KYUNGPOOK Math. J. 51(2011), 43-58
DOI 10.5666/KMJ.2011.51.1.043

Uniqueness of Certain Non-Linear Differential Polynomials Sharing 1-Points

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Abstract. Using the notion of weighted sharing of values we study the uniqueness of meromorphic functions when certain non-linear differential polynomials share the same 1-points. Though the main concern of the paper is to improve a result of Fang [5] but as a consequence of the main result we improve and supplement some former results of Lahiri-Sarkar [16], Fang-Fang[6] et. al.

1. Introduction definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

We shall use the standard notations of value distribution theory:

$$T(r,f), \ \ m(r,f), \ \ N(r,\infty;f), \ \ \overline{N}(r,\infty;f), \ldots$$

(see [8]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure. For any constant a, we define

$$\Theta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

In 1999 Lahiri [9] asked the following question.

What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 *CM?*

During the last couple of years a substantial amount of investigations have been carried out by several authors on the uniqueness of meromorphic functions

Received August 18, 2009; revised December 8, 2010; accepted December 24, 2010.
2000 Mathematics Subject Classification: 30D35.
Key words and phrases: Meromorphic function, derivative, non-linear differential polynomials, weighted sharing.

concerning non-linear differential polynomials and naturally several elegant results have been obtained in this aspect (see [2]-[7], [13]-[20]).

In 2001 Fang and Hong [7] proved the following result.

Theorem A. *Let f and g be two transcendental entire functions and $n(\geq 11)$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 CM, then $f \equiv g$.*

Also in 2002 Fang and Fang [6] improved and supplemented the above theorem by proving the following theorems.

Theorem B. *Let f and g be two non-constant entire functions and $n(\geq 8)$ be an integer . If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 CM, then $f \equiv g$.*

Theorem C. *Let f and g be two non-constant entire functions and $n(\geq 17)$ be an integer . If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 IM, then $f \equiv g$.*

In 2004 Lin and Yi [19] further improved *Theorem B* as follows.

Theorem D. *Let f and g be two transcendental entire functions and $n(\geq 7)$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 CM, then $f \equiv g$.*

In the same year Qiu and Fang [20] independently proved *Theorem D* resorting to a new technique than that was adopted in [19] and replace the value 1-by a non zero finite constant a.

The following example shows that the above theorems are not valid when f and g are two meromorphic functions.

Example 1.1.

$$f(z) = \frac{(n+2)}{(n+1)} \frac{e^z + \ldots + e^{(n+1)z}}{1 + e^z + \ldots + e^{(n+1)z}}$$

and

$$g(z) = \frac{(n+2)}{(n+1)} \frac{1 + e^z + \ldots + e^{nz}}{1 + e^z + \ldots + e^{(n+1)z}}$$

Clearly $f(z) = e^z g(z)$. Also $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 CM but $f \not\equiv g$.

We note that in the above example $\Theta(\infty; f) = \Theta(\infty; g) = 0$.

So to replace entire functions by meromorphic functions in the above mentioned theorems definitely some extra conditions are required.

For meromorphic function Lin and Yi [19] proved the following result.

Theorem E. *Let f and g be two non-constant meromorphic functions such that $\Theta(\infty; f) > \frac{2}{n+1}$ and $n(\geq 11)$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$, share the value 1-CM, then $f \equiv g$.*

To state the next results we require the following definition known as weighted sharing of values which measure how close a shared value is to be shared IM or to be shared CM.

Definition 1.1([10, 11])**.** Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of values Lahiri and Sarkar [16] proved the following theorem for the uniqueness of non-linear differential polynomials which is also an improvement of *Theorem E.*

Theorem F. *Let f and g be two non-constant meromorphic functions such that $\Theta(\infty; f)+\Theta(\infty; g) > \frac{4}{n+1}$ and $n(\geq 11)$ be an integer . If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share $(1, 2)$, then $f \equiv g$.*

Lahiri and Sarkar [16] also gave the following example to show that the condition $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$ is sharp in *Theorem F.*

Example 1.2. Let $f = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $g = h\frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$ and $h = \frac{\alpha^2(e^z-1)}{e^z-\alpha}$ where $\alpha = exp(\frac{2\pi i}{n+2})$ and n is a positive integer.

