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Degenerate Weakly (k_1, k_2)-Quasiregular Mappings

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Abstract. In this paper, we first give the definition of degenerate weakly (k_1, k_2)-quasiregular mappings by using the technique of exterior power and exterior differential forms, and then, by using Hodge decomposition and Reverse Hölder inequality, we obtain the higher integrability result: for any q_1 satisfying

$$0 < k_1\binom{n}{l}^{3/2}n^{l/2} \times 2^{n+1}l \times 100^{n^2}\left[2^l(2^{n+3l}+1)\right](l-q_1) < 1$$

there exists an integrable exponent $p_1 = p_1(n, l, k_1, k_2) > l$, such that every degenerate weakly (k_1, k_2)-quasiregular mapping $f \in W^{1,q_1}_{loc}(\Omega, R^n)$ belongs to $W^{1,p_1}_{loc}(\Omega, R^n)$, that is, f is a degenerate (k_1, k_2)-quasiregular mapping in the usual sense.

Denote by $\Lambda^l = \Lambda^l(R^n), l = 1, 2, \cdots, n$ the linear space of l-exterior forms (also called l-covectors) in R^n. It is a linear space of alternating l-tensors. Set $\Lambda^0 = R$ and $\Lambda^l(R^n) = 0$ for $l < 0$ or $l > n$. The dimension of $\Lambda^l(R^n)$ is $\binom{n}{l}$. The Hodge star operator $* : \Lambda^l(R^n) \to \Lambda^{n-l}(R^n)$ is defined for $\alpha, \beta \in \Lambda^l$ by

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$\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mathrm{vol}$, where vol be the volume element in R^n. Thus the Hodge star is a linear isometry between Λ^l and Λ^{n-l}. For $l = 0$ and $l = n$ this formula reads $*1 = \mathrm{vol}$ and $*\mathrm{vol} = 1$.

Let $\Omega \subset R^n$ be a domain. A differential l-form α on Ω is simply a locally integrable function or Schwarz distribution on Ω with values in $\Lambda^l = \Lambda^l(R^n)$. We write $\alpha \in D'(\Omega, \Lambda^l)$. If we denote by $x_1, x_2, \cdots, x_n$ the coordinate in R^n, then the differential form $\alpha : \Omega \to \Lambda^l(R^n)$ may be written uniquely as

$$\alpha(x) = \sum_{1 \le i_1 < \cdots < i_l \le n} \alpha_{i_1 \cdots i_l}(x) dx_I = \sum_{1 \le i_1 < \cdots < i_l \le n} \alpha_{i_1 \cdots i_l}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

here $\alpha_{i_1 \cdots i_l}(x)$ be functions or distributions and $I = (i_1, i_2, \cdots, i_l)$ be ordered l-tuples. The coindex J of I is an ordered $(n-l)$-tuple, consisting of index in $N = (1, 2, \cdots, n)$ but not in I. The exterior derivative $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ is a linear operator, determined uniquely by the following conditions:

(i) for $l = 0$, df is the differential of f.

(ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^l \alpha \wedge \beta$ for $\alpha \in D'(\Omega, \Lambda^l)$ and $\beta \in D'(\Omega, \Lambda^k)$.

(iii) $d(d\alpha) = 0$ (Poincaré's Lemma).

The formal adjoint d^* of d is called the Hodge codifferential, is given by

$$d^* = (-1)^{nl+1} * d* : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l).$$

The spaces of exact and coexact l-forms are defined, respectively, by

$$\ker(d) = \{\omega \in D'(\Omega, \Lambda^l) : d\omega = 0\}$$

$$\ker(d^*) = \{\omega \in D'(\Omega, \Lambda^l) : d^*\omega = 0\}$$

Let G be an $n \times n$ matrix. The l-exterior power of G is a linear operator

$$G^l_\# : \Lambda^l(R^n) \to \Lambda^l(R^n)$$

defined by

$$G^l_\#(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_l) = G\alpha_1 \wedge G\alpha_2 \wedge \cdots \wedge G\alpha_l,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_l \in \Lambda^1(R^n)$. We have the obvious relations $G^n_\# = \det G, G^1_\# = G$. The linear transform $G^l_\#$ can be expressed as an $\binom{n}{l} \times \binom{n}{l}$ matrix whose entries are $l \times l$ minors of G and denoted by $G^l_\# = (\det A^I_J) \in R^{\binom{n}{l} \times \binom{n}{l}}$, where I, J be ordered l-tuples and

