum_r(g_{pi}W_{rjkm}+g_{pj}W_{irkm}+g_{pk}W_{ijrm}+g_{pm}W_{ijkr})(S_{hrq}-A_qS_{hr})\\ &+\sum_r(g_{hi}W_{rjkm}+g_{hj}W_{irkm}+g_{hk}W_{ijrm}+g_{hm}W_{ijkr})(S_{prq}-A_qS_{pr})\\ &+\frac{\kappa}{n}\left(\frac{a}{(n-1)b}+2\right)A_q[(g_{hi}W_{pjkm}+g_{hj}W_{ipkm}+g_{hk}W_{ijpm}+g_{hm}W_{ijkp})\\ &-(g_{pi}W_{hjkm}+g_{pj}W_{ihkm}+g_{pk}W_{ijhm}+g_{pm}W_{ijkh})]=0, \qquad a,b\neq 0.\end{aligned}$$

The equation (3.11) can be written as

$$\begin{aligned}(3.12)\quad & [(S_{hiq}-A_qS_{hi})W_{pjkm}+(S_{hjq}-A_qS_{hj})W_{ipkm}\\ &+(S_{hkq}-A_qS_{hk})W_{ijpm}+(S_{hmq}-A_qS_{hm})W_{ijkp}]\\ &-[(S_{piq}-A_qS_{pi})W_{hjkm}+(S_{pjq}-A_qS_{pj})W_{ihkm}\\ &+(S_{pkq}-A_qS_{pk})W_{ijhm}+(S_{pmq}-A_qS_{pm})W_{ijkh}]\\ &-\sum_s(g_{pi}W_{sjkm}+g_{pj}W_{iskm}+g_{pk}W_{ijsm}+g_{pm}W_{ijks})(S_{hsq}-A_qS_{hs})\\ &+\sum_s(g_{hi}W_{sjkm}+g_{hj}W_{iskm}+g_{hk}W_{ijsm}+g_{hm}W_{ijks})(S_{psq}-A_qS_{ps})\\ &+\frac{\kappa}{n}\left(\frac{a}{(n-1)b}+2\right)A_q[(g_{hi}W_{pjkm}+g_{hj}W_{ipkm}+g_{hk}W_{ijpm}+g_{hm}W_{ijkp})\\ &-(g_{pi}W_{hjkm}+g_{pj}W_{ihkm}+g_{pk}W_{ijhm}+g_{pm}W_{ijkh})]=0, \qquad a,b\neq 0.\end{aligned}$$

If we transvect (3.12) with g^{hp}, then it can be easily shown that the term multiplying by $\frac{\kappa}{n}\left(\frac{a}{(n-1)b}+2\right)A_q$ will vanish. Then setting $i=p=r$ in the resulting equation and then taking summation with respect to r we obtain by virtue of (3.4) and (3.5) that

$$\begin{aligned}(3.13)\quad & \sum_r[(n-2)(S_{hrq}-A_qS_{hr})W_{rjkm}+(S_{jrq}-A_qS_{jr})W_{rhkm}\\ &+(S_{krq}-A_qS_{kr})W_{rjhm}+(S_{mrq}-A_qS_{mr})W_{rjkh}\\ &+\sum_s\{g_{hk}(S_{rsq}-A_qS_{rs})W_{rjms}-g_{hm}(S_{rsq}-A_qS_{rs})W_{rjks}\}]=0.\end{aligned}$$

Suppose that M is quasi-conformally recurrent and M has harmonic quasi-conformal curvature tensor. Then it satisfies $\sum_r W_{rjkmr} = 0$. Hence we have

$$\sum_r A_r W_{rjkm} = 0. \tag{3.14}$$

Putting $m = q$ in (3.13), summing up with respect to m and taking account of (3.2) and (3.14), we get

$$\begin{aligned}\sum_{r,s} \{(n-2)S_{hrs}W_{rjks} + S_{jrs}W_{rhks} + S_{krs}W_{rjhs} - A_s S_{rs} W_{rjkh}\} \\ + \sum_{r,s,t} g_{hk} S_{rst} W_{rjts} - \sum_{r,s} (S_{rsh} - A_h S_{rs}) W_{rjks} = 0.\end{aligned} \tag{3.15}$$