Clearly $T(r, f) = (n + 1)T(r, h) + O(1)$ and $T(r, g) = (n + 1)T(r, h) + O(1)$. Further we see that $h \neq \alpha, \alpha^2$ and a root of $h = 1$ is not a pole of f and g. Hence $\Theta(\infty; f) = \Theta(\infty; g) = \frac{2}{n+1}$. Also $f^{n+1}(\frac{f}{n+1} - \frac{1}{n+1}) \equiv g^{n+1}(\frac{g}{n+1} - \frac{1}{n+1})$ and $f^n(f-1)f^{'} \equiv g^n(g-1)g^{'}$ but $f \not\equiv g$.

In 2002 Fang [5] first considered the uniqueness of entire functions corresponding to more generalized non-linear differential polynomials and proved the following result.

Theorem G. *Let f and g be two non-constant entire functions and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

In the paper we will prove two theorems the second of which will not only improve *Theorem G* by reducing the lower bound of n and at the same time relaxing the nature of sharing the value 1 but also improve and supplement *Theorem C.* Our first theorem will improve and supplement *Theorem F.* Following theorems are the main results of the paper.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions and $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$. Suppose for two non zero constants a and b $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 3k+9$ or if $l = 1$ and $n \geq 4k+10$ or if $l = 0$ and $n \geq 9k+18$, then $f \equiv g$ or $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} \equiv 1$. When $k = 1$ the possibility $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} \equiv 1$ does not occur.*

Putting $n = m+1$, $a = \frac{1}{m+2}$ $b = -\frac{1}{m+1}$ and $k = 1$ in the above theorem we can immediately deduce the following corollary.

Corollary 1.1. *Let f and g be two non-constant meromorphic functions and $m(\geq 1)$, $l(\geq 0)$ be two integers such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{m+1}$. Suppose for two non zero constants a and b $f^m(f-1)f^{'}$ and $g^m(g-1)g^{'}$ share $(1, l)$. If $l \geq 2$ and $m \geq 11$ or if $l = 1$ and $m \geq 13$ or if $l = 0$ and $m \geq 26$, then $f \equiv g$.*

Remark 1.1. Since *Theorem F* can be obtained as a special case of *Theorem 1.1*, clearly *Theorem 1.1* improves and supplements *Theorem F*.

Theorem 1.2. *Let f and g be two non-constant entire functions and $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose for two non zero constants a and b $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 2k+6$ or if $l = 1$ and $n \geq \frac{5k}{2} + 7$ or if $l = 0$ and $n \geq 5k+12$, then $f \equiv g$.*

Putting $n = m+1$, $a = \frac{1}{m+2}$ $b = -\frac{1}{m+1}$ and $k = 1$ in the above theorem we can immediately deduce the following corollary.

Corollary 1.2. *Let f and g be two non-constant entire functions and $m(\geq 1)$, $l(\geq 0)$ be two integers. Suppose for two non zero constants a and b $f^m(f-1)f^{'}$ and $g^m(g-1)g^{'}$ share $(1, l)$. If $l \geq 2$ and $m \geq 7$ or if $l = 1$ and $m \geq 9$ or if $l = 0$ and $m \geq 16$, then $f \equiv g$.*

Remark 1.2. Clearly *Corollary 1.2* improve and supplement *Theorem C.*

Though we use the standard notations and definitions of the value distribution theory available in [8], we explain some definitions and notations which are used in the paper.

Definition 1.2([16]). Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)$ ($\overline{N}(r, a; f \mid\geq p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r, a; f \mid\leq p)$ ($\overline{N}(r, a; f \mid\leq p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.3(11, cf.[22])**.** For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.4. Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ $(\overline{N}(r, a; f \mid \geq p \mid g \neq b))$ the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

Definition 1.5(cf.[1], 2)**.** Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.

Definition 1.6(cf.[1], 2)**.** Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}\ (r, 1; g)$ the reduced counting function of those 1-points of f and g such that $p > q = k$. $\overline{N}_{g>k}\ (r, 1; f)$ is defined analogously.