$$\det A^I_J = \det \begin{bmatrix} A^{i_1}_{j_1} \cdots A^{i_1}_{j_l} \\ \cdots \\ A^{i_l}_{j_1} \cdots A^{i_l}_{j_l} \end{bmatrix}$$

If A, B are two matrices, then we define $\langle A, B \rangle = Tr(B^T A), |A|^2 = \langle A, A \rangle$, here B^T is the transpose of B. A useful inequality of $G^l_\#$ is (See [6,P220])

$$\binom{n}{k}^l |G^l_\#|^{2k} \le \binom{n}{l}^k |G^k_\#|^{2l}, \tag{1}$$

where $1 \le k \le l \le n$. Take $k = 1$ in the above inequality yields

(2) $$\binom{n}{1}^l |G^l_{\#}|^2 \le \binom{n}{l} |G|^{2l}$$

Moreover, if $l = n$, then

(3) $$n^{n/2} |\det G| \le |G|^n$$

For mapping $f = (f^1, f^2, \cdots, f^n) \in W^{1,q}_{loc}(\Omega, R^n), (1 \le q < \infty)$, denoted by $Df(x) = \left(\frac{\partial f^i}{\partial x_j}\right)_{1 \le i,j \le n}$ and $J_f(x) = \det Df(x)$ the Jacobi matrix and the Jacobian of f, respectively. In [3], the author gave the following definition.

Definition 1. A mapping $f = (f^1, f^2, \cdots, f^n) \in W^{1,q}_{loc}(\Omega, R^n), (1 \le q < \infty)$ is called weakly (k_1, k_2)-quasiregular, $(1 \le k_1 < \infty, 0 \le k_2 < \infty)$, if $J_f(x) > 0$, a.e.Ω and

$$|Df(x)|^n \le k_1 n^{n/2} J_f(x) + k_2, \quad \text{a.e.}\Omega$$

If $q \ge n$, then f is called weakly (k_1, k_2)-quasiregular mapping.

We now give a more general definition.

Definition 2. A mapping $f = (f^1, f^2, \cdots, f^n)$ is called degenerate weakly (k_1, k_2)-quasiregular, $(1 \le k_1 < \infty, 0 \le k_2 < \infty)$, if

(i) $f \in W^{1,q}_{loc}(\Omega, R^n), 1 \le q < \infty$;

(ii) there exists l, $1 < l \le n$, such that $(Df)^l_{\#} = (\det A^I_J)_{\binom{n}{l} \times \binom{n}{l}} \ne 0, \det A^I_J > 0$, a.e., and $(Df)^k_{\#} = 0$, a.e., $k = l + 1, \cdots, n$.

(iii) $|Df(x)|^l \le k_1 \binom{n}{l}^{-1/2} n^{l/2} |(Df(x))^l_{\#}| + k_2$, a.e. Ω.

If $q \ge l$, then f is called degenerate (k_1, k_2)-quasiregular.

Remark 1. If $l = n$, then Definition 2 coincides with Definition 1. Definition 2 can be used in degenerate case since there are only the l minors of $Df(x)$ in (iii), here the word *degenerate* means $J_f(x) = 0$, a.e.Ω, and the rank of $Df(x)$ is $l : 1 < l \le n$. See [11] for some results of degenerate quasiregular mappings.

