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# Weakly $np-$ Injective Rings and Weakly $C2$ Rings

Junchao Wei* and Jianhua Chen
*Junchao Wei and Jianhua Chen School of Mathematics, Yangzhou University, Yangzhou, 225002, P. R. China*
*e-mail* : `jcweiyz@yahoo.com.cn` *and* `cjh_m@yahoo.com.cn`

Abstract. A ring $R$ is called left weakly $np-$ injective if for each non-nilpotent element $a$ of $R$, there exists a positive integer $n$ such that any left $R-$ homomorphism from $Ra^n$ to $R$ is right multiplication by an element of $R$. In this paper various properties of these rings are first developed, many extending known results such as every left or right module over a left weakly $np-$ injective ring is divisible; $R$ is left seft-injective if and only if $R$ is left weakly $np$-injective and $_RR$ is weakly injective; $R$ is strongly regular if and only if $R$ is abelian left pp and left weakly $np-$ injective. We next introduce the concepts of left weakly *pp* rings and left weakly $C2$ rings. In terms of these rings, we give some characterizations of (von Neumann) regular rings such as $R$ is regular if and only if $R$ is $n-$ regular, left weakly *pp* and left weakly $C2$. Finally, the relations among left $C2$ rings, left weakly $C2$ rings and left $GC2$ rings are given.

## 1. Introduction

Throughout $R$ denotes an associative ring with identity, and all modules are unitary. We write $_RM$ and $M_R$ to indicate a left and right $R-$module, respectively. For any nonempty subset $X$ of a ring $R$, $r(X)$ and $l(X)$ denote the right annihilator of $X$ and the left annihilator of $X$, respectively. If $X = \{a\}$, we usually abbreviate it to $l(a)$ and $r(a)$. As usual, $J(R)$, $Z_l(R)$, $N(R)$, $N_2(R)$ and $E(R)$ denote the Jacobson radical, the left singular ideal, the set of all nilpotent elements, the set of all non-nilpotents and the set of all idempotent elements of $R$, respectively. As a generalization of $AP-$ injective rings (cf. Page and Zhou, (1998)) $np-$ injective rings (cf. Ming, (1983)), we introduced the notion of left weakly $np-$ injective modules, that is, left $R-$ module $M$ is weakly $np-$ injective if for each $a \in N_2(R)$, there exists a positive integer $n$ such that any left $R-$ homomorphism from $Ra^n$ to $M$ is right multiplication by an element of $M$. If $_RR$ is left weakly $np-$ injective, then $R$ is called a left weakly $np-$ injective ring, which is a generalization of left

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$YJ-$ injective rings (cf. Chen and Ding, (1999)). Some important results which are known for $p-$ injective rings (cf. Nicholson and Yousif, (1995)) and $np-$ injective rings were shown to hold for right weakly $np-$ injective rings.

An important source of left $C2$ rings is given by Nicholson and Yousif in Nicholson and Yousif, (2001), and in Wei and Chen, (2007) and (2008), left $NC2$ rings are introduced, which is a generalization of left $C2$ rings. In this note, we introduce left $WC2$ rings, that is, a ring $R$ is left $WC2$ if for any projective principally left ideal $Ra$ with $a \in N_2(R)$, $Ra = Re$ for some $e \in E(R)$. The connections among left $WC2$ rings, left $FGF$ rings and quasi-Frobenius rings are considered, which generalize the Theorem 4.5 and Theorem 4.6 of Nicholson and Yousif, (2001).

(von Neumann) regular rings have been studied extensively by many authors. It is well known that a ring $R$ is regular if and only if every left $R-$module is $p-$injective. Recently, Ding and Chen, (1999) showed that a ring $R$ is regular if and only if every left $R-$module is $YJ-$injective. In this paper, we show that $R$ is regular ring if and only if $N_1(R) = \{0 \neq a \in R \mid a^2 = 0\}$ is regular and every left $R$-module is left weakly $np-$injective.

## 2. Weakly np-injective rings and modules

**Theorem 2.1.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is a left weakly np-injective ring.*

(2) *For any $a \in N_2(R)$, there exists $n \in Z^+$ such that $rl(a^n) = a^nR$.*

(3) *For any $a \in N_2(R)$, there exists $n \in Z^+$ such that $b \in a^nR$ for any $b \in R$, whence $l(a^n) \subseteq l(b)$.*

(4) *For any $a \in N_2(R)$, there exists $n \in Z^+$ such that $b \in a^nR$ for any $b \in R$, whence $l(a^n)b = 0$.*

(5) *For any $a \in N_2(R)$, there exists $n \in Z^+$ such that $Ext^1_R(R/Ra^n, R) = 0$.*

*Proof.* Similar to Lemma 1.1 of Nicholson and Yousif, (1995), we can easy show the Theorem. □

Call an idempotent $e$ of $R$ is left weakly corner element if $ReN = N$ for any left $R-$ submodule $N$ of $Re$. Clearly any central idempotent of a ring $R$ is left weakly corner element. Let $e \in E(R)$ such that $ReR = R$, then $e$ is also a left weakly corner element of $R$.

**Theorem 2.2.** *Let $R$ be a left weakly np-injective ring with $e \in E(R)$. If $e$ satisfies one of the following conditions, then $S = eRe$ is left weakly $np-$ injective.*

(1) *$e$ is a left weakly corner element of $R$.*

(2) *$e$ is contained in central of $R$.*

(3) *$ReR = R$.*

*Proof.* (1) Let $a \in N_2(S)$. Then $a \in N_2(R)$. Since $R$ is left weakly np-injective, by Theorem 2.1, there exists $n \in Z^+$ such that $r_Rl_R(a^n) = a^nR$. We claim that $r_Sl_S(a^n) = a^nS$. Let $x \in r_Sl_S(a^n)$, then $l_S(a^n) \subseteq l_S(x)$. For any $y \in l_R(a^n)$, then $ya^n = 0$, so we have $eRyea^n = 0$. Therefore $eRye \in l_S(a^n)$ and so $eRyx =$

$eRyex = 0$ because $l_S(a^n) \subseteq l_S(x)$. Since $e$ is left weakly corner element, $Ryx = Ryxe = ReRyxe = ReRyx = ReRyex = 0$. Hence $y \in l_R(x)$ and so $l_R(a^n) \subseteq l_R(x)$. Therefore $x \in r_R l_R(x) \subseteq r_R l_R(a^n) = a^n R$. Since $x = xe$, $x \in a^n Re = a^n eRe = a^n S$. Therefore $r_S l_S(a^n) = a^n S$.

(2) and (3) follow from (1). □

An element $a \in R$ is called left regular if $l(a) = 0$. We have the following theorem.

**Theorem 2.3.** *Let $R$ be a left weakly np-injective ring. Then*

(1) *Any left regular element of $R$ is right invertible.*

(2) $Z_l(R) \subseteq J(R)$.

