KYUNGPOOK Math. J. 51(2011), 37-42
DOI 10.5666/KMJ.2011.51.1.037

Zero-divisors of Semigroup Modules

Peyman Nasehpour
Universität Osnabrück, FB Mathematik/Informatik, 49069 Osnabrück, Germany
e-mail : peyman.nasehpour@mathematik.uni-osnabrueck.de
and nasehpour@gmail.com

Abstract. Let M be an R-module and S a semigroup. Our goal is to discuss zero-divisors of the semigroup module $M[S]$. Particularly we show that if M is an R-module and S a commutative, cancellative and torsion-free monoid, then the $R[S]$-module $M[S]$ has few zero-divisors of size n if and only if the R-module M has few zero-divisors of size n and Property (A).

1. Introduction

Let S be a commutative semigroup and M be an R-module. One can define the semigroup module $M[S]$ as an $R[S]$-module constructed from the semigroup S and the R-module M similar to the standard definition of semigroup rings. Obviously similar to semigroup rings, the zero-divisors of the semigroup module $M[S]$ are interesting to investigate ([6, p. 82] and [12]).

We write each element of $g \in M[S]$ as "polynomials" $g = m_1X^{s_1} + m_2X^{s_2} + \cdots + m_nX^{s_n}$, where $m_1, \cdots, m_n \in M$ and $s_1, \cdots, s_n$ are distinct elements of S and this representation of g is called the canonical form of g. We define the *content* $c(g)$ of $g \in M[S]$ to be the R-submodule of M generated by the coefficients of g.

Northcott gave a nice generalization of *Dedekind-Mertens Lemma* as follows: if S is a commutative, cancellative and torsion-free monoid and M is an R-module, then for all $f \in R[S]$ and $g \in M[S]$, there exists a natural number k such that $c(f)^k c(g) = c(f)^{k-1} c(fg)$ ([16]). Dedekind-Mertens Lemma has different versions with various applications ([1], [2], [3], [8], [9], [15], [18] and [19] and [20]). One of its interesting consequences is McCoy's Theorem on zero-divisors ([6, p. 96] and [14]): If M is a nonzero R-module and S is a commutative, cancellative and torsion-free monoid, then for all $f \in R[S]$ and $g \in M[S] - \{0\}$, if $fg = 0$, then there exists an $m \in M - \{0\}$ such that $f.m = 0$.

An R-module M is said to have *few zero-divisors of size* n, if $Z_R(M)$ is a finite union of n prime ideals $\mathbf{p}_1, \cdots, \mathbf{p}_n$ of R such that $\mathbf{p}_i \not\subseteq \mathbf{p}_j$ for all $i \neq j$. Also note

Received March 18, 2010; revised August 13, 2010; accepted December 27, 2010.
2000 Mathematics Subject Classification: 13A15, 13B25, 13C05, 20M14.
Key words and phrases: Dedekind-Mertens Lemma, Semigroup modules, Few zero-divisors, Property (A), McCoy's property, Primal modules.

that an R-module M has *Property (A)*, if each finitely generated ideal $I \subseteq Z_R(M)$ has a nonzero annihilator in M. We use McCoy's Theorem to prove that if M is an R-module and S a commutative, cancellative and torsion-free monoid, then the $R[S]$-module $M[S]$ has few zero-divisors of size n, if and only if the R-module M has few zero-divisors of size n and Property (A).

In this paper all rings are commutative with identity and all modules are unital. Unless otherwise stated, our notation and terminology will follow as closely as possible that of Gilmer [6].

2. Zero-divisors of semigroup modules

Let us recall that if R is a ring and $f = a_0 + a_1X + \cdots + a_nX^n$ is a polynomial on the ring R, then content of f is defined as the R-ideal, generated by the coefficients of f, i.e. $c(f) = (a_0, a_1, \cdots, a_n)$. The content of an element of a semigroup module is a natural generalization of the content of a polynomial as follows:

Definition 1. Let M be an R-module and S be a commutative semigroup. Let $g \in M[S]$ and put $g = m_1X^{s_1} + m_2X^{s_2} + \cdots + m_nX^{s_n}$, where $m_1, \cdots, m_n \in M$ and $s_1, \cdots, s_n \in S$. We define the content of g to be the R-submodule of M generated by the coefficients of g, i.e. $c(g) = (m_1, \cdots, m_n)$.