Qusiregular mappings begun to be studied by Yu. G. Reshetnyak in 1966. See also the monograph [9]. The main results in [9] are the discrete and openness for quasiregular mappings. A few years later, O. Martio, S. Rickman and J. Väsälä established the normal family and distributive theories of quasiregular mappings by using the method of modulus of space curve families and the inequalities for modulus. See also [10]. In the 1990's, T. Iwaniec, G. Martin and C. Sbordone obtained the Liouville theorem, regularity and removability theories of quasiregular mappings in even dimensions ([7]) by using the result of *quasiconformal 4-manifolds* established by S. K. Donaldson and D. P. Sullivan. Then, they generated the regularity and removability results to all dimensions by using the method of harmonic analysis and Sobolev space method for partial differential equations ([8]).

Weakly (k_1, k_2)-quasiregular mappings are generalizations of quasiregular mappings ([3,12]). In [3], the authors gave its definition and obtained the regularity result. In the mean time, [3] obtained the integrable exponent estimate for very weak solutions. In this paper, we generalize the result of [3] to degenerate case.

Theorem. *For any q_1 satisfying $0 < k_1\binom{n}{l}^{3/2} n^{l/2} \times 2^{n+1} l \times 100^{n^2}\left[2^l(2^{n+3l}+1)\right](l-q_1) < 1$, there exists an integrable exponent $p_1 = p_1(n, l, k_1, k_2) > l$, such that for any degenerate weakly (k_1, k_2)-quasiregular mapping $f \in W^{1,q_1}_{loc}(\Omega, R^n)$, we have $f \in W^{1,p_1}_{loc}(\Omega, R^n)$, that is, f is degenerate (k_1, k_2)-quasiregular in the usual sense.*

Remark 2. The difficulty of this paper is that for degenerate quasiregular mapping f, if the rank of $Df(x)$ is $l : 1 < l < n$, then $J_f(x) = 0$, a.e., one can not obtain the higher integrability result by using the method of [3]. We will overcome this difficulty by using the technique of differential forms.

We give some lemmas that will be used in the proof of the theorem.

Lemma 1. *If $u \in W^{1,p}_{loc}(\Omega, R^n), 1 \le p < \infty$, then*

$$\|u - \overline{u_{B_r}}\|_{L^p(B_r)} \le 2^{1+n/p} r \|Du\|_{L^p(B_r)}$$

for any ball $B_r \subset\subset \Omega$. Here $\overline{u_{B_r}} = \fint_B u(x)dx$ is the integral mean of u over B_r.

Proof. This lemma follows if we take $\sigma = 1$ in [1, Lemma 1.5]. □

Lemma 2. *Suppose that $1 \le p < n$. If $u \in W^{1,p}_{loc}(\Omega, R^n)$, then*

$$\|u - \overline{u_{B_r}}\|_{L^{\frac{np}{n-p}}(B_r)} \le C(n) \frac{p}{n-p} \left(\frac{p}{p-1}\right)^{(n-p)/np} \|Du\|_{L^p(B_r)}$$

for any ball $B_r \subset\subset \Omega$.

This lemma comes from [5] Lemma 7.16 and 7.12. In the following, $C(n)$ will always denote this constant.

Lemma 3(Hodge decomposition). *Let $\omega \in L^{\tau(1-\varepsilon)}(\Omega, \Lambda^l), \tau \ge 7/4, \varepsilon < 1/2$. Consider the Hodge decomposition*

$$|\omega|^{-\varepsilon}\omega = d\alpha + d^*\beta, \;\; \alpha \in L^\tau_1(\Omega, \Lambda^{l-1}), \beta \in L^\tau_1(\Omega, \Lambda^{l+1})$$

If ω is closed, then

$$\|d^*\beta\|_\tau \le 100^{n^2} \tau\varepsilon \|\omega\|^{1-\varepsilon}_{\tau(1-\varepsilon)} \tag{4}$$

If ω is coclosed, then

$$\|d\alpha\|_\tau \le 100^{n^2} \tau\varepsilon \|\omega\|^{1-\varepsilon}_{\tau(1-\varepsilon)} \tag{5}$$