(3) *Every left or right $R$-module is divisible.*

(4) *If $P$ is a reduced principal left ideal of $R$, then $P = Re$, where $e = e^2 \in R$ and $R(1-e)$ is an ideal of $R$.*

*Proof.* (1) Let $c \in R$ such that $l(c) = 0$. Then $c \in N_2(R)$. By Theorem 2.1, there exists a positive integer $n$ such that $rl(c^n) = c^n R$. This shows that $R = c^n R$ which proves (1).

(2) If $z \in Z_l(R), a \in R$, then $l(1 - za) = 0$ implies $(1 - za)v = 1$ for some $v \in R$ by (1). This proves that $z \in J(R)$.

(3) If $c$ is a non-zero-divisor in $R$, then $cd = 1$ for some $d \in R$ by (1). Now $r(c) = 0$ implies $dc = 1$ and for any right $R$-module $M$, $M = Mdc \subseteq Mc \subseteq M$. Hence $M = Mc$ and we show that $M$ is divisible. Similarly, we can show that any left $R$-module is divisible.

(4) Let $P = Rc, c \in R$, be a non-zero reduced principal left ideal. Since $c^2 \in N_2(R)$ and $R$ is left weakly $np$-injective, there exists a positive integer $n$ such that $rl(c^{2n}) = c^{2n}R$. Hence $l(c) = l(c^{2n})$ shows that $cR \subseteq rl(c) = rl(c^{2n}) = c^{2n}R$. Therefore $c = c^{2n}b$ for some $b \in R$, which implies $c = cdc$, where $d = c^{2n-2}b$ ($P$ being reduced), whence $P$ is generated by the idempotent $e = dc$. Also, for any $a \in R$, $(ae - eae)^2 = 0$ implies $ae = eae$, whence $(1-e)Re = 0$. Therefore $(1-e)R \subseteq R(1-e)$ which establishes the last part of (4). □

An extension of left $R-$ modules

$(*)$: $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ (exact)

is said to be weakly pure if it has one of the following two equivalent properties:

(1) For every $d \in N_2(R)$, there exists a positive integer $n$ such that $A \cap d^n B = d^n A$.

(2) If $d^n c = 0$ for $c \in C$, then there exists $b \in B$ satisfying $\beta(b) = c$ and $d^n b = 0$.

(In (1) $A$ is identified with $\alpha(A) \subseteq B$.) It is easy to see that these are equivalent respectively to

($1'$) For every $d \in N_2(R)$, there exists a positive integer $n$ such that $R/d^n R \otimes A \xrightarrow{1\otimes\alpha} R/d^n R \otimes B$ is a monomorphism.

($2'$) $Hom(R/Rd^n, B) \xrightarrow{\beta^*} Hom(R/Rd^n, C)$ is an epimorphism.

Then, we have

**Proposition 2.4.** *Let $A$ be a left $R-$ module. Then the following conditions are equivalent.*

(1) *$A$ is left weakly np$-$ injective as $R-$ module.*

(2) *Every extension $(*)$ with $A$ as the kernel is weakly pure.*

(3) *For any $a \in N_2(R)$, there exists $n \in Z^+$ such that $r_A l_R(a^n) = a^n A$.*

(4) *For any $a \in N_2(R)$, there exists $n \in Z^+$ such that $b \in a^n A$ for any $b \in A$, whence $l_R(a^n)b = 0$.*

(5) *For any $a \in N_2(R)$, there exists $n \in Z^+$ such that $Ext^1_R(R/Ra^n, A) = 0$.*

As a corollary we have

**Corollary 2.5.** (1) *An extension of a left weakly np$-$ injective module by a left weakly np$-$ injective module yields always left weakly np$-$ injective.*

(2) *A direct product as well as a direct sum of left np$-$ injective modules is np$-$ injective.*

**Theorem 2.6.** *Let $R$ be a left weakly np- injective ring. If $Rb$ embeds in $Ra$, where $l(b) = 0$, then there exists a positive integer number $n$ such that $b^n R$ is an image of $aR$.*

*Proof.* If $\sigma : Rb \longrightarrow Ra$ is monic. Since $R$ is left weakly $np$- injective, there exists a positive integer $n$ such that any left $R$-homomorphism of $Rb^n$ into $R$ extends to one of $R$ into $R$. Let left $R$-homomorphism $f = \iota\sigma i : Rb^n \longrightarrow R$, where $i : Rb^n \hookrightarrow Rb$ and $\iota : Ra \hookrightarrow R$ are embedation maps. Hence $\sigma(b^n) = b^n v = ua$, where $v, u \in R$. Now let $\varphi : aR \longrightarrow b^n R$, via: $\varphi(ar) = uar = b^n vr$. Since $b^n v \in N_2(R)$, there exists a positive integer $m$ such that $(b^n v)^m R = rl((b^n v)^m)$. Since $l((b^n v)^m) = l(b^n v) = l(b^n) = l(b) = 0$, $(b^n v)^m R = rl((b^v)^m) = R$. Let $b^n = (b^n v)^m c$, where $c \in R$. Hence $\varphi(a(b^n v)^{m-1} c) = ua(b^n v)^{m-1} c = (b^n v)^m c = b^n$ and so $\varphi$ is an epic. □

According to Ming, (1983), a ring $R$ is called left $np-$ injective if, for each $a \in N_2(R)$, we have $rl(a) = aR$. Evidently, left $np-$ injective rings are weakly $np-$ injective. Similar to Theorem 2.6, we have the following corollary.

**Corollary 2.7.** *Let $R$ be a left np- injective ring. If $Rb$ embeds in $Ra$, where $l(b) = 0$, then $bR$ is an image of $aR$.*

On the other hand, we also have the following theorem.

**Theorem 2.8.** *Let $R$ be a left np- injective ring. If $Ra$ is an image of $Rb$, where $b \in N_2(R)$, then $aR$ embeds in $bR$.*

*Proof.* If $\sigma : Rb \longrightarrow Ra$ is epic. Since $R$ is left $np$- injective and $b \in N_2(R)$, $\sigma = \cdot v$, $v \in R$. Then $bv = ua$ for some $u \in R$. So define $\varphi : aR \longrightarrow bR$ by $ar \longmapsto uar = bvr$. Write $a = \sigma(sb) = sbv$, where $s \in R$. Then $\varphi(ar) = 0$ gives $0 = uar = bvr$, whence $ar = sbvr = 0$, and $\varphi$ is monic. □

A ring $R$ is called left zero-divisor power if, for each $0 \neq a \in R$, $l(a^n) = l(a)$ for all positive integer $n$ satisfying $a^n \neq 0$.

If $R$ is only a left weakly $np-$ injective ring, we don$'$t know whether the result in theorem 2.8 is right. But it is right if $R$ is also a left zero-divisor power ring.

A ring $R$ is called directly finite if $uv = 1$ in $R$ implies $vu = 1$. By Theorem 2.6, we have the following corollary.