Now

$$\sum_{r,s,t} g_{hk} S_{rst} W_{rjts} = \frac{1}{2} \sum_{r,s,t} g_{hk} (S_{rst} - S_{rts}) W_{rjts} = 0, \quad \text{by virtue of (2.5).}$$

From (2.5), it follows that

$$\sum_{r,s} (S_{rhs} - S_{rsh}) W_{rjks} = 0,$$

and hence (3.15) reduces to

$$\begin{aligned}\sum_{r,s} \{(n-3)S_{hrs}W_{rjks} + S_{jrs}W_{rhks} + S_{krs}W_{rjhs} - A_s S_{rs} W_{rjkh}\} \\ + \sum_{r,s} A_h S_{rs} W_{rjks} = 0.\end{aligned} \tag{3.16}$$

Since M is a quasi-conformally recurrent with harmonic quasi-conformal curvature tensor, the relation (3.5) holds. Replacing m and h by s in (3.5), we get

$$\sum_{r,s} (S_{jrs} W_{riks} - S_{krs} W_{rijs}) = 0. \tag{3.17}$$

From (3.16) and (3.17), it follows that

$$\sum_{r,s} \{(n-1)S_{hrs}W_{rjks} - A_s S_{rs} W_{rjkh}\} + \sum_{r,s} A_h S_{rs} W_{rjks} = 0.$$

The last relation together with the definition of quasi-conformally recurrent, yields

$$(n-1) \sum_{r,s} S_{hrs} W_{rjks} + \sum_{r,s} S_{rs} (W_{rjkhs} - W_{rjksh}) = 0.$$

Here we note that the indices j and k in the first and third terms are symmetric with each other, because S_{hrs} and S_{rs} are symmetric with respect to the indices r and s. Hence if we take the skew-symmetric part to the above equation, then it follows that

$$\begin{aligned} 0 &= \sum_{r,s} S_{rs}(W_{rjkhs} - W_{rkjhs}) \\ &= \sum_{r,s} S_{rs}(W_{rjkhs} + W_{rkhjs}) \\ &= -2\sum_{r,s} S_{rs}W_{rhjks}. \end{aligned}$$

Consequently,

$$\sum_{r,s} S_{hrs}W_{rjks} = \sum_{r,s} S_{rsh}W_{rjks} = 0. \tag{3.18}$$

Transvecting (3.13) to $A_mA_hA_q$, summing up with respect to m, h and q and then taking account of (3.14) and (3.18), we obtain

$$\|A\|^2 \sum_{r,s} S_{rs}W_{rjks} = 0,$$

where $\|A\|^2 = \|\sum_r A_rA_r\|$. This implies that M is quasi-conformally symmetric or M satisfies

$$\sum_{r,s} S_{rs}W_{rjks} = 0. \tag{3.19}$$

In view of (3.13), (3.18) and (3.19), we get

$$\begin{aligned} &\sum_r [(n-2)(S_{hrq} - A_qS_{hr})W_{rjkm} + (S_{jrq} - A_qS_{jr})W_{rhkm} \\ &+(S_{krq} - A_qS_{kr})W_{rjhm} + (S_{mrq} - A_qS_{mr})W_{rjkh}] = 0. \end{aligned}$$

By virtue of Lemma 3.2 we obtain

$$\begin{aligned} &\sum_r \{(S_{krq} - A_qS_{kr})W_{rjhm} + (S_{mrq} - A_qS_{mr})W_{rjkh}\} \\ &= \sum_r \{(S_{krq} - A_qS_{kr})W_{rjhm} - (S_{krq} - A_qS_{kr})W_{rjhm} - (S_{hrq} - A_qS_{hr})W_{rjmk}\} \\ &= -\sum_r (S_{hrq} - A_qS_{hr})W_{rjmk}. \end{aligned}$$