Definition 1.7([10, 11])**.** Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F^{(k+2)}}{F^{(k+1)}} - \frac{2F^{(k+1)}}{F^{(k)} - 1} \right) - \left(\frac{G^{(k+2)}}{G^{(k+1)}} - \frac{2G^{(k+1)}}{G^{(k)} - 1} \right). \tag{2.1}$$

Lemma 2.1([8]). *Let f be a non-constant meromorphic function, k a positive integer and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, \infty; f) + N(r, 0; f) + N\left(r, c; f^{(k)}\right) - N\left(r, 0; f^{(k+1)}\right) + S(r, f) \\ &\leq \overline{N}(r, \infty; f) + N_{k+1}(r, 0; f) + \overline{N}\left(r, c; f^{(k)}\right) - N_0\left(r, 0; f^{(k+1)}\right) + S(r, f), \end{aligned}$$

where $N_0\left(r, 0; f^{(k+1)}\right)$ is the counting function of the zeros of $f^{(k+1)}$ which are not the zeros of $f(f^{(k)} - c)$.

Following lemma was proved in [15] for $p = 2$ and the general form is stated in [23].

Lemma 2.2([23]) *Let f be a non-constant meromorphic function and p, k be positive integers, then*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.3([1]). *If f, g be two non-constant meromorphic functions such that they share $(1,1)$. Then*

$$\begin{aligned} & 2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \\ \leq\ & N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Lemma 2.4([2]). *Let f, g share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_{\oslash}(r,0;f^{'}) + S(r,f),$$

where $N_{\oslash}(r,0;f^{'})$ is the counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)$.

Lemma 2.5([2]). *Let f and g be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\begin{aligned} & \overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \\ \leq\ & N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Lemma 2.6([2]). *Let f, g share $(1,0)$. Then*

$$\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$$

Lemma 2.7([2]). *Let f, g share $(1,0)$. Then*

$$\begin{aligned} (i) \quad & \overline{N}_{f>1}(r,1;g) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\oslash}(r,0;f^{'}) + S(r,f) \\ (ii) \quad & \overline{N}_{g>1}(r,1;f) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_{\oslash}(r,0;g^{'}) + S(r,g). \end{aligned}$$

Lemma 2.8([21]). *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r,f) + O(1)$.*

Lemma 2.9. *Let f and g be two non-constant meromorphic functions. Then*

$$f^{n-1}\left[a(n+1)f+nb\right]f^{'}\ \ g^{n-1}\left[a(n+1)g+nb\right]g^{'}\not\equiv 1,$$

where $n\geq 12$ is an integer.

Proof. We omit the proof as it can be proved in the line of proof *Lemma 2.7* in [17]. □

Lemma 2.10. *Let f and g be two non-constant entire functions. Then*

$$[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)}\not\equiv 1,$$

where a and b are nonzero complex numbers; n, k be two positive integers and $n(>k)$.

Proof. We omit the proof since the proof can be found in the proof of *Theorem 2* in [5]. □

Lemma 2.11. *Let f and g be two non-constant meromorphic functions such that*

$$\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b)\equiv g^n(ag+b)$$

implies $f\equiv g$, where a, b are non-zero constants.

Proof. We omit the proof since it can be carried out in the line of *Lemma 6* in [12]. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let $F=f^n(af+b)$ and $G=g^n(ag+b)$. It follows that $F^{(k)}$ and $G^{(k)}$ share $(1,l)$.
Case 1 Let $H\not\equiv 0$.
Subcase 1.1 $l\geq 1$
From (2.1) we get

$$\begin{aligned}(3.1)\qquad N(r,\infty;H)\ \leq\ &\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right)\\ &+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)\\ &+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right),\end{aligned}$$

where $\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)$ is the reduced counting function of those zeros of $F^{(k+1)}$ which are not the zeros of $F^{(k)}\left(F^{(k)}-1\right)$ and $\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)$ is similarly defined.