**Corollary 2.9.** *Let $R$ be a left weakly $np-$ injective ring. Then $R$ is directly finite if and only if every left regular element is invertible.*

Let $E(M)$ be an injective hull of ${}_RM$. $M$ is called left weakly injective (cf. Nicholson and Yousif, (1995)) if for any finite generated submodule ${}_RN \subseteq E(M)$, there exists ${}_RX \cong M$ and ${}_RN \subseteq_R X \subseteq E(R)$. Clearly, left injective modules are left weakly injective. If ${}_RR$ is left weakly injective, we call $R$ is left weakly injective ring.

**Lemma 2.10.** *Let $R$ be a left weakly $np-$ injective ring. If ${}_RR$ is essential in ${}_RX$, where ${}_RX \cong_R R$, then $X = R$.*

*Proof.* Let $f :_R R \longrightarrow_R X$ be the isomorphism and $f(1) = b \in X$. Then $Rb = Im(f) = X$. Since $1 \in R \subseteq X$, let $1 = ub, u \in R$. Hence ${}_RR = R1 = Rub$ and $l(u) = 0$. Since $R$ is left weakly $np-$ injective, there exists a $d \in R$ such that $ud = 1$ by Theorem 2.3. Let $e = du$, then $e^2 = e$ and $Ru = Re$, so we have $R = Rub = Reb$. It is clear that $X = Rb = (Re \oplus R(1-e))b = Reb + R(1-e)b$. If $x \in Reb \cap R(1-e)b$, then there exist $r_1, r_2 \in R$ such that $x = r_1eb = r_2(1-e)b$, so $f^{-1}(x) = r_1e = r_2(1-e)$. Hence $f^{-1}(x) = 0$ and then $x = 0$, so $X = Rb = Reb \oplus R(1-e)b = R \oplus R(1-e)b$. Since ${}_RR$ is essential in ${}_RX$, $R(1-e)b = 0$, and so $X = Reb = R$. □

The following theorem is a generalization of Nicholson and Yousif, (1995, Theorem 1.3).

**Theorem 2.11.** *Ring $R$ is left seft-injective if and only if $R$ is left weakly np-injective and left weakly injective.*

*Proof.* We only show "if" part, in other word, we show that $E({}_RR) \subseteq R$. Let $a \in E({}_RR)$. Since $R + Ra \subseteq E({}_RR)$ and $R$ is left weakly injective, there exists $X \subseteq E({}_RR)$ such that $R + Ra \subseteq X$ and ${}_RX \cong_R R$. Since $R$ is left weakly $np-$ injective, $X = R$ by lemma 2.10. Hence $R = E({}_RR)$. □

The next theorem extends Chen and Ding, (1999, Theorem 3.1).

**Theorem 2.12.** *Let $R$ be a semiprime left weakly $np-$ injective ring. Then every maximal left (respectively, right) annihilator is a maximal left (respectively, right) ideal of $R$ which is generated by an idempotent.*

*Proof.* Let $L$ be a maximal left (respectively, right) annihilator. Then $L = l(a)$ (respectively, $r(a)$) for some $0 \neq a \in R$. Since $R$ is semiprime, $Z_l(R) \cap l(Z_l(R)) = 0$. Claim: $a \notin Z_l(R)$. Otherwise, $a \notin l(Z_l(R))$. That is $aZ_l(R) \neq 0$. Take $x \in Z_l(R)$ such that $ax \neq 0$. Since $l(ax) = l(a)$ by the maximality of $L$. Thus $l(x) \cap Ra = 0$, a contradiction. Therefore, $a \notin Z_l(R)$. Then there exists a nonzero left ideal $I$ of

$R$ such that "$l(a) \cap I = 0$" maximal. Let $0 \neq b \in I$, then $ba \neq 0$. If there exists a minimal positive integer $n$ such that $(ba)^n = 0$, then $(ba)^{n-1}b \in l(a) \cap I = 0$ and so $b \in l((ab)^{n-1})$. Since $l(a) = l((ab)^{n-1})$, $b \in l(a)$, a contradiction. Hence $ba \in N_2(R)$. Since $R$ is left weakly $np-$ injective, there exists a positive integer $m$ such that $rl((ba)^m) = (ba)^m R$ by Theorem 2.1. Since $l((ba)^{m-1}b) = l((ba)^m)$, $(ba)^{m-1}b = (ba)^m c$ for some $c \in R$. Hence $(ba)^{m-1}b(1-ac) = 0$. Let $d = a - aca$, then $l(a) \subsetneq l(d)$, since $(ba)^{m-1}b \notin l(a)$ and in $l(d)$. Hence $d = 0$ by the maximality of $l(a)$. Therefore $L = l(ac)$ with $ac$ is idempotent. So we can assume that $a = e$ is an idempotent. To see $L$ is a maximal left ideal, we show that $Re$ is a minimal left ideal of $R$. Since $R$ is semiprime, it suffices to show that $eRe$ is a division ring. Let $0 \neq d \in eRe$. Since $l(e) = l(d)$, $d$ is not nilpotent. Hence $d^s R = rl(d^s)$ for some positive integer $s$. Since $l(d^s) = l(e)$, $d^s R = eR$. Write $e = d^s t$ where $t \in R$. Then $e = d(d^{s-1}te)$ with $d^{s-1}te \in eRe$. So, $eRe$ is a division ring. □

## 3. Weakly $C2$ rings

A ring $R$ is called left $C2$ (cf. Nicholson and Yousif, (2001)) if, every left ideal $T$ is isomorphic to a summand of $_RR$ then $T$ is a summand.

A ring $R$ is called left generalized $C2$ (or $GC2$) (cf. Zhou, (2002)) if, for any left ideal $L$ of $R$ with $_RL \cong_R R$, $L$ is a summand of $R$.

A ring $R$ is called left $NC2$ (cf. Wei and Chen, (2007)) if for any $a \in N(R)$ with $_RRa$ projective, $Ra$ is a summand of $R$.

Call a ring $R$ left weakly $C2$ (or $WC2$) if for any $a \in N_2(R)$ with $_RRa$ projective, $Ra$ is a summand of $R$.

Clearly, A ring $R$ is left $C2$ if and only if $R$ is left $NC2$ and left $WC2$, and a left $C2$ ring is always left $GC2$. On the other hand, we can easy to show that left $np-$ injective rings are left $WC2$. Evidently, if $R$ is a local left $WC2$ ring, then $R$ is left $C2$ ring. Now let $R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix}$. Then $R$ is a left $WC2$ ring but not left $C2$ ring. On the other hand, the referee shows that left $WC2$ rings are left $GC2$ (Proof. If $f: R \longrightarrow Ra$ is a isomorphism with $a = f(1)$, then $l(a) = 0$. So $a \in N_2(R)$, and $Ra$ is a direct summand of $R$ for $R$ is left $WC2$. Hence $R$ is left $GC2$).

