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Some Global Estimates for the Jacobians of Quasiregular Mappings

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Abstract. Some global estimates for the Jacobians of quasiregular mappings $f = (f^1, f^2, \cdots, f^n)$ of the Sobolev class $W^{1,n}(\Omega, R^n)$ in $L^\varphi(\mu)$-domains and John domains are established.

1. Introduction

Throughout this paper, we always assume that Ω is an open subset of R^n, $n \geq 2$, and $f = (f^1, f^2, \cdots, f^n) \in W^{1,n}(\Omega, \mathrm{R}^n)$ be a mapping. The distributional differential $Df = \left(\frac{\partial f^i}{\partial x_j}\right)_{1\leq i,j\leq n} : \Omega \to GL(n)$ of f is an integrable function on Ω with values in the space $GL(n)$ of all $n \times n$-matrices. Denote by $|Df(x)| = \max\{|Df(x)h| : h \in S^{n-1}\}$ and $J(x, f) = \det Df(x)$ the norm of the Jacobian matrix $Df(x)$ and the Jacobian determinant of f, respectively.

Definition 1.1. A mapping $f \in W^{1,n}(\Omega, \mathrm{R}^n)$ is said to be K-quasiregular, $1 \leq K < \infty$, if

$$|Df(x)|^n \leq KJ(x, f), \quad \text{a.e. } \Omega$$

We say that f is orientation preserving (reversing), if its Jacobian determinant $J(x, f)$ is nonnegative (nonpositive) almost everywhere in Ω. From Definition 1.1 we know that any quasiregular mappings are orientation preserving.

This paper mainly deals with the Jacobians of quasiregular mappings. The

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Jacobian has played a crucial role in many fields such as the calculus of variations, geometric function theory, nonlinear elasticity and the geometric theory of nonlinear PDEs. The present knowledge of Jacobians will tell us something more, such as the regularity and topological behavior of the related mappings. Higher integrability properties of the Jacobian first showed up in [3], where Gehring invented the well-known reverse Hölder inequalities (Gehring's Lemma) and used these inequalities to establish the $L^{1+\varepsilon}$-integrability of the Jacobian of a quasiconformal mapping, $\varepsilon > 0$. In 1990, Müller [8] proved that the Jacobian of an orientation preserving mapping $f \in W^{1,n}(\Omega, \mathrm{R}^n)$ belongs to the Zygmund class $L \log L(E)$ for each compact $E \subset \Omega$. This result can be stated as the following proposition (See also [4]).

Proposition 1.1. *Let $\Omega \subset R^n$ be a bounded domain and f be an orientation preserving mapping of Sobolev class $W^{1,n}(\Omega, R^n)$, then*

$$\int_E J(x,f) \log \left(e + \frac{J(x,f)}{\int_\Omega J(y,f)dy} \right) dx \le C \int_\Omega |Df(x)|^n dx, \tag{1.1}$$

for each compact $E \subset \Omega$, where C is a constant.

Iwaniec and Sbordone showed in [5] the following estimate for the Jacobians of mappings in the Orlicz-Sobolev class over a compact subset of Ω in 1992.

Proposition 1.2. *Let $\Omega \subset R^n$ be a bounded domain and $f : \Omega \to R^n$ be an orientation preserving mapping of Orlicz-Sobolev class $D^n \log^{-1}(\Omega, R^n)$. Then, the Jacobian of f is locally integrable. Moreover, for each compact $E \subset \Omega$, the following estimate holds*

$$\int_E J(x,f)dx \le C \left(\int_\Omega \frac{|Df(x)|^n}{\log\left(e + \frac{|Df(x)|}{|Df|_\Omega} \right)} dx \right), \tag{1.2}$$

where $|Df|_\Omega$ denotes the integral mean of $|Df|$ over Ω and C is a constant.

We denote by B a ball and $\sigma B(\sigma > 0)$ the ball with the same center as B and with $\mathrm{diam}(\sigma B) = \sigma \mathrm{diam}(B)$. Also, $\bar{B}$ denotes the closure of B. For measurable set $E \subset \mathrm{R}^n$, we write $|E|$ or $m(E)$ for the n-dimensional Lebesgue measure of E. We call w a weight if $w \in L^1_{loc}(\mathrm{R}^n)$ and $w > 0$, a.e.. For a function u, we denote the μ-average over a set E by

$$u_{E,\mu} = \frac{1}{\mu(E)} \int_E u d\mu.$$

If μ is the n-dimensional Lebesgue measure, we write $u_E = u_{E,\mu}$. We say that a weight w satisfies the A_r-condition, where $r > 1$, and write $w \in A_r(\Omega)$, if

$$\sup_B \left(\frac{1}{|B|} \int_B w dx \right) \left(\frac{1}{|B|} \int_B w^{1/(1-r)} dx \right)^{r-1} < \infty,$$

where the supremum is taken over all balls $B \subset \Omega$. The weight w is said to be in $A_1(\Omega)$ if there is a constant C, such that

$$\frac{1}{|B|}\int_B w(x)dx \le C\text{essinf}_B w(x),$$

for all balls $B \subset \Omega$.