Let z_0 be a simple zero of $F^{(k)}-1$. Then z_0 is a simple zero of $G^{(k)}-1$ and a zero of H. So

$$N\left(r,1;F^{(k)}\mid=1\right)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G) \tag{3.2}$$

While $l\geq 2$, using (3.1) and (3.2) we get

$$\begin{aligned} &\overline{N}\left(r,1;F^{(k)}\right) \\ \leq\ & N\left(r,1;F^{(k)}\mid=1\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right) \\ \leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r.0;F^{(k)}\mid\geq 2)+\overline{N}\left(r.0;G^{(k)}\mid\geq 2\right) \\ & +\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right)+\overline{N}_\otimes(r,0;F^{(k+1)}) \\ & +\overline{N}_\otimes\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G). \end{aligned} \tag{3.3}$$

So from *Lemmas 2.1* and *2.8* we have

$$\begin{aligned} & T(r,F)+T(r,G) \\ \leq\ & 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+1}(r,0;F)+N_{k+1}(r,0;G) \\ & +\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)+\overline{N}_\otimes\left(r,0;F^{(k+1)}\right) \\ & +\overline{N}_\otimes\left(r,0;G^{(k+1)}\right)+\overline{N}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right) \\ & +\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right)-N_0\left(r,0;F^{(k+1)}\right)-N_0\left(r,0;G^{(k+1)}\right). \\ & +S(r,F)+S(r,G) \end{aligned} \tag{3.4}$$

We note that

$$\begin{aligned} & N_{k+1}(r,0;F)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}_\otimes\left(r,0;F^{(k+1)}\right) \\ \leq\ & N_{k+1}(r,0;F)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\mid F=0\right) \\ & +\overline{N}\left(r,0;F^{(k)}\mid\geq 2\mid F\neq 0\right)+\overline{N}_\otimes\left(r,0;F^{(k+1)}\right) \\ \leq\ & N_{k+1}(r,0;F)+\overline{N}\left(r,0;F\mid\geq k+2\right)+\overline{N}_0\left(r,0;F^{(k+1)}\right) \\ \leq\ & N_{k+2}(r,0;F)+\overline{N}_0\left(r,0;F^{(k+1)}\right). \end{aligned} \tag{3.5}$$

Clearly similar expression holds for G. Also

$$
\begin{aligned}
(3.6) \qquad & \overline{N}\left(r,1;F^{(k)}\mid\geq 2\right)+\overline{N}_{*}\left(r,1;F^{(k)},G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
\leq\ & \overline{N}\left(r,1;G^{(k)}\mid= 2\right)+2\overline{N}_{L}\left(r,1;F^{(k)}\right)+2\overline{N}_{L}\left(r,1;G^{(k)}\right)\\
& +\overline{N}_{E}^{(3}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
\leq\ & N\left(r,1;G^{(k)}\right)\\
\leq\ & T\left(r,G^{(k)}\right)+O(1)\\
\leq\ & T(r,G)+k\overline{N}(r,\infty;G)+S(r,G).
\end{aligned}
$$

Using *Lemma 2.8*, (3.5) and (3.6) in (3.4) we obtain for $\varepsilon>0$

$$
\begin{aligned}
(3.7) \qquad & (n+1)T(r,f)\\
=\ & T(r,F)+O(1)\\
\leq\ & N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+2\overline{N}(r,\infty;F)\\
& +(k+2)\overline{N}(r,\infty;G)+S(r,F)+S(r,G)\\
\leq\ & N_{k+2}\left(r,0;f^{n}\right)+N_{k+2}(r,0;af+b)+N_{k+2}\left(r,0;g^{n}\right)\\
& +N_{k+2}(r,0;ag+b)+2\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)\\
& +S(r,f)+S(r,g)\\
\leq\ & (5+k-2\Theta(\infty;f)+\varepsilon)\,T(r,f)+(5+2k-(2+k)\Theta(\infty;g)\\
& +\varepsilon)\,T(r,g)+S(r,f)+S(r,g)\\
\leq\ & (10+3k-2\Theta(\infty;f)-2\Theta(\infty;g)-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}\\
& +2\varepsilon)\,T(r)+S(r,f)+S(r,g).
\end{aligned}
$$

In a similar way we can obtain

$$
\begin{aligned}
(3.8) \qquad & (n+1)T(r,g)\\
\leq\ & (10+3k-2\Theta(\infty;f)-2\Theta(\infty;g)-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}\\
& +2\varepsilon)\,T(r)+S(r,f)+S(r,g).
\end{aligned}
$$

So from (3.7) and (3.8) we get

$$
\begin{aligned}
(3.9) \quad & (n-3k-9+2\Theta(\infty;f)+2\Theta(\infty;g)+k\min\{\Theta(\infty;f),\Theta(\infty;g)\}-2\varepsilon)\,T(r)\\
\leq\ & S(r).
\end{aligned}
$$