Similar to Nicholson and Yousif, (2001, Proposition 3.3), we have the following proposition.

**Proposition 3.1.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is a left $WC2$ ring.*

(2) *For each $a \in N_2(R)$ and each $R-$isomorphism $Ra \longrightarrow Re, e^2 = e \in R$, extends to $R \longrightarrow R$.*

(3) *For each $a \in N_2(R)$ and if $l(a) = l(e), e^2 = e \in R$, then $e \in aR$.*

(4) *For each $a \in N_2(R)$ and if $l(a) = l(e), e^2 = e \in R$, then $eR = aR$.*

(5) *For each $a \in N_2(R)$ and $aR \subseteq eR \subseteq rl(a), e^2 = e \in R$, then $eR = aR$.*

(6) *For each $a \in N_2(R)$ and if $Ra$ is projective, then $Ra$ is a direct summand of $_RR$.*

Similar to Theorem 2.3, for left $WC2$ rins, we have the following results:

**Proposition 3.2.** *Let $R$ be a left $WC2$ ring, then:*
(1) *Every left regular element of $R$ is right invertible.*
(2) *Every left or right $R-$module is divisible.*
(3) $Z_l(R) \subseteq J(R)$.

Recall that a ring $R$ is directly finite if and only if $R/J(R)$ is directly finite if and only if every left $R-$ epic: $R \longrightarrow R$ is monic.

**Theorem 3.3.** *Let $R$ be a left $WC2$ ring. Then the following conditions are equivalent:*
(1) *$R$ is directly finite.*
(2) *$R/Z_l(R)$ is directly finite.*
(3) *Every monomorphism ${}_RR \longrightarrow_R R$ is an isomorphism.*
*If $R$ is also a local ring, then also equivalent to*
(4) *$R$ is left $C2$.*
(5) $J(R) = \{a \in R | l(a) \neq 0\}$.

*Proof.* (1) $\Rightarrow$ (2) Let $\bar{a}\bar{b} = \bar{1}$ in $R/Z_l(R)$. Then $ab = 1 + x$ for some $x \in Z_l(R)$. By hypothesis and Proposition 3.2(3), $x \in J(R)$, so we have $1 = (1+x)^{-1}ab$. By (1), $b(1+x)^{-1}a = 1$. It follows that $\bar{b}\bar{a} = \bar{b}\bar{1}\bar{a} = \bar{b}\overline{(1+x)^{-1}}\bar{a} = \bar{1}$.

(2) $\Rightarrow$ (3) Let $f :_R R \longrightarrow_R R$ be monic and $a = f(1)$. Then $a \neq 0$ and $l(a) = 0$. By hypothesis and Proposition 3.2(1), there exists a $d \in R$ such that $ad = 1$. Hence $\bar{a}\bar{d} = \bar{1}$ in $R/Z_l(R)$, and so $\bar{d}\bar{a} = \bar{1}$ by (2). Then we have $da = 1 + y$, where $y \in Z_l(R) \subseteq J(R)$ and so $f((1+y)^{-1}d) = (1+y)^{-1}da = 1$. Showing that $f$ is an isomorphism.

(3) $\Rightarrow$ (1) Let $ab = 1$ in $R$. Define $f :_R R \longrightarrow_R R$ by $f(r) = ra$. Then $f$ is monic and, by (3), $1 = ca$ for some $c \in R$. Therefore, $c = c(ab) = (ca)b = b$ and so $ba = 1$.

By Nicholson and Yousif, (2001, Corollary 3.5), we yield (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5). □

**Theorem 3.4.** *Let $R$ be a left $WC2$ ring. then*
(1) *If $a \in R$ and $e \in E(R)$ is central with $f : Re \longrightarrow Ra$ being a left $R-$isomorphism, then there exists $g^2 = g \in R$ such that $Ra = Rg$.*
(2) *Let $e, f \in E(R)$ and $f$ be central, if $Re \cap Rf = 0$, then there exists $g^2 = g \in R$ such that $Re \oplus Rf = Rg$.*
(3) *If $R$ is an abelian ring, then $R$ is left $C2$.*

*Proof.* (1) Let $f(e) = b$, then $Ra = Im(f) = Rb$ and $eb = ef(e) = f(e) = b$. If there exists a positive integer $n > 1$ such that $b^n = 0$, then $f(b^{n-1}e) = b^n = 0$, and so $b^{n-1}e = 0$. Since $e$ is central, $eb^{n-1} = 0$ and so $b^{n-1} = 0$. Repeating the above process, we have $b = 0$, which is a contradiction. Hence $b \in N_2(R)$. Since $R$ is left $WC2$, $Ra$ is a summand of $R$.

(2) Let $L \subseteq_R R$ satisfy $R = Re \oplus L$, then $Re \oplus Rf = Re \oplus ((Re \oplus Rf) \cap L)$. Since $Rf \cong (Re \oplus Rf)/Re \cong (Re \oplus Rf) \cap L$, by (1), $(Re \oplus Rf) \cap L = Rh$ for

some $h \in E(R)$. Let $R = ((Re \oplus Rf) \cap L) \oplus K$, where $K \subseteq_R R$. Since $L = L \cap R = L \cap (((Re \oplus Rf) \cap L) \oplus K) = ((Re \oplus Rf) \cap L) \oplus (L \cap K)$, $R = Re \oplus L = Re \oplus ((Re \oplus Rf) \cap L) \oplus (L \cap K) = Re \oplus Rf \oplus (L \cap K)$ and so $Re \oplus Rf = Rg$ for some $g \in E(R)$.

(3) It is an immediate result of (1). □

**Proposition 3.5.** *Let $R$ be a left weakly $np-$ injective ring. If $R$ satisfies one of the following conditions, then $R$ is left $WC2$.*

(1) *$R$ is an abelian ring.*

(2) *$R$ is a left zero-disivor power ring.*

*Proof.* Let $a \in N_2(R)$ and $e \in E(R)$ such that $Ra \cong Re$. Then, clearly, there exists an idempotent $g \in R$ such that $a = ga$ and $l(a) = l(g)$. Since $R$ is left weakly $np-$ injective, $rl(a^n) = a^n R$ for some $n \geq 1$ by Theorem 2.1. If $R$ is a left zero-divisor power ring, $l(a^n) = l(a)$. Now we assume that $R$ is an abelian ring. Let $x \in l(a^n)$, then $xa^{n-1} \in l(a) = l(g)$. Hence $xa^{n-1} = xga^{n-1} = xa^{n-1}g = 0$, so we have $x \in l(a^{n-1})$. Repeating the above process, we have $x \in l(a)$. Hence $l(a^n) = l(a)$. However, we have $gR = rl(g) = rl(a) = rl(a^n) = a^n R \subseteq aR = gaR \subseteq gR$, this shows that $gR = aR$. Therefore $R$ is left $WC2$. □

Recall that a ring $R$ is left morphic (see, Nicholson and Campos, (2004)) if, for each $a \in R$, $R/Ra \cong l(a)$, equivalently, if for each $a \in R$, there exists $b \in R$ such that $Ra = l(b)$ and $Rb = l(a)$. In Nicholson and Campos, (2004) it is proved that left morphic rings are right $p-$injective, and hence right $C2$. Furthermore, we have.

