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A Voronovskaya Type Theorem on Modified Post-Widder Operators Preserving x^2

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Abstract. In this paper we obtain a Voronvskaya type theorem for modified Post-Widder operators.

1. Introduction

The Post-Widder operators

$$P_n(f;x) \equiv P_n(f(t);x) := \int_0^\infty f(t)p_n(x,t)dt.x \in I, n \in N, \tag{1}$$

$$p_n(x,t) := \frac{(n/x)^n t^{(n-1)}}{(n-1)!} exp\left(\frac{-nt}{x}\right) \tag{2}$$

$I = (0,\infty), N = \{1,2,...\}$ were examined in many papers and monographs(e.g.[2]), for real-valued functions f bounded on I. It is known (Chapter 9, [2]) thatP_n are well defined also for functions $e_k(x) = x^k, k \in N_0 = N \cup 0$, for $x \in I$ and $n \in N$. Denoting by

$$\varphi_x(t) := t - x \text{ for } t \in I \text{ and a fixed } x \in I, \tag{3}$$

we have

$$P_n\big(\varphi_x^2(t;x)\big) = \frac{x^2}{n} \text{ for } x \in I, n \in N. \tag{4}$$

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In papers [6] was examined approximation properties certain modified Post-Widder operators for differentiable functions in polynomial weighted spaces. In [7] were investigated modified Post-Widder operators P_n^* preserving the function $e_2(x) = x^2$ and was proved that these operators have better approximation properties than classical Post-Widder operators. The similar results were given for certain positive linear operators in the paper [3], [4] and [5].

The purpose of this paper is to give a Voronovskaya type theorem for modified Post-Widder operators P_n^* preserving $e_2(x) = x^2$ in polynomial weighted spaces which is given in [7]. These operators have better approximation properties than P_n given by (1). The definition and some properties of operators P_n^* will be given in section 2. The main theorems will be given section 3.

2. The definition and elementary properties of P_n^*

2.1. We introduce for $f \in C_r, r \in N_0$, the following modified Post-Widder operators P_n^*

$$P_n^*(f;x) := \int_0^\infty f(t)p_n(u_n(x),t)dt = P_n(f;u_n(x)).x \in I, n \in N, \tag{5}$$

where $P_n(f)$ and p_n are given by (1) and (2) and

$$u_n(x) := \sqrt{\frac{n}{n+1}}x, \quad 0 < u_n(x) < x, \text{ for } x \in I, n \in N. \tag{6}$$

From (5)-(6) we immediately obtain the following lemma:

Lemma 2.1. *Let $e_k(x) = x^k$ for $k \in N_0$ and $x \in I$. Then for all $x \in I$ and $n \in N$ we have*

$$P_n^*(e_0;x) = 1, P_n^*(e_1;x) = u_n(x), P_n^*(e_2;x) = x^2. \tag{7}$$

The formulas (7) show that P_n^* preserve the functions e_0 and e_2.

Now, fix $b > 0$ and consider the lattice homomorphism $T_b : C[0,+\infty) \to C[0,b]$ defined by $T_b(f) := f \,\|_[0,b]$ for every $f \in C[0,+\infty)$. In this case, we see that, for each $i = 0, 1, 2$.

$$\lim_{n\to\infty} T_b\left(P_n^*(e_i)\right) = T_b(e_i)$$

uniformly on [0,b]. On the other hand, with the universal Korovkin type property with respect to monotone operators (see Theorem 4.1.4 (vi) of[1], p.199) we have the following : Let X be a compact set and H be a confinal subspace of $C(X)$. If E a Banach lattice, $S : (X) \to E$ is a lattice homorphism and if $\{L_n\}$ is a sequence of positive linear operators from $C(X)$ into E such that $\lim_{n\to\infty}(L_n)(h) = S(h)$ for all $h \in H$, then $\lim_{n\to\infty}(L_n(f) = f)$ provided that f belongs to the Korovkin closure of H.

Convergence theorem

Theorem 1. *Let the sequence* $\{P_n^*\}$ *of positive linear operators given by* (5)*, and the sequence* $u_n(x)$ *defined by* (6)*. Then,*
(i) P_n^* *is a positive linear operators on* $C_r(I)$
(ii) $P_n^*(e_2;x) = e_2(x) = x^2$
(iii) $\lim_{n\to\infty} (P_n^*)f \to f$ *uniformly* $[0,b]$ *for any* $b > 0$.

Hence using limit and applying [1] the theorem follows.