Since $n\geq 3k+9$, $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n}$ and $\varepsilon>0$ be arbitrary, (3.9) gives a contradiction. While $l=1$, using *Lemmas 2.2*, *2.3* and *2.4*, (3.1) and (3.2) we get

$$
\begin{aligned}
(3.10)\quad & \overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
\leq\ & N\left(r,1;F^{(k)}\mid=1\right)+\overline{N}_L\left(r,1;F^{(k)}\right)\\
& +\overline{N}_L\left(r,1;G^{(k)}\right)+\overline{N}^{(2}_E\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
\leq\ & N\left(r,1;F^{(k)}\mid=1\right)+N\left(r,1;G^{(k)}\right)-\overline{N}_L\left(r,1;F^{(k)}\right)\\
& -\overline{N}_L\left(r,1;G^{(k)}\right)+\overline{N}_{F^{(k)}>2}\left(r,1;G^{(k)}\right)\\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)\\
& +\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right)-\overline{N}_L\left(r,1;F^{(k)}\right)-\overline{N}_L\left(r,1;G^{(k)}\right)\\
& +\frac{1}{2}\overline{N}\left(r,0;F^{(k)}\right)+\frac{1}{2}\overline{N}\left(r,\infty;F^{(k)}\right)+T\left(r,G^{(k)}\right)+\overline{N}_\otimes\left(r,0;F^{(k+1)}\right)\\
& +\overline{N}_\otimes\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G)\\
\leq\ & \left(\frac{k}{2}+\frac{3}{2}\right)\overline{N}(r,\infty;F)+(k+1)\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)\\
& +\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)+\frac{1}{2}N_{k+1}(r,0;F)+T(r,G)+\overline{N}_\otimes\left(r,0;F^{(k+1)}\right)\\
& +\overline{N}_\otimes\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).
\end{aligned}
$$

So in view of *Lemmas 2.1*, *2.8*, (3.5) and (3.10) we get for $\varepsilon>0$

$$
\begin{aligned}
(3.11)\quad & (n+1)T(r,f)\\
=\ & T(r,F)+O(1)\\
\leq\ & \left(\frac{k}{2}+\frac{5}{2}\right)\overline{N}(r,\infty;F)+(k+2)\overline{N}(r,\infty;G)+\frac{1}{2}N_{k+1}(r,0;F)\\
& +N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+S(r,F)+S(r,G)\\
\leq\ & \left(2k+\frac{13}{2}-(\frac{k}{2}+2)\Theta(\infty;f)-\frac{1}{2}\Theta(\infty;f)+\varepsilon\right)T(r,f)\\
& +\left(2k+5-(\frac{k}{2}+2)\Theta(\infty;g)-\frac{k}{2}\Theta(\infty;g)+\varepsilon\right)T(r,g)\\
& +S(r,f)+S(r,g)\\
\leq\ & \left(4k+\frac{23}{2}-(\frac{k}{2}+\frac{5}{2})\left(\Theta(\infty;f)+\Theta(\infty;g)\right)+2\varepsilon\right)T(r)\\
& +S(r).
\end{aligned}
$$

In a similar manner we can get

$$(3.12)\qquad \begin{aligned}&(n+1)T(r,g)\\ &\leq \left(4k+\frac{23}{2}-(\frac{k}{2}+\frac{5}{2})\left(\Theta(\infty;f)+\Theta(\infty;g)\right)+2\varepsilon\right)T(r)+S(r).\end{aligned}$$

Combining (3.11) and (3.12) we get

$$(3.13)\qquad \left(n-4k-\frac{21}{2}+(\frac{k}{2}+\frac{5}{2})\left(\Theta(\infty;f)+\Theta(\infty;g)\right)-2\varepsilon\right)T(r)\leq S(r).$$

Since $n\geq 4k+10$, $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n}$ and $\varepsilon>0$ be arbitrary, (3.13) implies a contradiction.