**Theorem 3.6.** *Let $R$ be a left morphic ring. Then $R$ is left $C2$.*

*Proof.* Let $a \in R$ and $\sigma : Ra \cong Re, e^2 = e \in R$. Then there exists an idempotent $f$ of $R$ such that $a = fa$ and $l(a) = l(f) = R(1-f)$. Since $R$ is a left morphic ring, $Ra = l(d)$ and $Rd = l(a)$ for some $d \in R$. Write $d = dud$, where $u \in R$ with $ud = 1 - f$. Set $g = du$, then $g^2 = g$ and $dR = gR$. Hence $Ra = l(d) = l(g) = R(1-g)$ is a direct summand. □

A ring $R$ is called left *Johns* (cf. Faith and Menal, (1992)) if it is left noetherian and every left ideal is an annihilator, and $R$ is called strongly left *Johns* (cf. Faith and Menal, (1994)) if the matrix ring $M_n(R)$ is left *Johns* for every $n \geq 1$. It is an open question whether or not strongly left *Johns* rings are quasi-Frobenius. A ring $R$ is called a left $CEP$ if every cyclic left $R-$module can be essentially embedded in a projective module. These rings are known to be left artinian (cf. Pardo and Asensio, (1997)). A ring $R$ is called left (right, resp.) perfect if $R$ satisfies the descending chain condition for cyclic right (left, resp.) ideals. It is well known that $R$ is left perfect if and only if $R/J(R)$ is semisimple and $J(R)$ is right $T-$ nilpotent, where a ring $R$ is called right $T-$ nilpotent if for every family $\{a_1, a_2, a_3, \cdots\} \subseteq R$, there exists a positive integer $n$ such that $a_1 a_2 \cdots a_n = 0$. In Nicholson and Yousif, (2001, Theorem 4.5) it is proved that a ring $R$ is left $CEP$ if and only if $R$ is left *Johns* and left $C2$, and Nicholson and Yousif, (2001, Theorem 4.6) proved that $R$

is a strongly left *Johns* left $C2$ ring if and only if $R$ is a quasi-Frobenius ring. We will show that left *Johns* rings and left $CEP$ rings are same whence $R$ is a left $WC2$ ring.

Since left *Johns* rings are right principally injective (cf. Nicholson and Yousif, (1995)), left *Johns* rings are right $AGP-$injective and left noetherian, by Zhou, (2003, Theorem 2.1), $J(R)$ is nilpotent. Hence we have the following theorem, which is a generalization of Nicholson and Yousif, (2001, Theorem 4.5).

**Theorem 3.7.** *$R$ is a left $CEP$ ring if and only if $R$ is a left Johns left $WC2$ ring.*

*Proof.* $\Longrightarrow$ It is an immediate consequence of Nicholson and Yousif, (2001, Theorem 4.5).

$\Longleftarrow$ Since $R$ is left *Johns*, $J(R)$ is nilpotent. Since $R$ is left noetherian, $R$ is directly finite. By Theorem 3.3, every left $R-$ monic $R \longrightarrow R$ is epic because $R$ is left $WC2$. By Camps and Dicks, (1993, Theorem 5), $R$ is semilocal, so, $R$ is semiprimary. Therefore $R$ is left artinian. Then, by Nicholson and Yousif, (1998, Proposition 3.3), $R$ is left $CEP$. □

The following theorem is a generalization of Nicholson and Yousif, (2001, Theorem 4.6).

**Theorem 3.8.** *$R$ is a quasi-Frobenius ring if and only if $R$ is a strongly left Johns left $WC2$ ring.*

*Proof.* $\Longrightarrow$ It is obvious by Nicholson and Yousif, (2001, Theorem 4.6).

$\Longleftarrow$ By hypothesis, and using theorem 3.7, $R$ is left $CEP$. By Nicholson and Yousif, (2001, Theorem 4.5), $R$ is left $C2$. By Nicholson and Yousif, (2001, Theorem 4.6), $R$ is quasi-Frobenius. □

Since quasi-Frobenius rings are left self-injective, quasi-Frobenius rings are left $np-$ injective. Since left $np-$ injective rings are left $WC2$, by Theorem 3.8, we have the following corollary.

**Corollary 3.9.** *$R$ is a quasi-Frobenius ring if and only if $R$ is a strongly left Johns left $np-$ injetive ring.*

By Proposition 3.5, we yield the following corollary.

**Corollary 3.10.** *Let $R$ be an abelian ring. Then $R$ is a quasi-Frobenius ring if and only if $R$ is a strongly left Johns left weakly $np-$ injetive ring.*

**Theorem 3.11.** *Let $R$ be a directly finite left weakly $np-$ injective ring, then:*

(1) *$R$ is left $GC2$.*

(2) *If ${}_RR$ is of finite Goldie dimension, then $R$ is semilocal.*

(3) *If $Soc({}_RR) \subseteq l(J)$, then $R$ is left Noetherian if and only if $R$ is left artinian.*

*Proof.* (1) Suppose $\sigma : Ra \cong R$, where $a \in R$. Write $\sigma(a) = d$, and $\sigma(ca) = 1$ for some $c, d \in R$. Hence $1 = cd$, and so $dc = 1$ because $R$ is directly finite. Since $l(a) = l(d) = 0$, $a \in N_2(R)$. Since $R$ is left $np-$ injective, there exists a positive integer $m$ such that $rl(a^m) = a^m R$. Hence $R = rl(a) = rl(a^m) = a^m R = aR$. Since $R$ is directly finite, $Ra = R$. Hence $R$ is a left $GC2$ ring.

(2) Let $\sigma : R \longrightarrow R$ be a monomorphism. Then, by (1), $R = \sigma(R) \oplus L$ for some $L \subseteq R$. Since ${}_RR$ has finite Goldie dimension, $L = 0$ . So $\sigma$ is an isomorphism. Therefore, ${}_RR$ satisfies the assumptions in Camps-Dicks, (1993, Theorem 5), and so $R \cong End(R)$ is semilocal.