Lemma 2.2. *For every* $x \geq 0$, *we have*

$$P_n^*(\varphi_x;x) = x\sqrt{\frac{n}{n+1}} - x, \tag{8}$$

$$P_n^*(\varphi_x^2;x) = 2x^2\left[1 - \sqrt{\frac{n}{n+1}}\right], \tag{9}$$

$$P_n^*(\varphi_x^3;x) = 2x^3\left[2\sqrt{\frac{n}{n+1}} + \frac{1}{\sqrt{n(n+1)}} - 1\right], \tag{10}$$

$$P_n^*(\varphi_x^4;x) = x^4\left[\frac{(n+2)(n+3)}{n(n+1)} - 8\sqrt{\frac{n+1}{n}} + 7\right]. \tag{11}$$

Proof. By linearity of P_n^* we have

$$P_n^*(\varphi_x^2(t;x)) = P_n^*(e_2;x) - 2xP_n^*(e_1;x) + x^2P_n^*(e_0;x)$$

$$= x^2 - 2xu_n + x^2$$

$$= 2x(x - u_n).$$

by (6) we get

$$P_n^*(\varphi_x^2(t;x)) = 2x^2\left[1 - \sqrt{\frac{n}{n+1}}\right]$$

similarly we obtain other results. □

3. A Voronovskaya type theorem

In this section, we prove a Voronovskaya type theorem for the operators P_n^* given by (5).

We first need the following lemmas.

Lemma 3.1. $\lim_{n\to\infty} n\,(x - u_n(x)) = \frac{x}{2}$.

Proof.

$$\begin{aligned}\lim_{n\to\infty} n\,(x - u_n(x)) &= \left[x - x\sqrt{n/n+1}\right] \\ &= \frac{nx}{n+1+\sqrt{n^2+n}} \\ &= \frac{x}{2}.\end{aligned}$$

□

Lemma 3.2. $\lim_{n\to\infty} n^2 P_n^*\left(\varphi_x^4\; x\right) = 3x^4$ *uniformly with respect to* $x \in [0,b]$ *with* $b > 0$.

Proof. Then by using Lemma(2.2) and after some calculations, we may write that

$$\begin{aligned}n^2 P_n^*\left(\varphi_x^4; x\right) &= x^4\left[\frac{n(n+1)(n+3)}{n+1} - 8n\sqrt{n(n+1)} + 7n^2\right] \\ &= x^4\left[\frac{4n^2+6n}{n+1} - \frac{8n^2}{n+\sqrt{n(n+1)}}\right] \\ &= x^4\left[\frac{6n}{n+1} - \frac{12n^3+16n^2}{(n+1)(n+\sqrt{n^2+n}(\sqrt{n^2+n}+n+2))}\right] \\ &= 3x^4\end{aligned}$$

□

Theorem 2. *For every* $f \in C_r$ *such that* $f', f'' \in C_r$ *we have*

$$\lim_{n\to\infty} n\left[P_n^*(f;x) - f(x)\right] = x/2\left[xf''(x) - f'(x)\right]$$

uniformly with respect to $x \in [0,b], (b > 0)$.

Proof. Let $f, f', f'' \in C_r$. Define

$$\psi(y,x) = \begin{cases} \frac{f(y)-f(x)-(y-x)f'(x)-\frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2}, & \text{if } y \neq x; \\ 0, & \text{if } y = x. \end{cases}$$

Then by assumption we have $\psi(x,x) = 0$ and the function $\psi(.,x)$ belongs to C_r. Hence, by Taylor's theorem we get

$$f(y) = f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2}f"(x) + (y-x)^2\psi(y,x).$$

Now from Lemma (2.2)

(12) $n\left[P_n^*(f;x) - f(x)\right] = n(x - u_n(x))(xf"(x) - f'(x)) + nP_n^*\left(\varphi_x^2(y)\psi(y,x);x\right)$

If we apply the Cauchy-Schwarz inequality for the second term on the right hand side of (12), then we conclude that

$$n\left|P_n^*(\varphi_x^2(y)\psi(y,x);x)\right| \leq (n^2 P_n^*(\varphi_x^4(y);x))^{\frac{1}{2}}(P_n^*(\psi^2(y,x);x))^{\frac{1}{2}} \tag{13}$$

Let $\eta(y,x) := \psi^2(y,x)$. In this case, observe that $\eta(x,x) = 0$ and $\eta(.,x) \in C_r$. Then it follows from Theorem 1. that

$$\lim_{n\to\infty} P_n^*\left(\psi^2(y,x);x\right) = \lim_{n\to\infty} P_n^*\left(\eta(y,x);x\right) = \eta(x,x) = 0 \tag{14}$$

uniformly with respect to $x \in [0,b], (b > 0)$. Now considering (13) and (14), and also using lemma 3.2, we immediately see that

$$\lim_{n\to\infty} nP_n^*\left(\varphi_x^2(y)\psi(y,x);x\right) = 0 \tag{15}$$

uniformly with respect to$x \in [0,b]$. Then taking limit as $n \to \infty$ in (12) and using (15) we have

$$\lim_{n\to\infty} n\left[P_n^*(f;x) - f(x)\right] = x/2\left[xf''(x) - f'(x)\right\},$$

uniformly with respect to $x \in [0,b]$.

The proof is completed. □

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