Theorem 2. *Let S be a commutative monoid and M be a nonzero R-module. Then the following statements are equivalent:*

1. *S is a cancellative and torsion-free monoid.*
2. *For all $f \in R[S]$ and $g \in M[S]$, there is a natural number k such that $c(f)^kc(g) = c(f)^{k-1}c(fg)$.*
3. *(McCoy's Property) For all $f \in R[S]$ and $g \in M[S] - \{0\}$, if $fg = 0$, then there exists an $m \in M - \{0\}$ such that $f.m = 0$.*
4. *For all $f \in R[S]$, $\mathrm{Ann}_M(c(f)) = 0$ if and only if $f \notin Z_{R[S]}(M[S])$.*

Proof. (1) → (2) has been proved in [16].

For (2) → (3), assume that $f \in R[S]$ and $g \in M[S] - \{0\}$, such that $fg = 0$. So there exists a natural number k such that $c(f)^kc(g) = c(f)^{k-1}c(fg) = (0)$. Take t the smallest natural number such that $c(f)^tc(g) = (0)$ and choose m a nonzero element of $c(f)^{t-1}c(g)$. It is easy to check that $f.m = 0$.

For (3) → (1), we prove that if S is not cancellative or not torsion-free then (1) cannot hold. For the moment, suppose that S is not cancellative, so there exist $s, t, u \in S$ such that $s + t = s + u$ while $t \neq u$. Put $f = X^s$ and $g = (qX^t - qX^u)$, where q is a nonzero element of M. Then obviously $fg = 0$, while $f.m \neq 0$ for all $m \in M - \{0\}$. Finally suppose that S is cancellative but not torsion-free. Let $s, t \in S$ be such that $s \neq t$, while $ns = nt$ for some natural n. Choose the natural number k minimal so that $ks = kt$. Then we have $0 = qX^{ks} - qX^{kt} = (\sum_{i=0}^{k-1} X^{(k-i-1)s+it})(qX^s - qX^t)$, where q is a nonzero element of M.

Since S is cancellative, the choice of k implies that $(k-i_1-1)s+i_1t \neq (k-i_2-1)s+i_2t$ for $0 \leq i_1 < i_2 \leq k-1$. Therefore $\sum_{i=0}^{k-1} X^{(k-i-1)s+it} \neq 0$, and this completes the proof. (3) $\leftrightarrow$ (4) is obvious. □

Corollary 3. *Let M be an R-module and S be a commutative, cancellative and torsion-free monoid. Then the following statements hold:*

1. *R is a domain if and only if $R[S]$ is a domain.*
2. *If $\boldsymbol{p}$ is a prime ideal of R, then $\boldsymbol{p}[S]$ is a prime ideal of $R[S]$.*
3. *If $\boldsymbol{p}$ is in $\mathrm{Ass}_R(M)$, then $\boldsymbol{p}[S]$ is in $\mathrm{Ass}_{R[S]}(M[S])$.*

Definition 4. Let M be an R-module and P be a proper R-submodule of M. P is said to be a *prime submodule* (*primary submodule*) of M, if $rx \in P$ implies $x \in P$ or $rM \subseteq P$ (there exists a natural number n such that $r^nM \subseteq P$), for each $r \in R$ and $x \in M$.

Corollary 5. *Let M be an R-module and S be a commutative, cancellative and torsion-free monoid. Then the following statements hold:*

1. *(0) is a prime (primary) submodule of M if and only if (0) is a prime (primary) submodule of $M[S]$.*
2. *If P is a prime (primary) submodule of M, then $P[S]$ is a prime (primary) submodule of $M[S]$.*

In [5], it has been defined that a ring R has *few zero-divisors*, if $Z(R)$ is a finite union of prime ideals. We give the following definition and prove some interesting results about zero-divisors of semigroup modules. Modules having (very) few zero-divisors, introduced in [15], have also some interesting homological properties [17].

Definition 6. An R-module M has *very few zero-divisors*, if $Z_R(M)$ is a finite union of prime ideals in $\mathrm{Ass}_R(M)$.