From the above two equations, we obtain

$$\sum_r \{(n-1)(S_{hrq} - A_qS_{hr})W_{rjkm} + (S_{jrq} - A_qS_{jr})W_{rhkm}\} = 0,$$

which implies that

$$\sum_r (S_{jrq} - A_q S_{jr}) W_{rhkm} = -(n-1) \sum_r (S_{hrq} - A_q S_{hr}) W_{rjkm}.$$

Hence it follows that

$$\sum_r (S_{jrq} - A_q S_{jr}) W_{rhkm} = 0. \tag{3.20}$$

Transvecting $S_{hiq} - A_q S_{hi}$, or W_{pjkm} to (3.12) and applying the equation (3.19) and (3.20), we can obtain

$$\|\nabla S - A \otimes S\|^2 \, W = 0, \;\text{ or }\; \|W\|^2 \, (\nabla S - A \otimes S) = 0 \tag{3.21}$$

on $M - M'$, which means that M is Ricci recurrent or M is quasi-conformally flat. This proves the theorem. □

As a particular case of Theorem 3.1, we can state the following:

Corollary 3.1([14]). *A conformally recurrent manifold with harmonic conformal curvature tensor and constant scalar curvature, is any one of the following:*
(i) *conformally symmetric,*
(ii) *conformally flat,*
(iii) *Ricci recurrent.*

Again by virtue of Theorem A, Theorem 3.1 can be stated as follows:

Theorem 3.2. *Let M be a quasi-conformally recurrent manifold with harmonic quasi-conformal curvature tensor such that $a + b \neq 0$ and $a + (n-2)b \neq 0$. Then the manifold is any one of the following:*
(i) *quasi-conformally symmetric,*
(ii) *conformally flat,*
(iii) *manifold of constant curvature,*
(iv) *vanishing scalar curvature,*
(v) *Ricci recurrent.*

Lemma 3.3. *If a Riemannian manifold M is quasi-conformally symmetric, then it is either quasi-conformally flat or locally symmetric, provided $a + (n-2)b \neq 0$.*

Proof. If the manifold is not quasi-conformally flat, then from (2.3) it follows that

$$\begin{aligned} W_{ijklm} &= aR_{ijklm} + b[g_{jk}S_{ilm} - g_{ik}S_{jlm} + g_{il}S_{jkm} - g_{jl}S_{ikm}] \\ &\quad -\frac{\kappa_m}{n}\left(\frac{a}{n-1} + 2b\right)(g_{il}g_{jk} - g_{jl}g_{ik}). \end{aligned} \tag{3.22}$$

If the manifold is quasi-conformally symmetric, then the last relation yields

$$\begin{aligned} aR_{ijklm} &+ b[g_{jk}S_{ilm} - g_{ik}S_{jlm} + g_{il}S_{jkm} - g_{jl}S_{ikm}] \\ &- \frac{\kappa_m}{n}\left(\frac{a}{n-1} + 2b\right)(g_{il}g_{jk} - g_{jl}g_{ik}) = 0. \end{aligned}$$

Contracting over i and l we have

$$S_{jkm} = \frac{1}{n}\kappa_m g_{jk}, \quad \text{if } a + (n-2)b \neq 0.$$

Again taking contraction over j and m we have $(n-2)\kappa_k = 0$, which implies $\kappa_k = 0$ as $(n > 3)$. Hence from above we have $\nabla S = 0$ and hence $\nabla R = 0$ for $a \neq 0$. This proves the lemma. □