Lemma 4([4], Reverse Hölder inequality). *Let $0 < 2r < r_0 \le dist(x_0, \partial\Omega), x_0 \in \Omega$. If for functions $g(x) \in L^p(B_{2r})(1 < p < \infty), h(x) \in L^t(B_{2r}), t > p$, it holds*

$$\fint_{B_r} |g(x)|^p dx \le \theta \fint_{B_{2r}} |g(x)|^p dx + C\left(\fint_{B_{2r}} |g(x)|^s dx\right)^{p/s} + \fint_{B_{2r}} |h(x)|^p dx, \tag{6}$$

here $1 \le s < p, 0 \le \theta < 1$*, then there exist exponent* $p' = p'(\theta, p, n, C) > p$*, such that* $g(x) \in L^{p'}_{loc}(\Omega)$*, and*

$$\left(\fint_{B_r} |g(x)|^{p'} dx\right)^{1/p'} \le C_1 \left\{\left(\fint_{B_{2r}} |g(x)|^p dx\right)^{1/p} + \left(\fint_{B_{2r}} |h(x)|^p dx\right)^{1/p}\right\}, \tag{7}$$

here C_1 *depends only on* n, C, p, θ, r_0*.*

Lemma 5. *Let* $F = (F^1, F^2, \cdots, F^n) \in W_0^{1,l-\varepsilon}(\Omega, R^n), 1 \le l \le n, 0 < \varepsilon < 1/2$*, then*

$$\int_\Omega |dF^{i_1}|^{-\varepsilon} dF^{i_1} \wedge \cdots \wedge dF^{i_l} \wedge dx_J \le 2l \times 100^{n^2} \varepsilon \int_\Omega |DF|^{l-\varepsilon} dx$$

for any ordered $(n-l)$-tuple J.

Proof. Firstly, if $|dF^{i_1}| = 0$, then we take $|dF^{i_1}|^{-\varepsilon} dF^{i_1}$ to be 0, the result is obvious. Else, consider the following Hodge decomposition

$$|dF^{i_1}|^{-\varepsilon} dF^{i_1} = d\alpha + d^*\beta. \tag{8}$$

Since $F = 0$ on $\partial\Omega$, then by the uniqueness of Hodge decomposition, $\alpha = \beta = 0$ on $\partial\Omega$. By Poincaré's Lemma we know that dF^{i_1} is a closed form. By Lemma 3, we obtain

$$\|d^*\beta\|_{\frac{l-\varepsilon}{1-\varepsilon}} \le 100^{n^2} \frac{l-\varepsilon}{1-\varepsilon} \varepsilon \|dF^{i_1}\|^{1-\varepsilon}_{l-\varepsilon} \le 2l \times 100^{n^2} \varepsilon \|dF^{i_1}\|^{1-\varepsilon}_{l-\varepsilon}$$

It is obvious that $dF^{i_2} \wedge \cdots \wedge dF^{i_l} \in L^{\frac{l-\varepsilon}{l-1}}(\Omega, \Lambda^{l-1})$. Since $\frac{l-\varepsilon}{1-\varepsilon}$ and $\frac{l-\varepsilon}{l-1}$ are Hölder conjugate exponents, then Stokes theorem yields

$$\begin{aligned} &\int_\Omega d\alpha \wedge dF^{i_2} \wedge \cdots \wedge dF^{i_l} \wedge dx_J = \int_\Omega d(\alpha \wedge dF^{i_2} \wedge \cdots \wedge dF^{i_l} \wedge dx_J) \\ &= \int_{\partial\Omega} \alpha \wedge dF^{i_2} \wedge \cdots \wedge dF^{i_l} \wedge dx_J = 0 \end{aligned} \tag{9}$$