Remark 7. *Examples of modules having very few zero-divisors.* If R is a Noetherian ring and M is an R-module such that $\mathrm{Ass}_R(M)$ is finite, then obviously M has very few zero-divisors. For example $\mathrm{Ass}_R(M)$ is finite if M is a finitely generated R-module [13, p. 55]. Also if R is a Noetherian quasi-local ring and M is a balanced big Cohen-Macaulay R-module, then $\mathrm{Ass}_R(M)$ is finite [4, Proposition 8.5.5, p. 344].

Remark 8. Let R be a ring and consider the following three conditions on R:

1. R is a Noetherian ring.
2. R has very few zero-divisors.
3. R has few zero-divisors.

Then, (1) $\rightarrow$ (2) $\rightarrow$ (3) and none of the implications are reversible.

Proof. For (1) $\rightarrow$ (2) use [13, p. 55]. It is obvious that (2) $\rightarrow$ (3).

Suppose k is a field, $A = k[X_1, X_2, X_3, \cdots, X_n, \cdots]$ and $\mathbf{m} = (X_1, X_2, X_3, \cdots, X_n, \cdots)$ and at last $\mathbf{a} = (X_1^2, X_2^2, X_3^2, \cdots, X_n^2, \cdots)$. Since A is a domain, A has very few zero-divisors while it is not a Noetherian ring. Also consider the ring $R = A/\mathbf{a}$. It is easy to check that R is a quasi-local ring with the only prime ideal $\mathbf{m}/\mathbf{a}$ and $Z(R) = \mathbf{m}/\mathbf{a}$ and finally $\mathbf{m}/\mathbf{a} \notin \mathrm{Ass}_R(R)$. Note that $\mathrm{Ass}_R(R) = \emptyset$ [15]. □

Theorem 9. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ has very few zero-divisors, if and only if the R-module M has very few zero-divisors.*

Proof. ($\leftarrow$): Let $Z_R(M) = \mathbf{p}_1 \cup \mathbf{p}_2 \cup \cdots \cup \mathbf{p}_n$, where $\mathbf{p}_i \in \mathrm{Ass}_R(M)$ for all $1 \leq i \leq n$. We will show that $Z_{R[S]}(M[S]) = \mathbf{p}_1[S] \cup \mathbf{p}_2[S] \cup \cdots \cup \mathbf{p}_n[S]$. Let $f \in Z_{R[S]}(M[S])$, so there exists an $m \in M - \{0\}$ such that $f.m = 0$ and so $c(f).m = (0)$. Therefore $c(f) \subseteq Z_R(M)$ and this means that $c(f) \subseteq \mathbf{p}_1 \cup \mathbf{p}_2 \cup \cdots \cup \mathbf{p}_n$ and according to the Prime Avoidance Theorem, we have $c(f) \subseteq \mathbf{p}_i$, for some $1 \leq i \leq n$ and therefore $f \in \mathbf{p}_i[S]$. Now let $f \in \mathbf{p}_1[S] \cup \mathbf{p}_2[S] \cup \cdots \cup \mathbf{p}_n[S]$, so there exists an i such that $f \in \mathbf{p}_i[S]$, so $c(f) \subseteq \mathbf{p}_i$ and $c(f)$ has a nonzero annihilator in M and this means that f is a zero-divisor of $M[S]$. Note that by Corollary 3, $\mathbf{p}_i[S] \in \mathrm{Ass}_{R[S]}(M[S])$ for all $1 \leq i \leq n$.

($\rightarrow$): Let $Z_{R[S]}(M[S]) = \cup_{i=1}^n Q_i$, where $Q_i \in \mathrm{Ass}_{R[S]}(M[S])$ for all $1 \leq i \leq n$. Therefore $Z_R(M) = \cup_{i=1}^n (Q_i \cap R)$. Without loss of generality, we can assume that $Q_i \cap R \nsubseteq Q_j \cap R$ for all $i \neq j$. Now we prove that $Q_i \cap R \in \mathrm{Ass}_R(M)$ for all $1 \leq i \leq n$. Consider $g \in M[S]$ such that $Q_i = \mathrm{Ann}(g)$ and $g = m_1 X^{s_1} + m_2 X^{s_2} + \cdots + m_n X^{s_n}$, where $m_1, \cdots, m_n \in M$ and $s_1, \cdots, s_n \in S$. It is easy to see that $Q_i \cap R = \mathrm{Ann}(c(g)) \subseteq \mathrm{Ann}(m_1) \subseteq Z_R(M)$ and by the Prime Avoidance Theorem, $Q_1 \cap R = \mathrm{Ann}(m_1)$. □