Subcase 1.2 $l=0$. Here (3.2) changes to

$$(3.14)\qquad N_E^{1)}\left(r,1;F^{(k)}\mid=1\right)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G)$$

Using *Lemmas 2.2*, *2.5*, *2.6*, *2.7* and (3.1) and (3.14) we get

$$(3.15)\qquad \begin{aligned}
&\overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
\leq\ & N_E^{1)}\left(r,1;F^{(k)}\right)+\overline{N}_L\left(r,1;F^{(k)}\right)+\overline{N}_L\left(r,1;G^{(k)}\right)\\
&+\overline{N}_E^{(2}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
\leq\ & N_E^{1)}\left(r,1;F^{(k)}\right)+N\left(r,1;G^{(k)}\right)-\overline{N}_L\left(r,1;G^{(k)}\right)\\
&+\overline{N}_{F^{(k)}>1}\left(r,1;G^{(k)}\right)+\overline{N}_{G^{(k)}>1}\left(r,1;F^{(k)}\right)\\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)\\
&+\overline{N}_*\left(r,1;F^{(k)},G(k)\right)+T\left(r,G^{(k)}\right)-\overline{N}_L\left(r,1;G^{(k)}\right)\\
&+\overline{N}_{F^{(k)}>1}\left(r,1;G^{(k)}\right)+\overline{N}_{G^{(k)}>1}\left(r,1;F^{(k)}\right)\\
&+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G)\\
\leq\ & (2k+3)\overline{N}(r,\infty;F)+(2k+2)\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)\\
&+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)+2N_{k+1}(r,0;F)+N_{k+1}(r,0;G)+T(r,G)\\
&+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).
\end{aligned}$$

So in view of *Lemmas 2.1*, *2.8*, (3.5) and (3.15) we get for $\varepsilon > 0$

$$\begin{aligned}
(3.16)\quad & (n+1)T(r,f) \\
= \;& T(r,F) + O(1) \\
\leq \;& (2k+4)\overline{N}(r,\infty;f) + (2k+3)\overline{N}(r,\infty;g) + 2N_{k+1}(r,0;F) \\
& +N_{k+1}(r,0;G) + N_{k+2}(r,0;F) + N_{k+2}(r,0;G) + S(r,f) + S(r,g) \\
\leq \;& (9k+19-(2k+3)\Theta(\infty;f)-(2k+3)\Theta(\infty;g)-\min\{\Theta(\infty;f),\Theta(\infty;g)\} \\
& +2\varepsilon)\,T(r) + S(r).
\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}
(3.17)\quad & (n+1)T(r,g) \\
= \;& T(r,G) + O(1) \\
\leq \;& (9k+19-(2k+3)\Theta(\infty;f)-(2k+3)\Theta(\infty;g)-\min\{\Theta(\infty;f),\Theta(\infty;g)\} \\
& +2\varepsilon)\,T(r) + S(r).
\end{aligned}$$

Combining (3.16) and (3.17) we get

$$\begin{aligned}
(3.18)\qquad & (n-9k-18+(2k+3)\Theta(\infty;f)+(2k+3)\Theta(\infty;g) \\
& \qquad\qquad + \min\{\Theta(\infty;f),\Theta(\infty;g)\} - 2\varepsilon)T(r) \\
& \leq S(r).
\end{aligned}$$

Since $n \geq 9k+18$, $\Theta(\infty;f)+\Theta(\infty;g) > \frac{4}{n}$ and $\varepsilon > 0$ be arbitrary, (3.18) implies a contradiction.

Case 2 Next we suppose that $H \equiv 0$. Then by integration we get from (2.1)

$$(3.19)\qquad \frac{1}{F^{(k)}-1} \equiv \frac{bG^{(k)}+a-b}{G^{(k)}-1},$$

where a, b are constants and $a \neq 0$. From (3.19) it is clear that $F^{(k)}$ and $G^{(k)}$ share $(1,\infty)$ and hence they share $(1,2)$. So in this case always $n \geq 3k+9$. We now consider the following subcases.

Subcase 2.1 Let $b \neq 0$ and $a \neq b$.