Here we have used the fact that $\alpha = 0$ on $\partial\Omega$. By (9), (10), Hölder inequality and Hadamard inequality, we have

$$\begin{aligned} &\left|\int_\Omega |dF^{i_1}|^{-\varepsilon} dF^{i_1} \wedge \cdots \wedge dF^{i_l} \wedge dx_J\right| \\ = &\left|\int_\Omega (d\alpha + d^*\beta) \wedge dF^{i_2} \wedge \cdots \wedge dF^{i_l} \wedge dx_J\right| \\ = &\left|\int_\Omega d^*\beta \wedge dF^{i_2} \wedge \cdots \wedge dF^{i_l} \wedge dx_J\right| \\ \le &\|d^*\beta\|_{\frac{l-\varepsilon}{1-\varepsilon}} \|dF^{i_2} \wedge \cdots \wedge dF^{i_l} \wedge dx_J\|_{\frac{l-\varepsilon}{l-1}} \\ \le &2l \times 100^{n^2} \varepsilon \|dF^{i_1}\|^{1-\varepsilon}_{l-\varepsilon} \|dF^{i_2}\|_{l-\varepsilon} \cdots \|dF^{i_l}\|_{l-\varepsilon} \\ \le &2l \times 100^{n^2} \varepsilon \int_\Omega |DF|^{l-\varepsilon} dx. \end{aligned}$$

Proof of Theorem. Let $f \in W^{1,l-\varepsilon}_{loc}(\Omega, R^n)(0 < \varepsilon < 1/2)$ be a degenerate weakly (k_1, k_2)-quasiregular mapping. If we set $\Omega_1 = \{x \in \Omega : |Df(x)| \geq 1\}$, $\Omega_2 = \{x \in \Omega : |Df(x)| < 1\}$, then $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \phi$. Let $x_0 \in \Omega$ be arbitrary. $B = B(x_0, r) \subset B(x_0, \frac{3}{2}r) = \frac{3}{2}B \subset B(x_0, 2r) = 2B \subset\subset \Omega$ be concentric balls in Ω.

$$\begin{aligned}\int_B |Df(x)|^{l-\varepsilon}dx &= \int_{B\cap\Omega_1} |Df(x)|^{l-\varepsilon}dx + \int_{B\cap\Omega_2} |Df(x)|^{l-\varepsilon}dx \\ &\leq \int_{B\cap\Omega_1} |Df(x)|^{l-\varepsilon}dx + |B| \end{aligned}\tag{10}$$

By the definition of degenerate weakly (k_1, k_2)-quasiregular mapping, we know that

$$\begin{aligned}&\int_{B\cap\Omega_1} |Df(x)|^{l-\varepsilon}dx \\ \leq &\int_{B\cap\Omega_1} |Df(x)|^{-\varepsilon}\left[k_1\tbinom{n}{l}^{-1/2}n^{l/2}|(Df(x))^l_{\#}| + k_2\right]dx \\ \leq &\; k_1\tbinom{n}{l}^{-1/2}n^{l/2}\int_B |Df(x)|^{-\varepsilon}|(Df(x))^l_{\#}|dx + k_2|B| \end{aligned}\tag{11}$$

Since $(Df(x))^l_{\#} = (\det A^I_J)_{\binom{n}{l}\times\binom{n}{l}}$, and $\det A^I_J > 0$, a.e.Ω, then

$$|(Df(x))^l_{\#}| = \left(\sum_{I,J}(\det A^I_J)^2\right)^{1/2} \leq \sum_{I,J}\det A^I_J$$

So, by Hadamard inequality, we have

$$\begin{aligned}\int_B |Df(x)|^{-\varepsilon}|(Df(x))^l_{\#}|dx &\leq \int_B |Df(x)|^{-\varepsilon}\sum_{I,J}\det A^I_J dx \\ &= \sum_{I,J}\int_B |df^{i_1}(x)|^{-\varepsilon}\det A^I_J dx\end{aligned}\tag{12}$$

Here i_1 is the first index of $I = (i_1, i_2, \cdots, i_l)$.