In [11], it has been defined that a ring R has *Property (A)*, if each finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator. We give the following definition:

Definition 10. An R-module M has *Property (A)*, if each finitely generated ideal $I \subseteq Z_R(M)$ has a nonzero annihilator in M.

Remark 11. If the R-module M has very few zero-divisors, then M has Property (A).

Theorem 12. *Let S be a commutative, cancellative and torsion-free monoid and M be an R-module. The following statements are equivalent:*

1. *The R-module M has Property (A).*
2. *For all $f \in R[S]$, f is $M[S]$-regular if and only if $c(f)$ is M-regular.*

Proof. (1) $\rightarrow$ (2): Let the R-module M have Property (A). If $f \in R[S]$ is $M[S]$-regular, then $f.m \neq 0$ for all nonzero $m \in M$ and so $c(f).m \neq (0)$ for all nonzero $m \in M$ and according to the definition of Property (A), $c(f) \nsubseteq Z_R(M)$. This means that $c(f)$ is M-regular. Now let $c(f)$ be M-regular, so $c(f) \nsubseteq Z_R(M)$ and this means that $c(f).m \neq (0)$ for all nonzero $m \in M$ and hence $f.m \neq 0$ for all nonzero $m \in M$. Since S is a commutative, cancellative and torsion-free monoid, f is not a zero-divisor of $M[S]$, i.e. f is $M[S]$-regular.

(2) $\to$ (1): Let I be a finitely generated ideal of R such that $I \subseteq Z_R(M)$. Then there exists an $f \in R[S]$ such that $c(f) = I$. But $c(f)$ is not M-regular, therefore according to our assumption, f is not $M[S]$-regular. Therefore there exists a nonzero $m \in M$ such that $f.m = 0$ and this means that $I.m = (0)$, i.e. I has a nonzero annihilator in M. □

Let, for the moment, M be an R-module such that the set $Z_R(M)$ of zero-divisors of M is a finite union of prime ideals. One can consider $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$ such that $\mathbf{p}_i \nsubseteq \cup_{j=1 \wedge j \neq i}^n \mathbf{p}_j$ for all $1 \leq i \leq n$. Obviously we have $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$. Also, it is easy to check that, if $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$ and $Z_R(M) = \cup_{k=1}^m \mathbf{q}_k$ such that $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$ and $\mathbf{q}_k \nsubseteq \mathbf{q}_l$ for all $k \neq l$, then $m = n$ and $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\} = \{\mathbf{q}_1, \cdots, \mathbf{q}_n\}$, i.e. these prime ideals are uniquely determined (Use the Prime Avoidance Theorem). This is the base for the following definition:

Definition 13. An R-module M is said to have *few zero-divisors of size* n, if $Z_R(M)$ is a finite union of n prime ideals $\mathbf{p}_1, \cdots, \mathbf{p}_n$ of R such that $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$.

Theorem 14. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ has few zero-divisors of size n, if and only if the R-module M has few zero-divisors of size n and Property (A).*

Proof. ($\leftarrow$): By considering the R-module M having Property (A), similar to the proof of Theorem 9, we have if $Z_R(M) = \cup_{i=1}^n \mathbf{p}_i$, then $Z_{R[S]}(M[S]) = \cup_{i=1}^n \mathbf{p}_i[S]$. Also it is obvious that $\mathbf{p}_i[S] \subseteq \mathbf{p}_j[S]$ if and only if $\mathbf{p}_i \subseteq \mathbf{p}_j$ for all $1 \leq i, j \leq n$. These two imply that the $R[S]$-module $M[S]$ has few zero-divisors of size n.