Lemma 3.4. *If a Riemannian manifold M is quasi-conformally recurrent with harmonic quasi-conformal curvature tensor such that $a+b \neq 0$ and $a+(n-2)b \neq 0$, then the following relation holds*

$$W \otimes (\nabla R - A \otimes R) = 0.$$

Proof. From Theorem 3.1 we get

$$A \otimes W \otimes (\nabla S - A \otimes S) = 0.$$

Let $M_1 = \{x \in M : A(x) = 0\}$. Then $M_1 \subset M$. First we suppose that M_1 is not empty. If IntM_1 is empty, then the non-vanishing 1-form A implies that $W = 0$ or $\nabla S - A \otimes S = 0$, which yields $\kappa_m = A_m \kappa$. Since in the manifold under consideration the scalar curvature κ is constant, that is $\kappa_m = 0$ and the 1-form A is non-vanishing, it follows that $\kappa = 0$. Hence using $W_{ijklm} = A_m W_{ijkl}$, (2.3) and (3.22) we obtain $\nabla R - A \otimes R = 0$. Consequently by the continuity of W and $\nabla R - A \otimes R$, the conclusion holds in such a case.

Next suppose that IntM_1 is not empty. Then $A = 0$ on IntM_1 and hence M is quasi-conformally symmetric. Therefore by Lemma 3.3, we have $W = 0$ or $\nabla R = 0$ on IntM_1. Thus we have $W = 0$ or $\nabla R = A \otimes R$ on IntM_1. Hence we have $W \otimes (\nabla R - A \otimes R) = 0$ on M.

We now suppose that M_1 is empty. Then non-vanishing 1-form A implies that $W = 0$ or $\nabla S - A \otimes S = 0$. If $\nabla S - A \otimes S = 0$, then we also have $\nabla R - A \otimes R = 0$, because M is quasi-conformally recurrent. This proves the lemma. □

Theorem 3.3. *Let M be a quasi-conformally recurrent manifold with harmonic quasi-conformal curvature tensor and non-vanishing scalar curvature such that $a + b \neq 0$ and $a + (n-2)b \neq 0$. Then the manifold is either quasi-conformally flat or locally symmetric.*

Proof. Let $M'' = \{x \in M : (\nabla R - A \otimes R)(x) = 0\}$. Then $M'' \subset M$ and we have $(\nabla\kappa - A\kappa)(x) = 0$ on M''. We suppose that the manifold under consideration has non-zero scalar curvature. Since in the manifold under consideration the scalar curvature κ is constant, we must have κ is a non-zero constant, and hence we get $A(x) = 0$ on M''. Hence by virtue of Lemma 3.4, it follows that $A = 0$ or $W = 0$, that is, $A \otimes W = 0$ on M.

We now consider an open subset M^* of M such that $M^* = \{x \in M : W(x) = 0\}$. Then on such an open subset we have $\nabla W = 0$ and the inner product $g(W, W) = 0$, where g denotes the Riemannian metric tensor on M. By the continuity of $g(W,W)$, if M^* is not empty, then $g(W,W) = 0$ on M, namely $W = 0$ on M. Hence M is quasi-conformally flat with non-vanishing scalar curvature as the scalar curvature can not be zero by our supposition. If M^* is empty, then $A \otimes W = 0$ implies that $A = 0$ on M and in this case M is quasi-conformally symmetric. This completes the proof of our theorem. □

As a particular case we can state the following:

Corollary 3.2([14]). *If a Riemannian manifold M is conformally recurrent with harmonic conformal curvature tensor such that the scalar curvature is a non-zero constant, then the manifold is either conformally flat or locally symmetric.*

If the manifold is non-vanishing scalar curvature, then by virtue of Theorem A, Theorem 3.3 can be stated as follows:

Theorem 3.4. *Let M be a quasi-conformally recurrent manifold with harmonic quasi-conformal curvature tensor such that $a + b \neq 0$ and $a + (n-2)b \neq 0$. Then the manifold is any one of the following:*
(i) *conformally flat,*
(ii) *manifold of constant curvature,*
(iii) *locally symmetric.*

In [8] Goldberg and Okumura proved that in an $n(> 3)$-dimensional compact conformally flat manifold if the length of the Ricci tensor is constant and less than $\kappa/\sqrt{n-1}$, then the manifold is of constant curvature. Improving this result we can state and prove the following.