Note that the inequalities (1.1) and (1.2) are local estimates. A natural question now arises: does there exist a domain Ω such that the above estimates (1.1) and (1.2) become global weighted estimates on Ω? In paper [6], Iwaniec and Ding proved that a John domain or an $L^\varphi(\mu)$-domain Ω made the estimates (1.1) and (1.2) became global ones on Ω. In this paper, we continue to consider this question and obtain some global estimates for the Jacobians of quasiregular mappings in $L^\varphi(\mu)$-domains and John domains.

In the following, $C = C(*, \cdots, *)$ denotes a constant depending only on the quantities appearing in parentheses. In a given context, the same letter C will be used to denote different constants depending on the same set of arguments.

2. Global estimate in $L^\varphi(\mu)$-domains

Definition 2.1([6]). Let φ be an increasing convex function on $[0,\infty)$ with $\varphi(0) = 0$. We call a proper subdomain $\Omega \subset \mathrm{R}^n$ an $L^\varphi(\mu)$-domain, if $\mu(\Omega) < \infty$ and there exists a constant C, such that

$$\int_\Omega \varphi(\tau|u - u_{B_0,\mu}|)d\mu \le C \sup_{4B\subset\Omega}\int_B \varphi(\sigma|u - u_{B,\mu}|)d\mu, \tag{2.1}$$

for some ball $B_0 \subset \Omega$ and all u such that $\varphi(|u|) \in L^1_{loc}(\Omega;\mu)$, where the measure μ is defined by $d\mu = w(x)dx$, $w(x)$ is a weight and τ, σ are constants with $0 < \tau < 1, 0 < \sigma < 1$, and the supremum is taken over all balls $B \subset \Omega$ with $4B \subset \Omega$.

The factor 4 in Def. 2.1 is for convenience and in fact these domains are independent of this expansion factor, see [2]. In order to prove the global estimate for the Jacobians of quasiregular mappings in $L^\varphi(\mu)$-domains, we need a lemma from [6].

Lemma 2.2. *Let φ be a strictly increasing convex function on $[0,\infty)$ with $\varphi(0) = 0$. Assume that u is a function in $D \subset R^n$ such that $\varphi(|u|) \in L^1(D;\mu)$ and $\mu(\{x \in D : |u - c| > 0\}) > 0$ for any constant c. Then, for any positive constant a, we have*

$$\int_D \varphi\left(\frac{a}{2}|u - u_{D,\mu}|\right)d\mu \le \int_D \varphi(a|u|)d\mu \le C\int_D \varphi\left(2a|u - c|\right)d\mu, \tag{2.2}$$

where C is a positive constant.

The following lemma shows us the weakly reverse Hölder inequality for the Jacobians of quasiregular mappings.

Lemma 2.3. *Let $f : \Omega \to R^n$ be a K-quasiregular mapping of Sobolev class $W^{1,n}(\Omega, R^n)$. Then, for any ball $B \subset 2B \subset \Omega$,*

$$\left(\frac{1}{|B|}\int_B J^s(x,f)dx\right)^{\frac{1}{s}} \le C\left(\frac{1}{|2B|}\int_{2B} J(x,f)dx\right) \le C\left(\frac{1}{|2B|}\int_{2B}|Df(x)|^n dx\right), \tag{2.3}$$

where $C = C(n,K) > 0$ and $s > 1$ are some constants.

Remark. The factor 2 in (2.3) is not essential. In fact, one can derive by the same method as in the proof of Lemma 2.3 that

$$\left(\frac{1}{|B|}\int_B J^s(x,f)dx\right)^{\frac{1}{s}} \le C\left(\frac{1}{|\sigma B|}\int_{\sigma B} J(x,f)dx\right) \le C\left(\frac{1}{|\sigma B|}\int_{\sigma B}|Df(x)|^n dx\right) \tag{2.3´}$$

for any $\sigma > 1$, where $C = C(n,K,\sigma) > 0$ and $s > 1$ are some constants.