(3) If $R$ is left Noetherian, then $R/Soc({}_RR)$ is left Noetherian, and so $R/Soc({}_RR)$ has $ACC$ on left annihilator. Hence $(r(Soc({}_RR) \cap J(R)) + Soc({}_RR))/Soc({}_RR)$ is nilpotent in $R/Soc({}_RR)$. Because $Soc({}_RR) \subseteq l(J)$, $J + Soc({}_RR)/Soc({}_RR)$ is nilpotent. Hence there exists a positive integer $n$ such that $J^n \subseteq Soc({}_RR)$, and so $J^{n+1} \subseteq JS_l = 0$. By (2) $R$ is a semilocal ring, and so $R$ is a semiprimary ring. Hence $R$ is left artinian. □

A ring $R$ is called left finite embedded (cf. Nicholson and Yousif, (2000)) if, $Soc({}_RR)$ is finite generated and left essential in ${}_RR$, and $R$ is said to be right $Kasch$ if for any maximal right ideal $M$ of $R$, $l(M) \neq 0$.

A ring $R$ is called left minsymmetric if, whence $Rk$ is a simple left ideal of $R$, then $kR$ is also simple as right ideal. This is a large class of rings, including the left mininjective rings (see, Nicholson and Yousif, (1997)). If $R$ is left minsymmetric, then $S_l \subseteq S_r$. If $S_l$ is also an essential left ideal of $R$, then $J(R) \subseteq Z_l(R)$. Hence the next proposition is a generalization of Nicholson and Yousif, (2000, Lemma 1).

**Proposition 3.12.** *Suppose $R$ is a left finite embedded, left minsymmetric ring. Then the following conditions are equivalent:*

(1) *$R$ is a right Kasch ring.*
(2) *$R$ is a left $C2$ ring.*
(3) *$R$ is a left $WC2$ ring.*
(4) *$R$ is a left $GC2$ ring.*
(5) *$Z_l(R) = J(R)$.*
(6) *$Z_l(R) \subseteq J(R)$.*

## 4. Weakly *pp* rings and weakly regular rings

Call a ring $R$ left weakly almost *pp* if, for each $a \in N_2(R)$, $l(a)$ is generated by a family idempotents $e_i, i \in I$ of $R$, that is $l(a) = \Sigma_{i \in I} Re_i$, and $R$ is said to be left weakly *pp* if, for each $a \in N_2(R)$, $Ra$ is projective as left $R-$ module, or equivalently, $l(a) = Re$ for some $e^2 = e \in R$. Clearly left pp rings are left weakly pp and left weakly pp rings are left weakly almost pp. According to Wei and Chen, (2007), a ring $R$ is called left $NPP$ if for any $a \in N(R)$, $Ra$ is projective as left $R-$ module.

**Theorem 4.1.** (1) *$R$ is a left pp ring if and only if $R$ is a left $NPP$ ring and left weakly pp ring.*

(2) *The following conditions are equivalent for a ring $R$:*

(a) *$R$ is a left weakly pp ring.*

(b) *Every factor module of an injective left $R-$ module is $np-$ injective.*

(c) *Every sum of two injective submodules of a left $R-$ module is $np-$ injective.*

(d) *Every sum of two isomorphic injective submodules of a left $R-$ module is $np-$ injective.*

(e) *Every factor module of a $np-$ injective left $R-$ module is $np-$ injective.*

(3) *The following conditions are equivalent for a ring $R$:*

(a) *$R$ is an abelian left weakly almost pp ring.*

(b) *For any $a \in N_2(R)$ and each $x \in l(a)$, there exists $e \in E(R)$ such that $e \in l(a)$ and $x = ex = xe$.*

(c) *For any $a \in N_2(R)$, $l(a) = l(a^2) = l(a^3) = \cdots = l(a^n) = \cdots = \Sigma_{i\in I} Re_i \subseteq r(a)$, where $\{e_i | i \in I\} \subseteq E(R)$.*

(d) *For any $a \in N_2(R)$, $l(a) = \Sigma_{i\in I} Re_i \subseteq r(a)$ and $l(a) \cap Ra = 0$.*

*Proof.* (1) is obvious.

(2) Similar to Theorem 2.1 of Wei and Chen, (2008).

(3) $(d) \Longrightarrow (a)$ is obvious.

$(a) \Longrightarrow (b)$ Assume that $a \in N_2(R)$. Since $R$ is left weakly almost pp, $l(a) = \Sigma_{i\in I} Re_i$, where $\{e_i | i \in I\} \subseteq E(R)$. For any $x \in l(a)$, there exists positive integer $n$ such that $x \in \Sigma_{i=1}^{n} Re_i$. Since $R$ is abelian, there exists a $e \in E(R)$ such that $\Sigma_{i=1}^{n} Re_i = Re$. Therefore $x = xe = ex$.

$(b) \Longrightarrow (c)$ We first claim that $R$ is abelian. Let $e \in E(R)$. For any $x \in R$, set $h = ex - exe$. Then $he = 0$ and $eh = h$. Since $e \in N_2(R)$, $h \in l(e)$, by (b), there exists a $g \in E(R)$ such that $g \in l(e)$ and $h = gh = hg$. Since $ge = 0$, $g = g(1-e)$. Therefore $h = gh = g(1-e)h = gh - geh = gh - gh = 0$, this implies $R$ is abelian. Next, let $a \in N_2(R)$ and let $\{e_i | i \in I\}$ are the set of all idempotents containing in $l(a)$. Clearly, $l(a) = \Sigma_{i\in I} Re_i$. For any $n \geq 1$ and $x \in l(a^{n+1})$, then $xa^n \in l(a)$. By (b), there exists a $e \in E(R)$ such that $e \in l(a)$ and $xa^n = exa^n = xa^n e$. Clearly, $xa^n e = xea^n = 0$ because $R$ is abelian. Hence $xa^n = 0$ and so $x \in l(a^n)$. This implies that $l(a^n) = l(a^{n+1})$ for any positive integer $n$. Finally we assume that $y \in l(a)$, then there exists a $e^2 = e \in l(a)$ such that $y = ye = ey$. Hence $ay = aey = eay = 0$, so we have $y \in r(a)$. Therefore $l(a) \subseteq r(a)$.

$(c) \Longrightarrow (d)$ Let $a \in N_2(R)$ and $x \in l(a) \cap Ra$. Then $xa = 0$ and $x = ba$ for some $b \in R$. Clearly $b \in l(a^2)$. By (c), $b \in l(a)$, so we have $x = ba = 0$. □

The following theorem is similar to Chen and Ding, (2001, Theorem 2.9).

**Theorem 4.2.** *Let $R$ be a ring, then the following conditions are equivalent:*

(1) *$R$ is a left weakly pp left $np-$ injective ring.*

(2) *For each $a \in N_2(R)$, $aR = eR$, where $e^2 = e \in R$.*

(3) *For each $a \in N_2(R)$, $Ra = Rg$, where $g^2 = g \in R$.*

(4) *$R$ is a right weakly pp right $np-$ injective ring.*

(5) *$R$ is a left weakly pp left $WC2$ ring.*

*Proof.* (2) $\Leftrightarrow$ (3) and (1) $\Longrightarrow$ (5) are clear.