To estimate the right hand side of (13), we take $\phi(x) \in C^\infty_0(\frac{3}{2}B), \psi(x) \in C^\infty_0(2B)$ (ϕ, ψ be zero outside $\frac{3}{2}, 2B$) to be test functions satisfying

1) $0 \leq \phi \leq 1$, $\phi \equiv 1$, if $x \in B$, $|\bigtriangledown \phi| \leq \frac{4}{r}$,

2) $0 \leq \psi \leq 1$, $\psi \equiv 1$, if $x \in \frac{3}{2}B$, $|\bigtriangledown \psi| \leq \frac{4}{r}$

We introduce the auxiliary function $F \in W^{1,l-\varepsilon}_0(\Omega, R^n)$ to be

$$F = (\psi(f^{i_1} - c^{i_1}), \cdots, \psi(f^{i_{l-1}} - c^{i_{l-1}}), \phi(f^{i_l} - c^{i_l}), \phi(f^J - c^J))$$

Here J is the coindex of I and $c = (c^{i_1}, c^{i_2}, \cdots, c^{i_l}, c^J)c^{i_k}, k = 1, 2, \cdots, n$ are some constants to be determined. Let K be any ordered $(n-l)$-tuple. We have in $\frac{3}{2}B$

$$\begin{aligned}&\phi|df^{i_1}|^{-\varepsilon}df^{i_1}\wedge\cdots\wedge df^{i_l}\wedge dx_K \\ =&|dF^{i_1}|^{-\varepsilon}dF^{i_1}\wedge\cdots\wedge dF^{i_l}\wedge dx_K - (f^{i_l} - c^{i_l})|df^{i_1}|^{-\varepsilon}df^{i_1}\wedge\cdots\wedge df^{i_{l-1}}\wedge d\phi\wedge dx_K\end{aligned}$$

Applying Lemma 5 to the auxiliary function F and notice that $|f^{i_l} - c^{i_l}| \leq |f - c|, l = 1, 2, \cdots, n$, we have

$$\begin{aligned} & \left| \int_B |df^{i_1}|^{-\varepsilon} df^{i_1} \wedge \cdots \wedge df^{i_l} \wedge dx_K \right| \\ \leq\ & 2l \times 100^{n^2} \varepsilon \int_{2B} |DF|^{l-\varepsilon} dx + \int_{2B} |\bigtriangledown \phi||f - c||Df|^{l-1-\varepsilon} dx \\ \leq\ & 2l \times 100^{n^2} \varepsilon I_1 + I_2 \end{aligned} \tag{13}$$

Firstly, we estimate I_1. By the definition of F and the conditions 1) and 2), we can easily derive the estimate

$$|DF| \leq 2(4r^{-1}|f - c| + |Df|)$$

Therefore

$$\int_{2B} |DF|^{l-\varepsilon} dx \leq 2^{l-\varepsilon} \left[4^{l-\varepsilon} r^{\varepsilon - l} \int_{2B} |f - c|^{l-\varepsilon} dx + \int_{2B} |Df|^{l-\varepsilon} dx \right]$$

Take $c = \overline{f_{2B}} = ⨍_{2B} |f| dx$. Applying Lemma 1 to the first term in the right hand side yields

$$I_1 = \int_{2B} |DF|^{l-\varepsilon} dx \leq 2^{l-\varepsilon}(2^{n+3l-3\varepsilon} + 1) \int_{2B} |Df|^{l-\varepsilon} dx \leq 2^l (2^{n+3l} + 1) \int_{2B} |Df|^{l-\varepsilon} dx. \tag{14}$$

Secondly, we estimate I_2.

$$I_2 = \int_{2B} |\bigtriangledown \phi||f - \overline{f_{2B}}||Df|^{l-1-\varepsilon} dx \leq \frac{4}{r} \int_{2B} |f - \overline{f_{2B}}||Df|^{l-1-\varepsilon} dx. \tag{15}$$