Proof. Since f is a K-quasiregular mapping of Sobolev class $W^{1,n}(\Omega, \mathrm{R}^n)$, we obtain from [7, P355] that

$$\frac{1}{|B|}\int_B |Df|^n dx \le C(n,K)\left(\frac{1}{|2B|}\int_{2B}|Df|^{\frac{n^2}{n+1}}dx\right)^{\frac{n+1}{n}}.$$

This is a reverse Hölder's inequality for $|Df|$. By Gehring's lemma, there exists $\varepsilon > 0$, such that

$$\frac{1}{|B|}\int_B |Df|^{n+\varepsilon} dx \le C(n,K)\left(\frac{1}{|2B|}\int_{2B}|Df|^n dx\right)^{\frac{n+\varepsilon}{n}}. \tag{2.4}$$

The distortion inequality

$$J(x,f) \le |Df|^n \le KJ(x,f), \quad \text{a.e. } \Omega \tag{2.5}$$

together with (2.4) yields

$$\begin{aligned}\frac{1}{|B|}\int_B J^{\frac{n+\varepsilon}{n}}(x,f)dx &\le \frac{1}{|B|}\int_B |Df|^{n+\varepsilon}dx \\ &\le C(n,K)\left(\frac{1}{|2B|}\int_{2B}|Df|^n dx\right)^{\frac{n+\varepsilon}{n}} \le C(n,K)\left(\frac{1}{|2B|}\int_{2B}J(x,f)dx\right)^{\frac{n+\varepsilon}{n}},\end{aligned}$$

which implies the first inequality of (2.3) with $s = \frac{n+\varepsilon}{n} > 1$. The second inequality of (2.3) is a simple consequence of (2.5). This completes the proof of Lemma 2.3.□

With the above lemmas in hand, we can now prove the global estimates for the Jacobian of a quasiregular mapping in $L^\varphi(\mu)$-domains, which can be considered as a generalization of [6, Theorem 2.10].

Theorem 2.4. *Let Ω be an $L^{\varphi}(\mu)$-domain with*

$$\varphi(t) = t\log^{1-\alpha}\left(e + \frac{t}{\int_\Omega J(y,f)dy}\right), \quad 0<\alpha<1,$$

and $f:\Omega \to R^n$ be a quasiregular mapping of Sobolev class $W^{1,n}(\Omega,R^n)$. Then, there is a constant C, such that

$$\int_\Omega J(x,f)\log^{1-\alpha}\left(e+\frac{J(x,f)}{J_\Omega}\right)w^{\alpha(s-1)/s}dx \le C\int_\Omega |Df(x)|^n dx,$$

where α is a constant with $0<\alpha<1$, s is the exponent in the reverse Hölder inequality (2.3) and $w(x)\in A_1(\Omega)$ is a weight.

Proof. Similar to the proof of Theorem 2.10 in [6], we can easily derive from Lemma 2.2 with $a=1, c=J_{B_{0,\mu}}$, the fact that $\varphi(t)$ is a convex function, (2.1) and Hölder's inequality that

$$\begin{aligned}
(2.6) \qquad & \int_\Omega J(x,f)\log^{1-\alpha}\left(e+\frac{J(x,f)}{J_\Omega}\right)d\mu \\
\le & C\int_\Omega J(x,f)\log^{1-\alpha}\left(e+\frac{J(x,f)}{\int_\Omega J(y,f)dy}\right)d\mu \\
\le & C\int_\Omega |J(x,f)-J_{B_{0,\mu}}|\log^{1-\alpha}\left(e+\frac{|J(x,f)-J_{B_{0,\mu}}|}{\int_\Omega J(y,f)}\right)d\mu \\
\le & C\sup_{B\subset\Omega}\int_B J(x,f)\log^{1-\alpha}\left(e+\frac{J(x,f)}{\int_\Omega J(y,f)dy}\right)d\mu \\
\le & C\sup_{B\subset\Omega}\left(\int_B J(x,f)\log\left(e+\frac{J(x,f)}{\int_\Omega J(y,f)dy}\right)dx\right)^{1-\alpha}\left(\int_B J(x,f)w^{(s-1)/s}dx\right)^{\alpha}.
\end{aligned}$$

By Proposition 1.1, one can see that

$$\begin{aligned}
(2.7) \qquad & \int_B J(x,f)\log\left(e+\frac{J(x,f)}{\int_\Omega J(y,f)dy}\right)dx \\
\le & \int_{\overline{B}} J(x,f)\log\left(e+\frac{J(x,f)}{\int_\Omega J(y,f)dy}\right)dx \le C\int_\Omega |Df(x)|^n dx.
\end{aligned}$$

Note that

$$\int_B J(x,f)w^{(s-1)/s}dx \le \left(\int_B J^s(x,f)dx\right)^{1/s}\left(\int_B wdx\right)^{(s-1)/s}.$$