(5) $\Rightarrow$ (2) Let $a \in N_2(R)$. Since $R$ is left weakly *pp*, $l(a) = Re$ for some $e \in E(R)$. Hence $Ra \cong Re$. Since $R$ is left $WC2$, $Ra = Rg$ for some $g \in E(R)$.

(3) $\Rightarrow$ (1). Let $a \in N_2(R)$. Since $Ra = Rg, g^2 = g$, $Ra$ is projective in ${}_RR$. Hence $R$ is left weakly *pp*. Let $a = ag, g = ba, b \in R$ and let $e = ab$. Then $e^2 = e$ and $aR = eR$. Now because $l(a) = R(1-e)$, $rl(a) = eR$. Hence $aR = eR = rl(a)$ and so $R$ is left *np*- injective.

Similarly, we can show (3) $\Longleftrightarrow$ (4). □

We call a ring $R$ $W-$ regular if it satisfies the conditions in Theorem 4.2. According to Wei and Chen, (2007), a ring $R$ is called $n-$ regular if for every $a \in N(R)$, $a = aba$ for some $b \in R$. Clearly, $R$ is regular if and only if $R$ is $W-$ regular and $n-$ regular.

Similar to Wei and Chen, (2001, Theorem 2.18), we have the following theorem.

**Theorem 4.3.** *The following conditions are equivalent for a ring $R$.*

(1) *$R$ is a $w-$ regular ring.*

(2) *Every left $R-$module is $np-$ injective.*

(3) *Every cyclic left $R-$module is $np-$ injective.*

Call a right $R-$ module $M$ $w-$ flat if, for any $a \in N_2(R)$ and the inclusion mapping $\iota : Ra \longrightarrow R$, mapping $1_M \otimes \iota : M \otimes_R Ra \longrightarrow M \otimes_R R$ is always monomorphism. Clearly, right $R-$ module $M$ is flat if and only if $M$ is $Nflat$ (cf. Wei and Chen, (2008)) and $w-$flat.

Similar to Wei and Chen, (2008, Theorem 4.5 and Theorem 4.7), we have the following theorem.

**Theorem 4.4.** (1) *Right $R-$ module $B$ is $w-$ flat if and only if $B^* \stackrel{\text{def}}{=} Hom_Z(B, Q/Z)$ is $np-$ injective, where $Q$ is the field of real number.*

(2) *The following conditions are equivalent for a ring $R$:*

(a) *$R$ is a $w-$ regular ring.*

(b) *Every right $R$- module is $w-$ flat.*

(c) *Every cyclic right $R$- module is $w-$ flat.*

Recall that a ring $R$ is strongly regular if for any $a \in R$, $a \in a^2R$. It is well known that $R$ is a strongly regular if and only if $R$ is an abelian ring and regular ring. On the other hand, $R$ is regular if and only if $R$ is left pp and left $C2$. Hence by theorem 3.4 and Proposition 3.5, we have:

**Corollary 4.5.** *The following conditions arequivalent for a ring $R$:*

(1) *$R$ is strongly regular*

(2) *$R$ is abelian left pp and left weakly $np-$ injective.*

(3) *$R$ is abelian left pp and left $np-$ injective.*

(4) *$R$ is abelian left $PP$ and left $WC2$.*

According to Chen and Ding, (2001), an element $a$ of a ring $R$ is called generalized $\Pi-$ regular if there exists a positive integer $n$ such that $a^n = a^nba$ for some

$b \in R$. A ring $R$ is called generalized $\Pi-$ regular if, every element of $R$ is generalized $\Pi-$regular. In Chen and Ding, (2001) it is shown that a ring $R$ is regular if and only if $N_1(R) = \{0 \neq a \in R \mid a^2 = 0\}$ is regular and $R$ is generalized $\Pi-$ regular if and only if every cyclic left $R-$ module is $YJ-$ injective. We generalize the result as follows.

**Theorem 4.6.** *The following conditions are equivalent for a ring $R$ with $N_1(R) = \{0 \neq a \in R \mid a^2 = 0\}$ is regular.*

(1) *$R$ is a regular ring.*
(2) *$R$ is a left pp left weakly np$-$ injective ring.*
(3) *Every left $R$- module is left weakly np$-$ injective.*
(4) *Every cyclic left $R$- module is left weakly np$-$ injective.*
(5) *Every principal left ideal of $R$ is left weakly np$-$ injective.*
(6) *Every proper principal left ideal of $R$ is left weakly np$-$ injective.*
(7) *Every essential left ideal of $R$ is left weakly np$-$ injective.*

*Proof.* (1) $\Longrightarrow$ (2), (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6) and (3) $\Longrightarrow$ (7) are evident.

(2) $\Longrightarrow$ (1) Let $0 \neq a \in R$. If $a$ is nilpotent, then there exists a minimal positive integer $n$ such that $a^n = 0$ and $a^{n-1} \neq 0$. If $n = 2$, then $a \in N_1(R)$ and so $a$ is regular. If $n > 2$, then $a^{n-1}$ is regular, Hence $a$ is generalized $\Pi-$ regular. Hence we can assume that $a$ is not nilpotent. Since $R$ is left weakly $np-$ injective, there exists a positive integer $m$ such that $rl(a^m) = a^m R$. Because $R$ is left $pp$, $l(a^m) = Re, e^2 = e \in R$. Hence $a^m R = (1-e)R$ and so $a$ is $\Pi-$ regular. Therefore, we always have $a$ is generalized $\Pi-$ regular.

(6)$\Longrightarrow$ (1) Let $0 \neq a \in R$. If $Ra = R$, we are done. Hence we can assume that $Ra \neq R$ and $a$ is not nilpotent. By (6), $Ra$ is left weakly $np-$ injective, then there exists a positive integer $n$ such that any homomorphism of $Ra^n$ into $Ra$ can be extended to one of $R$ into $Ra$. Hence there exists a $c \in R$ such that $a^n = a^n ca$ and so $a$ is generalized $\Pi-$ regular.

(7)$\Longrightarrow$ (5) Let $0 \neq a \in R$. Then there exists a left ideal $L$ of $R$, respect to property "$Ra \cap L = 0$" maximal. Hence $Ra \oplus L$ is essential left ideal of $R$. By (7), $Ra \oplus L$ is left weakly $np-$ injective, then we can easy to show that $Ra$ is left weakly $np-$ injective. □

By Corollary 2.5(2) and Theorem 4.1(2), we have the following proposition.

**Proposition 4.6.** (1) *If $R$ is a left weakly pp ring, then every left $R-$ module possesses the largest np$-$ injective submodule.*

(2) *Among submodules $B$ of a left $R-$ module $M$ with np$-$ injective factor modules there exists the smallest one, which we denote by $WD(M)$.*

As an immediate result of Proposition 4.7, we have the following theorem.