If $b=-1$, then from (3.19) we have

$$F^{(k)} = \frac{-a}{G^{(k)}-a-1}.$$

Therefore

$$\overline{N}\left(r,a+1;G^{(k)}\right) = \overline{N}\left(r,\infty;F^{(k)}\right) = \overline{N}(r,\infty;f).$$

Since $a \neq b = -1$, from *Lemma 2.1* we have

$$\begin{aligned}
(n+1)T(r,g) &= T(r,G) + O(1) \\
&\leq \overline{N}(r,\infty;G) + N_{k+1}(r,0;G) + \overline{N}(r,a+1;G^{(k)}) + S(r,G) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;G) + S(r,G) \\
&\leq T(r,f) + (k+3)T(r,g) + S(r,g)
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.
So for $r \in I$ we have

$$(n-k-3)T(r,g) \leq S(r,g),$$

which is a contradiction for $n \geq 3k+9$.
If $b \neq -1$, from (3.19) we obtain that

$$F^{(k)} - \left(1+\frac{1}{b}\right) = \frac{-a}{b^2[G^{(k)} + (a-b)/b]}.$$

Therefore

$$\overline{N}\left(r,(b-a)/b;G^{(k)}\right) = \overline{N}\left(r,\infty;F^{(k)} - (1+1/b)\right) = \overline{N}(r,\infty;f)$$

Using *Lemma 2.1* and the same argument as used in the case when $b=-1$ we can get a contradiction.
Subcase 2.2 Let $b \neq 0$ and $a=b$.
If $b=-1$, then from (3.19) we have

$$F^{(k)}G^{(k)} \equiv 1,$$

that is

$$[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} \equiv 1,$$

which in view of *Lemma 2.9* is impossible when $k=1$.
If $b \neq -1$, from (3.19) we have

$$\frac{1}{F^{(k)}} = \frac{bG^{(k)}}{(1+b)G^{(k)}-1}.$$

Hence from *Lemma 2.2* we have

$$\begin{aligned}\overline{N}\left(r,1/(1+b);G^{(k)}\right) &= \overline{N}\left(r,0;F^{(k)}\right)\\ &\leq N_{k+1}(r,0;F) + k\overline{N}(r,\infty;f).\end{aligned}$$

From *Lemma 2.1* we have

$$\begin{aligned}&(n+1)T(r,g)+O(1)\\ =\ &T(r,G)\\ \leq\ &\overline{N}(r,\infty;G) + N_{k+1}(r,0;G) + \overline{N}\left(r,\frac{1}{b+1};G^{(k)}\right) + S(r,G)\\ \leq\ &k\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;F) + N_{k+1}(r,0;G) + S(r,G)\\ \leq\ &(2k+2)T(r,f) + (k+3)T(r,g) + S(r,g)\end{aligned}$$

For $r \in I$ we have

$$(n-3k-4)T(r,g) \leq S(r,g),$$

which is a contradiction for $n \geq 3k+9$.
Subcase 2.3 Let $b=0$. From (3.19) we obtain

$$F^{(k)} = \frac{G^{(k)}+a-1}{a}. \tag{3.20}$$

If $a-1 \neq 0$ then From (3.20) we obtain

$$\overline{N}\left(r, 1-a; G^{(k)}\right) = \overline{N}\left(r, 0; F^{(k)}\right).$$

We can similarly deduce a contradiction as in Subcase 2.2. Therefore $a=1$ and from (3.20) we obtain

$$F = G + p(z), \tag{3.21}$$

where $p(z)$ is a polynomial of degree at most $k-1$. We claim that $p(z) \equiv 0$. Otherwise noting that f is transcendental when $k \geq 2$, in view of *Lemma 2.8* we have

$$\begin{aligned} (n+1)T(r,f) &= T(r,F) + O(1) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;f) + \overline{N}(r,p;F) + S(r,F) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;f) + \overline{N}(r,0;G) + S(r,F) \\ &\leq 3T(r,f) + 2T(r,g) + S(r,f) \end{aligned} \tag{3.22}$$

Also from (3.21) we get

$$T(r,f) = T(r,g) + S(r,f),$$

which together with (3.22) implies a contradiction. Hence (3.21) becomes

$$F \equiv G.$$

So from *Lemma 2.11* we get $f \equiv g$. □

Proof of Theorem 1.2. We omit the proof since instead of *Lemma 2.9* using *Lemma 2.10* and proceeding in the same way the proof of the theorem can be carried out in the line of proof of *Theorem 1.1.* □

Acknowledgement The author is thankful to Prof. M.L.Fang for supplying him the pdf file of the paper [5].

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