($\to$): Note that $Z_R(M) \subseteq Z_{R[S]}(M[S])$. It is easy to check that if $Z_{R[S]}(M[S]) = \cup_{i=1}^n Q_i$, where Q_i are prime ideals of $R[S]$ for all $1 \leq i \leq n$, then $Z_R(M) = \cup_{i=1}^n (Q_i \cap R)$. Now we prove that the R-module M has Property (A). Let $I \subseteq Z_R(M)$ be a finite ideal of R. Choose $f \in R[S]$ such that $I = c(f)$. So $c(f) \subseteq Z_R(M)$ and obviously $f \in Z_{R[S]}(M[S])$ and according to McCoy's property, there exists a nonzero $m \in M$ such that $f.m = 0$. This means that $I.m = 0$ and I has a nonzero annihilator in M. Consider that by a similar discussion in ($\leftarrow$), the R-module M has few zero-divisors obviously not less than size n and this completes the proof. □

An R-module M is said to be *primal*, if $Z_R(M)$ is an ideal of R [5]. It is easy to check that if $Z_R(M)$ is an ideal of R, then it is a prime ideal and therefore the R-module M is primal if and only if M has few zero-divisors of size one.

Corollary 15. *Let M be an R-module and S a commutative, cancellative and torsion-free monoid. Then the $R[S]$-module $M[S]$ is primal, if and only if the R-module M is primal and has Property (A).*

Acknowledgment The author wishes to thank Prof. Winfried Bruns and the referee for their useful advice and Dr. Neil Epstein for his editorial comments.

References

[1] J. T. Arnold and R. Gilmer, *On the content of polynomials*, Proc. Amer. Math. Soc. **40**(1970), 556-562.

[2] D. D. Anderson and B. G. Kang, *Content formulas for polynomials and power series and complete integral closure*, J. Algebra, **181**(1996), 82-94.

[3] W. Bruns and A. Guerrieri, *The Dedekind-Mertens formula and determinantal rings*, Proc. Amer. Math. Soc. **127**(1999), 657-663.

[4] W. Bruns and J. Herzog, Cohen-Macaulay Rings, revised edn., Cambridge, 1998.

[5] J. Dauns, *Primal modules*, Comm. Algebra **25**(1997), 2409-2435.

[6] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York, 1972.

[7] R. Gilmer, Commutative Semigroup Rings, The University of Chicago Press, 1984.

[8] R. Gilmer, A. Grams and T. Parker, *Zero divisors in power series rings*, J. Reine Angew. Math. **278**(1975), 145-164.

[9] W. Heinzer and C. Huneke, *The Dedekind-Mertens Lemma and the content of polynomials*, Proc. Amer. Math. Soc. **126**(1998), 1305-1309.

[10] J. A. Huckaba, Commutative Rings with Zero Divisors, Marcel Dekker, 1988.

[11] J. A. Huckaba and J. M. Keller, *Annihilation of ideals in commutative rings*, Pac. J. Math. **83**(1979), 375-379.

[12] B. D. Janeway, *Zero divisors in commutative semigroup rings*, Comm. Algebra **12**(1984), 1877-1887.

[13] I. Kaplansky, Commutative Rings, Allyn and Bacon, Boston, 1970.

[14] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly, **49**(1942), 286-295.

[15] P. Nasehpour, *Content algebras over commutative rings with zero divisors*, arXiv:0807.1835v3, preprint.

[16] D. G. Northcott, *A generalization of a theorem on the content of polynomials*, Proc. Cambridge Phil. Soc. **55**(1959), 282-288.

[17] P. Nasehpour and Sh. Payrovi, *Modules having few zero-divisors*, to appear in Comm. Algebra.

[18] P. Nasehpour and S. Yassemi, *M-cancellation Ideals*, Kyungpook Math. J., **40**(2000), 259-263.

[19] J. Ohm and D. E. Rush, *Content modules and algebras*, Math. Scand. **31** (1972), 49-68.

[20] D. E. Rush, *Content algebras*, Canad. Math. Bull. Vol. **21**(3)(1978), 329-334.

[21] H. Tsang, Gauss' lemma, dissertation, University of Chicago, Chicago, 1965.

[22] O. Zariski and P. Samuel, Commutative Algebra, Vol. I, Van Nostrand, New York, 1958.