**Theorem 4.7.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is a left weakly pp ring.*
(2) *For every left $R-$ module $M$, $WD(M)$ is np$-$ injective.*
(3) *For any left $R-$ module $M$ with $WD(M) = M$, $M$ is np$-$ injective.*

Let $a \in N_2(R)$ such that ${}_R Ra$ be projective. Then, clearly, there exists an idempotent $e_a \in R$ such that $a = e_a a$ and $l(a) = l(e_a)$. Hence we can easy yield the following theorem.

**Theorem 4.8.** *Let $R$ be a left weakly pp ring. Then a left $R-$ module $M$ is left $np-$ injective if and only if $a_l : M \longrightarrow e_a M$ induced by the left operation of $a$, is epimorphic for every $a \in N_2(R)$.*

## 5. Application

Since division rings are regular, every module over division rings is $p-$ injective. Hence every left (right) module over division rings is left (right) weakly $np-$ injective. we now characterize division rings in terms of the following notion: $R$ is called a left $F-$ ring (cf. Ming, (1983)) if, for any maximal left ideal $M$ of $R$, any $b \in M$, $R/Mb$ is flat left $R-$ module. Division rings are left (right) $F-$ rings.

A ring $R$ is called left uniform if and only if every non-zero left ideal is an essential left ideal of $R$.

**Theorem 5.1.** *The following conditions are equivalent for a semiprime left uniform ring $R$:*

(1) *$R$ is a division ring.*

(2) *$R$ is a left $p-$ injective left $F-$ ring.*

(3) *$R$ is a left $YJ-$ injective left $F-$ ring.*

(4) *$R$ is a left weakly $np-$ injective left $F-$ ring.*

*Proof.* It is evident that (1) implies (2), which, in turn, implies (3) and (4).

Assume (4). If $b \in R, b \notin Z_l(R)$. Since $R$ is left uniform ring, $l(b) = 0$ which implies $bc = 1$ for some $c \in R$ by theorem 2.3. This shows that every maximal right ideal of $R$ is contained in $Z_l(R)$, whence $R$ is a local ring with $Z_l(R) = J(R)$. Since $R$ is a left $F-$ ring, $J(R)^2 = 0$ and so $J(R) = 0$, because $R$ is a semiprime ring. Hence $R$ is a division ring. □

$R$ is called a left $CAM-$ ring if, for any essential maximal left ideal $M$ of $R$ (if it exists), for any left subideal $I$ of $M$ which is either a complement left subideal of $M$ or a left annihilator ideal in $R$, $I$ is an ideal of $M$.

Left $CAM-$ rings generalize semisimple artinian rings. In Ming, (1983) it is shown that semiprime left $CAM$- ring $R$ is either semisimple artinian or reduced. If $R$ is also left $np-$ injective, then $R$ is either semisimple artinian or strongly regular ring. We yield the following theorem, because reduced left weakly $np-$ injective ring is left $np-$ injective ring.

**Theorem 5.2.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is either semisimple artinian or strongly regular.*

(2) *$R$ is a semiprime left $CAM-$ ring whose singular simple right modules are flat.*

(3) *$R$ is a semiprime left weakly $np-$ injective, left $CAM-$ ring.*

(4) *$R$ is a semiprime $MERT$ left $CAM-$ ring whose singular simple right $R-$ modules are $YJ-$ injective.*

(5) *$R$ is a semiprime $MERT$ left $CAM-$ ring whose singular simple right $R-$ modules are $np-$ injective.*

(6) *$R$ is a semiprime $MERT$ left $CAM-$ ring whose singular simple right $R-$ modules are weakly $np-$ injective.*

*Proof.* (1) implies (2) and (1)$\Longleftrightarrow$(3) are evident.

(2)$\Longrightarrow$(1) If $R$ is not a semisimple artinian ring, then $R$ is reduced. Let $0 \neq a \in R$, if $aR \oplus r(a) \neq R$, then $aR \oplus r(a) \subseteq M$ for some maximal right ideal $M$ of $R$. If $M$ is not essential right ideal of $R$, then $M = (1-e)R, e^2 = e \in R$. Because $R$ is reduced, $ae = ea = 0$ and $e \in r(a) \subseteq M = r(e)$, a contradiction. Hence $M$ is an essential right ideal of $R$ and so $R/M$ is a singular simple right $R-$ module. By (2), $R/M$ is flat, then there exists a $m \in M$ such that $a = ma$. But then $a = am$, because $R$ is reduced. Now we obtain $1 - m \in r(a)$, and so $1 \in M$, a contradiction. Hence $aR \oplus r(a) = R$ and then $R$ is a strongly regular ring.

(1) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6) are clear.

(6) $\Longrightarrow$ (1) We can assume directly that $R$ is reduced. Let $0 \neq a \in R$, if $aR \oplus r(a) \neq R$, then $aR \oplus r(a) \subseteq M$ for some essential maximal right ideal $M$ of $R$. Hence $R/M$ is a singular simple right $R-$ module. By (6), $R/M$ is weakly $np-$ injective, then there exists a positive integer $n$ and a $c \in R$ such that $1 - ca^n \in M$. But then $1 \in M$, because $R$ is a $MERT$ ring and $M$ is an ideal. It is a contradiction. Hence $aR \oplus r(a) = R$ and then $R$ is a strongly regular ring. □

A ring $R$ is called left $CM$ (cf. Ming, (1983)) if, for any essential maximal left ideal $M$ of $R$, every complement left subideal is an ideal of $M$, and $R$ is said to be left $PS$ ring (cf. Nicholson and Watters, (1988)) if $Soc(_RR)$ is projective left $R-$ module. Note that left finite embedded left $PS$ ring need not semiprime. We conclude with a few characteristic properties of semisimple artinian rings.

**Theorem 5.3.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is a semisimple artinian ring.*

(2) *$R$ is a left $CM$, left finite embedded and left $PS$ ring.*

(3) *$R$ is a semiprime left weakly $np-$ injective, left or right Goldie ring.*

*Proof.* (1) implies (2) and (3) are evident.

(2)$\Longrightarrow$ (1) Since $R$ is a left $PS$ left finite embedded ring, $Soc(_RR)$ is semisimple projective left $R-$ module. Since $R$ is a left $CM$ ring, $Soc(_RR)$ is injective as left $R$- module. Hence $Soc(_RR) = Re$, $e^2 = e \in R$. But then $Soc(_RR) = R$, because $Soc(_RR)$ is essential in $_RR$. Hence $R$ is semisimple artinian.

(3)$\Longrightarrow$ (1) Clearly, $R$ has left (or right) fraction ring $Q$, and $Q$ is semisimple artinian ring. If $Q$ is left fraction ring, then for every $x \in Q$, $x = a^{-1}b$, where $a, b \in R$ and $l(a) = r(a) = 0$. Since $R$ is left weakly $np-$ injective, there exists a $c \in R$ such that $ac = 1$ and then $ca = 1$. Hence $a^{-1} \in R$ and so $x \in R$. Thus $R = Q$ is a semisimple artinian ring. □

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