If we take $p' = \frac{n(l-\varepsilon)}{n-l+1+\varepsilon}, q' = \frac{n(l-\varepsilon)}{(n+1)(l-1-\varepsilon)}$, then $1 < p', q' < \infty, \frac{1}{p'} + \frac{1}{q'} = 1$, Hölder's inequality yields

$$\begin{aligned} & \int_{2B} |f - \overline{f_{2B}}||Df|^{l-1-\varepsilon} dx \\ \leq\ & \left(\int_{2B} |f - \overline{f_{2B}}|^{\frac{n(l-\varepsilon)}{n-l+1+\varepsilon}} dx \right)^{\frac{n-l+1+\varepsilon}{n(l-\varepsilon)}} \left(\int_{2B} |Df|^{\frac{n(l-\varepsilon)}{n+1}} dx \right)^{\frac{(n+1)(l-1-\varepsilon)}{n(l-\varepsilon)}}. \end{aligned}$$

Take $p'' = \frac{n(l-\varepsilon)}{n+1}, q'' = \frac{n(l-\varepsilon)}{n-l+1+\varepsilon}$, $1 < p'' < n, q'' = \frac{np''}{n-p''}$. The above inequality and

Lemma 2 yields

$$\begin{aligned}
&\int_{2B} |f-\overline{f_{2B}}||Df|^{l-1-\varepsilon} \\
\le\ & C(n)\left(\frac{p''}{n-p''}\right)\left(\frac{p''}{p''-1}\right)^{1/q''}\left(\int_{2B}|Df|^{p''}dx\right)^{1/p''}\left(\int_{2B}|Df|^{p''}dx\right)^{\frac{n-l-\varepsilon}{p''}} \\
\le\ & C(n)\frac{l-\varepsilon}{n-l+1+\varepsilon}\left(\frac{n(l-\varepsilon)}{n(l-\varepsilon)-n-1}\right)^{\frac{n-l+1+\varepsilon}{n(l-\varepsilon)}}\left(\int_{2B}|Df|^{\frac{n(l-\varepsilon)}{n+1}}dx\right)^{\frac{n+1}{n}} \\
\text{(16)}\quad \le\ & C(n)\frac{l}{n-l+1}\left(\frac{nl}{n(l-\frac{1}{2})-n-1}\right)^{\frac{n-l+2}{n(l-1)}}\left(\int_{2B}|Df|^{\frac{n(l-\varepsilon)}{n+1}}dx\right)^{\frac{n+1}{n}}.
\end{aligned}$$

Combining (14), (15), (16) with (17) yields

$$\begin{aligned}
\text{(17)}\quad & \left|\int_B |df^{i_1}|^{-\varepsilon} df^{i_1}\wedge\cdots\wedge df^{i_l}\wedge dx_K\right| \\
\le\ & 2l\times 100^{n^2}\varepsilon\left[2^l(2^{n+3l}+1)\right]\int_{2B}|Df|^{l-\varepsilon}dx \\
& +C(n)\frac{4}{r}\frac{l}{n-l+1}\left(\frac{nl}{n(l-\frac{1}{2})-l-1}\right)^{\frac{n-l+2}{n(l-1)}}\left(\int_{2B}|Df|^{\frac{n(l-\varepsilon)}{n+1}}dx\right)^{\frac{n+1}{n}}
\end{aligned}$$

Take the summation for all ordered l-tuples $I=(i_1,\cdots,i_l)$ and all ordered $(n-l)$-tuples $K=(k_1,\cdots,k_{n-l})$ yields

$$\begin{aligned}
\text{(18)}\quad & \sum_{I,J}\int_B |df^{i_1}|^{-\varepsilon}\det A_J^I dx \le \sum_{I,J}\left|\int_B |df^{i_1}|^{-\varepsilon} df^{i_1}\wedge\cdots\wedge df^{i_l}\wedge dx_K\right| \\
\le\ & \tbinom{n}{l}^2\times 2l\times 100^{n^2}\varepsilon\left[2^l(2^{n+3l}+1)\right]\int_{2B}|Df|^{l-\varepsilon}dx \\
& +C(n)\frac{4}{r}\tbinom{n}{l}^2\frac{l}{n-l+1}\left(\frac{nl}{n(l-\frac{1}{2})-l-1}\right)^{\frac{n-l+2}{n(l-1)}}\left(\int_{2B}|Df|^{\frac{n(l-\varepsilon)}{n+1}}\right)^{\frac{n+1}{n}}
\end{aligned}$$