Theorem 3.5. *Let M be a quasi-conformally recurrent manifold with harmonic quasi-conformal curvature tensor such that $a + b \neq 0$ and $a + (n-2)b \neq 0$. Then the manifold is either quasi-conformally flat or recurrent.*

Proof. Without loss of generality, we may suppose that A does not vanish identically on M. From Lemma 3.4, we have $W \otimes (\nabla R - A \otimes R) = 0$ on M. Let M_1, M_2 and M_3 be the subsets of M defined as follows:

$$\begin{aligned} M_1 &= \{x \in M : A(x) = 0\}, \\ M_2 &= \{x \in M : W(x) = 0\} \\ \text{and} \qquad M_3 &= \{x \in M : (\nabla R - A \otimes R)(x) = 0\}. \end{aligned}$$

We now consider the following cases:

Case I) We suppose that IntM_3 is not empty. Then we have $\nabla R - A \otimes R = 0$ on IntM_3, and hence we get

$$\nabla S - A \otimes S = 0$$

on IntM_3. Accordingly, by the assumption we get

$$\nabla g(S,S) = 2Ag(S,S) = 0,$$

where g denotes a Riemannian metric tensor on M. From this we have $g(S, S) = 0$ on M or otherwise $g(S, S) \neq 0$ on M and $A = 0$ on $\mathrm{Int}M_3$.

In the first case, we see that $S = 0$, which implies that the scalar curvature κ vanishes identically on M, which means that $W = aR$ on M and hence

$$\nabla W = a\nabla R = aA \otimes R$$

on M. This implies that M is recurrent.

In the other one, we have $A = 0$ on $\mathrm{Int}M_3$. From the construction of the set M_3 it follows that $\nabla R = A \otimes R$ on $\mathrm{Int}M_3$. Moreover we know that the subset M_3 is contained in M_1, because

$$\mathrm{Int}\ M_3 \subset \overline{\mathrm{Int}M_3} \subset \overline{M_1} = M_1,$$

where '—' denotes the closure. Hence, we have $M = M_1 \cup M_2$. From this together with the assumption of quasi-conformal recurrence we get $\nabla W = 0$ on M.

We suppose that there is a point x in M_1 at which $W(x) \neq 0$. Since the inner product $g(W, W)$ is constant on M, it is a non-zero constant, which yields that the subset M_2 is empty. Then M is only $M = M_1$. So we have $\nabla R = 0$ on M. We suppose that there does not exist a point x in M_1 at which $W(x) \neq 0$. So we see that $W = 0$ on M, which yields the conclusion.

Case II) Next we suppose that $\mathrm{Int}M_3$ is empty. From the construction of the sets M_1, M_2 and M_3 and Lemma 3.4 we know that $M_3 \supset M - M_1 \cup M_2$, the set M_3 is empty. So we have $M = M_1 \cup M_2$. Since $\nabla W = 0$ on M_1, the inner product $g(W, W)$ is constant on M. This implies that $W = 0$ or $\nabla W = 0$ on M. By virtue of Lemma 3.3, we have $W = 0$ or $\nabla R = 0$ on M. This completes the proof. □

As a particular case we can state the following:

Corollary 3.3([14]). *If a Riemannian manifold M is conformally recurrent with harmonic conformal curvature tensor, constant scalar curvature and if the length of the Ricci tensor is constant, then the manifold is either conformally flat or recurrent.*

Again by virtue of Theorem A, Theorem 3.5 can be stated as follows:

Theorem 3.6. *Let M be a quasi-conformally recurrent manifold with harmonic quasi-conformal curvature tensor such that $a + b \neq 0$ and $a + (n-2)b \neq 0$. Then the manifold is any one of the following:*
(i) *conformally flat,*
(ii) *manifold of constant curvature,*
(iii) *vanishing scalar curvature,*
(iv) *locally symmetric,*
(v) *recurrent.*

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