This inequality combined with (2.3) yields

$$\int_B J(x,f)w^{(s-1)/s}dx \le C|B|^{1/s}\left(\frac{1}{|2B|}\int_{2B}|Df|^n dx\right)\left(\int_B wdx\right)^{(s-1)/s}.$$

This inequality is equivalent to

$$\left(\int_B J(x,f)w^{(s-1)/s}dx\right)^{\alpha} \leq C|B|^{\alpha(1-s)/s}\left(\int_{2B}|Df|^n dx\right)^{\alpha}\left(\int_B wdx\right)^{\alpha(s-1)/s}. \tag{2.8}$$

for any ball B with $4B \subset \Omega$. Substituting (2.7) and (2.8) into (2.6) and using the condition that $w(x) \in A_1(\Omega)$, we obtain

$$\begin{aligned}
&\int_\Omega J(x,f)\log^{1-\alpha}\left(e+\frac{J(x,f)}{\int_\Omega J(y,f)dy}\right)w^{\alpha(s-1)/s}dx \\
&\leq C\sup_{B\subset\Omega}\left(\int_\Omega |Df(x)|^n dx\right)\left(\frac{1}{|B|}\int_B wdx\right)^{\alpha(s-1)/s} \\
&\leq C\int_\Omega |Df(x)|^n dx.
\end{aligned}$$

The proof of Theorem 2.4 has been completed. □

3. Global estimate in John domains

For the definition of John domain we refer to the paper [6]. Note that John domains are $L^\varphi(\mu)$-domains [1]. An important property for John domain is the covering Lemma [9].

Lemma 3.1. *Each John domain Ω has a modified Whitney cover of cubes $\nu = \{Q_i\}$, such that*

$$\bigcup_i Q_i = \Omega, \quad \text{and} \quad \sum_{Q\in\nu}\chi_{\sqrt{5/4}Q}(x) \leq N\chi_\Omega(x), \tag{3.1}$$

for all $x \in R^n$ and some $N > 1$, where χ_E is the characteristic function for a set E. Moreover, if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube does not need to be a member of ν) in $Q_i \cap Q_j$, such that $Q_i \cup Q_j \subset NR$. Also, there is a distinguished cube $Q_0 \in \nu$ which can be connected with every cube $Q \in \nu$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from ν and such that $Q \subset \rho Q_i, i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.

Theorem 3.2. *Let Ω be a bounded John domain and $f : \Omega \to R^n$ be a quasiregular mapping. Then, there is a constant C, such that*

$$\int_\Omega J^{\alpha s}(x,f)w^{1-\alpha}dx \leq C\left(\int_\Omega \frac{|Df(x)|^n}{\log\left(e+\frac{|Df(x)|}{|Df|_\Omega}\right)}dx\right)^{\alpha s}, \tag{3.2}$$

where $s > 1$ comes from Lemma 2.3 and α is a constant with $0 < \alpha < 1$ and the weight $w(x) \in A_1(\Omega)$.

Proof. Since f is quasiregular, then $|Df| \in L^n(\Omega)$. This implies that $f \in$

$D^n \log^{-1} D(\Omega, \mathrm{R}^n)$ and then we can use the result of Proposition 1.2. For any cube $Q \subset \Omega$, using Hölder's inequality, (2.3)' with $\sigma = \sqrt{\frac{5}{4}}$, and the condition $w(x) \in A_1(\Omega)$, one can derive that

$$\int_Q J^{\alpha s}(x,f) w^{1-\alpha} dx \le \left(\int_Q J^s(x,f)dx\right)^{\alpha} \left(\int_Q w dx\right)^{1-\alpha} \le C\left(\int_{\sqrt{\frac{5}{4}}Q} J(x,f)dx\right)^{\alpha s}. \tag{3.3}$$

Thus

$$\int_\Omega J^{\alpha s}(x,f) w^{1-\alpha} dx \le \sum_{Q\in\nu}\left(\int_Q J^{\alpha s}(x,f) w^{1-\alpha} dx\right)$$
$$\le C\sum_{Q\in\nu}\left(\int_{\sqrt{\frac{5}{4}}Q} J(x,f)dx\right)^{\alpha s}$$
$$\le C\sum_{Q\in\nu}\left(\int_{\sqrt{\frac{5}{4}}Q} \frac{|Df(x)|^n}{\log(e+|Df(x)|/|Df|_\Omega)}dx\right)^{\alpha s}$$
$$\le CN\left(\int_\Omega \frac{|Df(x)|^n}{\log(e+|Df(x)|/|Df|_\Omega)}dx\right)^{\alpha s}.$$

The proof of Theorem 3.2 has been completed. □

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