We thus arrive at

$$\begin{aligned}
\text{(19)}\quad & \int_B |Df(x)|^{l-\varepsilon}dx \\
\le\ & k_1\tbinom{n}{l}^{3/2}n^{l/2}\times 2l\times 100^{n^2}\varepsilon\left[2^l(2^{n+3l}+1)\right]\int_{2B}|Df|^{l-\varepsilon}dx \\
& +k_1\tbinom{n}{l}^{3/2}C(n)\frac{4}{r}\frac{l}{n-l+1}\left(\frac{nl}{n(l-\frac{1}{2})-l-1}\right)^{\frac{n-l+2}{n(l-1)}}\left(\int_{2B}|Df|^{\frac{n(l-\varepsilon)}{n+1}}\right)^{\frac{n+1}{n}} \\
& +(k_2+1)|B|.
\end{aligned}$$

Divided by $|B| = \omega_n r^n$ (ω_n be the volume of the unit ball in R^n) in both sides of the above inequality yields

$$\begin{aligned}
& \fint_B |Df(x)|^{l-\varepsilon} dx \\
\leq \quad & k_1 \tbinom{n}{l}^{3/2} n^{l/2} \times 2^{n+1} l \times 100^{n^2} \varepsilon \left[2^l (2^{n+3l}+1)\right] \fint_{2B} |Df|^{l-\varepsilon} dx \\
& + k_1 \tbinom{n}{l}^{3/2} C(n) 2^{n+3} \omega_n^{1/n} \frac{l}{n-l+1} \left(\frac{nl}{n(l-\frac{1}{2})-l-1}\right)^{\frac{n-l+2}{n(l-1)}} \left(\fint_{2B} |Df|^{\frac{n(l-\varepsilon)}{n+1}}\right)^{\frac{n+1}{n}} \\
& + (k_2+1) \\
(20) \quad = \quad & C_1(n,l,k_1)\varepsilon \fint_{2B} |Df|^{l-\varepsilon} dx + C_2(n,l,k_1) \left(\fint_{2B} |Df|^{\frac{n(l-\varepsilon)}{n+1}}\right)^{\frac{n+1}{n}} + (k_2+1),
\end{aligned}$$

here the values of $C_1(n,l,k_1), C_2(n,l,k_1)$ are obvious.

Take $\varepsilon_0 = 1/C_1(n,l,k_1) > 0$. Thus, if $\varepsilon < \varepsilon_0$, then $0 < C_1(n,l,k_1)\varepsilon = \theta < 1$. In this case, the above inequality becomes

$$\fint_B |Df(x)|^{l-\varepsilon} dx \leq \theta \fint_{2B} |Df|^{l-\varepsilon} dx + C_2(n,l,k_1) \left(\fint_{2B} |Df|^{\frac{n(l-\varepsilon)}{n+1}}\right)^{\frac{n+1}{n}} + (k_2+1)$$

The above inequality is a reverse Hölder inequality since $\frac{n(l-\varepsilon)}{n+1} < l - \varepsilon$. By Lemma 4, there exists $p' > p$, such that $|Df(x)| \in L_{loc}^{p'}(\Omega)$. Suppose $I = \{p \in [l-\varepsilon, l] : |Df(x)| \in L_{loc}^p(\Omega)\}$, it is obvious that $q_1 \in I$, and we know that I is closed by the uniform estimate of the above reverse Hölder inequality. By Lemma 4 again, we know that I is relatively open. Hence $I = [l-\varepsilon, l]$, that is, $|Df(x)| \in L_{loc}^l(\Omega)$. The above procedure is also true for $p = l$, by Lemma 4, there exist $p_1 = p_1(n,l,k_1,k_2) > l$, such that $|Df(x)| \in L_{loc}^{p_1}(\Omega)$. By Sobolev Imbedding theorem, $f \in W_{loc}^{1,p_1}(\Omega, R^n)$. This completes the proof of the theorem. □

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